33
(Axial-)vector two-point functions

The Wilson coefficients for the OPE of these correlators were first calculated by SVZ to leading order in α_s and in the quark mass terms. Calculations of the coefficients beyond the leading order exist in the literature. These results are collected here.

We shall be concerned with the two-point correlator for the vector $V^\mu_{ij} = \bar\psi_i \gamma^\mu \psi_j$ and axial-vector currents $A^\mu_{ij} = \bar\psi_i \gamma^\mu \gamma_5 \psi_j$:

$$\Pi^{\mu\nu}_{ij,V}(q) \equiv i \int d^4x \; e^{iqx} \langle 0|\mathcal{T} V^\mu(x)^j_i \left(V^\nu(0)^j_i\right)^\dagger |0\rangle \; ,$$
$$\Pi^{\mu\nu}_{ij,A}(q) \equiv i \int d^4x \; e^{iqx} \langle 0|\mathcal{T} A^\mu(x)^j_i \left(A^\nu(0)^j_i\right)^\dagger |0\rangle \; . \tag{33.1}$$

Here the indices $i,\ j$ correspond to the light quark flavours $u,\ d,\ s$. The vector (V) and axial-vector (A) correlators have the Lorentz decomposition:

$$\Pi^{\mu\nu}_{ij,V/A} = -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi^{(1)}_{ij,V/A}\left(q^2, m_i^2, m_j^2\right) + q^\mu q^\nu \Pi^{(0)}_{ij,V/A}\left(q^2, m_i^2, m_j^2\right) \; , \tag{33.2}$$

where m_i is the mass of the quark i; $\Pi^{(J)}_{ij}$ is the correlator associated to the hadrons of spin $J = 0, 1$. The (pseudo)scalar correlators $\Psi_{(5)}(q^2)^i_j$ is related to $\Pi^{(0)}_{ij}$ via the non-anomalous Ward identity in Eq. (2.17). It will be convenient to introduce the notation:

$$\Pi^{(1+0)}_{ij} \equiv \Pi^{(1)}_{ij} + \Pi^{(0)}_{ij} \; . \tag{33.3}$$

The result of the axial-vector current can be deduced from the one of the vector by the change m_j into $-m_j$ or, equivalently, by the change m_- into m_+ and vice-versa.

33.1 Exact two-loop perturbative expression in the $\overline{MS}$ scheme

The complete two-loop result for the vector correlator is:

$$\begin{aligned}\Pi^{(1+0)}_{ij,V} \equiv & -\frac{1}{3}\left[1 + \left(\frac{\alpha_s}{\pi}\right)\frac{15}{4}\right] + PK \\ & + \alpha l_i + \beta l_j + 2(\alpha - \beta)(\alpha Z_i - \beta Z_j) \\ & + \frac{2}{3}\left(\frac{\alpha_s}{\pi}\right)\left[\frac{1}{2}PL + \alpha l_i(1 + 2l_i) + \beta l_j(1 + 2l_j)\right.\end{aligned}$$

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$$
\begin{aligned}
& -\frac{1}{4}(1+2N_i+2N_j)\left(1+6\frac{(m_i-m_j)^2}{q^2}\right) \\
& +x_i f_i^2 + x_j f_j^2 + \frac{1}{2}(N_i-N_j)^2 \\
& -(\alpha-\beta)(G(x_i)-G(x_j)) - (3-2(\alpha+\beta))K^2 \\
& -\left(\frac{1}{2}+\alpha(2+l_i)+\beta(2+l_j)\right)\Bigg] , \qquad (33.4)
\end{aligned}
$$

with:

$$
\begin{aligned}
\alpha &\equiv -m_i^2/q^2 , \\
\beta &\equiv -m_j^2/q^2 , \\
P &\equiv 1-\alpha-\beta-2(\alpha-\beta) , \\
N_i &\equiv \alpha(1+f_i)(1+x_j f_j) , \\
N_j &\equiv \beta(1+f_j)(1+x_i f_i) , \\
Z_i &\equiv 1+l_i+\frac{2}{3}\left(\frac{\alpha_s}{\pi}\right)\left(5+5l_i+3l_i^2\right) , \\
G(x) &\equiv xF'(x) = \int_o^x dy \left(\frac{\log y}{1-y}\right)^2 = \sum_{n=1}^{\infty}[(1-n\log x)^2+1]x^n/n^2 , \qquad (33.5)
\end{aligned}
$$

where K has been defined in Eq. (32.5). The log-mass terms appearing there should be cancelled once one introduces the contributions of non-normal ordered condensates.

33.2 Three-loop expression including the m^2-terms

Including the m^2-term to order α_s^2, the correlator reads:

$$
\begin{aligned}
& (16\pi^2)\Pi_V^{(0+1)} \\
& = +\left[\frac{20}{3}+4\ln\frac{\nu^2}{-q^2}\right] \\
& \quad +\frac{\alpha_s}{\pi}\left[\frac{55}{3}-16\zeta(3)+4\ln\frac{\nu^2}{-q^2}\right] \\
& \quad +\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{41927}{216}-\frac{1658}{9}\zeta(3)+\frac{100}{3}\zeta(5)-\frac{3701}{324}n_f+\frac{76}{9}\zeta(3)n_f\right. \\
& \qquad +\frac{365}{6}\ln\frac{\nu^2}{-q^2}-44\zeta(3)\ln\frac{\nu^2}{-q^2}-\frac{11}{3}n_f\ln\frac{\nu^2}{-q^2} \\
& \qquad \left.+\frac{8}{3}\zeta(3)n_f\ln\frac{\nu^2}{-q^2}+\frac{11}{2}\ln^2\frac{\nu^2}{-q^2}-\frac{1}{3}n_f\ln^2\frac{\nu^2}{-q^2}\right]
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{m_-^2}{-q^2}\left[-6+\frac{\alpha_s}{\pi}\left(-12-12\ln\frac{\nu^2}{-q^2}\right)\right]\\
&+\frac{m_+^2}{-q^2}\left[-6+\frac{\alpha_s}{\pi}\left(-16-12\ln\frac{\nu^2}{-q^2}\right)\right]\\
&+\left(\frac{\alpha_s}{\pi}\right)^2\frac{m_-^2}{-q^2}\left[-\frac{4681}{24}-34\,\zeta(3)+115\,\zeta(5)+\frac{55}{12}n_f+\frac{8}{3}\zeta(3)n_f\right.\\
&\qquad\left.-\frac{215}{2}\ln\frac{\nu^2}{-q^2}+\frac{11}{3}n_f\ln\frac{\nu^2}{-q^2}-\frac{57}{2}\ln^2\frac{\nu^2}{-q^2}+n_f\ln^2\frac{\nu^2}{-q^2}\right]\\
&+\left(\frac{\alpha_s}{\pi}\right)^2\frac{m_+^2}{-q^2}\left[-\frac{19691}{72}-\frac{124}{9}\zeta(3)+\frac{1045}{9}\zeta(5)+\frac{95}{12}n_f-\frac{253}{2}\ln\frac{\nu^2}{-q^2}\right.\\
&\qquad\left.+\frac{13}{3}n_f\ln\frac{\nu^2}{-q^2}-\frac{57}{2}\ln^2\frac{\nu^2}{-q^2}+n_f\ln^2\frac{\nu^2}{-q^2}\right]\\
&+\left(\frac{\alpha_s}{\pi}\right)^2\frac{\sum_f m_f^2}{-q^2}\left[\frac{128}{3}-32\,\zeta(3)\right]. \qquad (33.6)
\end{aligned}
$$

$$
\begin{aligned}
&(16\pi^2)\Pi_A^{(0+1)}\\
&=+\left[\frac{20}{3}+4\ln\frac{\nu^2}{-q^2}\right]\\
&+\frac{\alpha_s}{\pi}\left[\frac{55}{3}-16\,\zeta(3)+4\ln\frac{\nu^2}{-q^2}\right]\\
&+\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{34525}{216}-\frac{1430}{9}\zeta(3)+\frac{100}{3}\zeta(5)+\frac{299}{6}\ln\frac{\nu^2}{-q^2}-36\,\zeta(3)\ln\frac{\nu^2}{-q^2}\right.\\
&\qquad\left.+\frac{9}{2}\ln^2\frac{\nu^2}{-q^2}\right]\\
&+\frac{m_-^2}{-q^2}\left[-6+\frac{\alpha_s}{\pi}\left(-12-12\ln\frac{\nu^2}{-q^2}\right)\right]\\
&+\frac{m_+^2}{-q^2}\left[-6+\frac{\alpha_s}{\pi}\left(-16-12\ln\frac{\nu^2}{-q^2}\right)\right]\\
&+\frac{m_-^2}{-q^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{4351}{24}-26\,\zeta(3)+115\,\zeta(5)-\frac{193}{2}\ln\frac{\nu^2}{-q^2}-\frac{51}{2}\ln^2\frac{\nu^2}{-q^2}\right]\\
&+\frac{m_+^2}{-q^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{17981}{72}-\frac{124}{9}\zeta(3)+\frac{1045}{9}\zeta(5)-\frac{227}{2}\ln\frac{\nu^2}{-q^2}-\frac{51}{2}\ln^2\frac{\nu^2}{-q^2}\right]\\
&+\sum_f m_f^2\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{128}{3}-32\,\zeta(3)\right]. \qquad (33.7)
\end{aligned}
$$

33.3 Dimension-four

The dynamic operators of dimension-four are the gluon and quark condensates. Let us start by giving the contributions coming from the normal ordered condensates, which are

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obtained from a direct calculation of the Feynman diagrams within the Wick's theorem. One obtains:

$$\left[\Pi^{(1)}_{ij,V/A}\right]^{(D=4)}_{\psi} = -\frac{1}{3m_i q^2}\langle\ :\bar\psi_i\psi_i:\ \rangle\left[1+2\frac{\left(m_j^2-m_i^2\right)}{q^2} - \frac{\left[q^2+m_j^2+m_i^2-2\left(m_j^2-m_i^2\right)^2/q^2\right]}{q^2-m_j^2+m_i^2}f(z_i)\right] + (i\longleftrightarrow j)\,, \tag{33.8}$$

$$\left[\Pi^{(1)}_{ij,V/A}\right]^{(D=4)}_{G} = \frac{1}{48\pi}\langle :\alpha_s GG:\rangle\frac{1}{q^4_\pm}\left[\frac{3(1+u^2)(1-u^2)^2}{2u^5}\log\frac{u+1}{u-1} - \frac{3u^4+2u^2+3}{u^4}\right], \tag{33.9}$$

where the result for the axial-vector can be obtained by the additionnal change of u into $1/u$.

Let us now use the previous results and truncate the series to the $D=4$ contributions but including radiative corrections. In so doing, we shall consider these quark and gluon operators defined previously in the $\overline{MS}$ scheme. The remaining $D=4$ operators are product of the running quark masses. In terms of the scale invariant condensates defined previously, the contributions to the correlators are:

$$\begin{aligned}
&Q^4\left[\Pi^{(1+0)}_{ij,V/A}(-Q^2)\right]^{(D=4)} \\
&\quad = \frac{1}{12}\left[1-\frac{11}{18}\left(\frac{\alpha_s}{\pi}\right)(Q)\right]\left\langle\frac{\alpha_s}{\pi}GG\right\rangle \\
&\qquad + \left[1-\left(\frac{\alpha_s}{\pi}\right)(Q) - \frac{13}{3}\left(\frac{\alpha_s}{\pi}\right)^2(Q)\right]\langle m_i\bar\psi_i\psi_i + m_j\bar\psi_j\psi_j\rangle \\
&\qquad \pm \left[\frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)(Q) + \frac{59}{6}\left(\frac{\alpha_s}{\pi}\right)^2(Q)\right]\langle m_j\bar\psi_i\psi_i + m_i\bar\psi_j\psi_j\rangle \\
&\qquad + \left[\frac{4}{27}\left(\frac{\alpha_s}{\pi}\right)(Q) + \left(-\frac{257}{486}+\frac{4}{3}\zeta(3)\right)\left(\frac{\alpha_s}{\pi}\right)^2(Q)\right]\sum_k\langle m_k\bar\psi_k\psi_k\rangle \\
&\qquad + \frac{3}{2\pi^2}\left[1+\left(\frac{76}{9}-\frac{4}{3}\zeta(3)\right)\left(\frac{\alpha_s}{\pi}\right)^2(Q)\right]m_i^2(Q)m_j^2(Q) \\
&\qquad + \frac{1}{4\pi^2}\left[-\frac{12}{7}\left(\frac{\alpha_s}{\pi}\right)^{-1}(Q)+1\right]\left[m_i^4(Q)+m_j^4(Q)\right] \\
&\qquad \mp \frac{4}{7\pi^2}m_i(Q)m_j(Q)\left[m_i^2(Q)+m_j^2(Q)\right] \\
&\qquad - \frac{1}{28\pi^2}\left[1-\left(\frac{65}{6}-16\zeta(3)\right)\left(\frac{\alpha_s}{\pi}\right)(Q)\right]\sum_k m_k^4(Q)\,,
\end{aligned} \tag{33.10}$$

and:

$$\begin{aligned}
&Q^4\left[\Pi^{(0)}_{ij,V/A}(-Q^2)\right]^{(D=4)} \\
&\quad = \langle (m_i\mp m_j)(\bar\psi_i\psi_i\mp\bar\psi_j\psi_j)\rangle
\end{aligned}$$

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$$+\frac{1}{4\pi^2}\left[-\frac{12}{7}\left(\frac{\alpha_s}{\pi}\right)^{-1}(Q)+\frac{11}{14}\right][m_i(Q)\mp m_j(Q)]\big[m_i^3(Q)\mp m_j^3(Q)\big]$$
$$\mp\frac{3}{4\pi^2}m_i(Q)m_j(Q)[m_i(Q)\mp m_j(Q)]^2\,. \tag{33.11}$$

33.4 Dimension-five

This contribution is due to the mixed quark-gluon condensate. It has been evaluated to all orders in the quark mass. In terms of the normal ordered condensate, it reads:

$$\begin{aligned}\big[\Pi^{(1)}_{ij,V/A}(-Q^2)\big]^{(D=5)}_{\text{mix}} &= -\langle\ :\bar\psi_i G\psi_i:\ \rangle\frac{1}{3m_i^3q^4q_\pm^2q_\pm^2}\\ &\times\bigg[[q^2+2m_j(m_j\mp m_i)]\big(q^2-m_j^2\big)^2-m_i^2q^2\big(q^2+m_j^2\big)\\ &-2m_jm_i^2(m_j\mp m_i)\big(2m_j^2-m_i^2\big)\\ &-\frac{P(q^2,m_i,m_j)}{q^2-m_j^2+m_i^2}f(z_i)\bigg]+(i\longleftrightarrow j)\,,\end{aligned} \tag{33.12}$$

with:

$$\begin{aligned}P(q^2,m_i,m_j) &= [q^2+2m_j(m_j\mp m_i)]\big(q^2-m_j^2\big)^3\\ &-m_j^2m_i^2q^2\big(4m_j^2\mp 6m_jm_i+m_i^2\big)\\ &-m_i^2q^4\big(q^2+m_j^2\big)\\ &+2m_jm_i^2(m_j\mp m_i)\big(3m_j^4-3m_j^2m_i^2+m_i^4\big)\,.\end{aligned} \tag{33.13}$$

33.5 Dimension-six

Here we shall consider the contributions which do not vanish for massless quarks. Then we shall neglect the triple gluon condensate contribution $g^3\langle f_{abc}G^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu}\rangle$, where the coefficient vanishes in the chiral limit. Therefore, we have:

$$\begin{aligned}&Q^6\big[\Pi^{(1+0)}_{ij,V/A}(-Q^2)\big]^{(D=6)}\\ &=-8\pi^2\left[1+\left(\frac{431}{96}-\frac{9}{8}L\right)\left(\frac{\alpha_s}{\pi}\right)(\nu)\right]\left(\frac{\alpha_s}{\pi}\right)(\nu)\left\langle\bar\psi_i\gamma_\mu\begin{pmatrix}\gamma_5\\1\end{pmatrix}T_a\psi_j\bar\psi_j\gamma_\mu\begin{pmatrix}\gamma_5\\1\end{pmatrix}T_a\psi_i(\nu)\right\rangle\\ &\quad+\frac{5\pi^2}{4}(3+4L)\left(\frac{\alpha_s}{\pi}\right)^2(\nu)\left\langle\bar\psi_i\gamma_\mu\begin{pmatrix}1\\\gamma_5\end{pmatrix}T_a\psi_j\bar\psi_j\gamma_\mu\begin{pmatrix}1\\\gamma_5\end{pmatrix}T_a\psi_i(\nu)\right\rangle\\ &\quad+\frac{2\pi^2}{3}(3+4L)\left(\frac{\alpha_s}{\pi}\right)^2(\nu)\left\langle\bar\psi_i\gamma_\mu\begin{pmatrix}1\\\gamma_5\end{pmatrix}\psi_j\bar\psi_j\gamma_\mu\begin{pmatrix}1\\\gamma_5\end{pmatrix}\psi_i(\nu)\right\rangle\\ &\quad-\frac{8\pi^2}{9}\left[1+\left(\frac{107}{48}-\frac{95}{72}L\right)\left(\frac{\alpha_s}{\pi}\right)(\nu)\right]\left(\frac{\alpha_s}{\pi}\right)(\nu)\\ &\quad\times\sum_k\langle(\bar\psi_i\gamma_\mu T_a\psi_i+\bar\psi_j\gamma_\mu T_a\psi_j)\bar\psi_k\gamma^\mu T_a\psi_k(\nu)\rangle\end{aligned}$$

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$$+\frac{5\pi^2}{54}(-7+6L)\left(\frac{\alpha_s}{\pi}\right)^2(\nu)\sum_k\langle(\bar\psi_i\gamma_\mu\gamma_5 T_a\psi_i+\bar\psi_j\gamma_\mu\gamma_5 T_a\psi_j)\bar\psi_k\gamma^\mu\gamma_5 T_a\psi_k(\nu)\rangle$$
$$+\frac{4\pi^2}{81}(-7+6L)\left(\frac{\alpha_s}{\pi}\right)^2(\nu)\sum_k\langle(\bar\psi_i\gamma_\mu\gamma_5\psi_i+\bar\psi_j\gamma_\mu\gamma_5\psi_j)\bar\psi_k\gamma^\mu\gamma_5\psi_k(\nu)\rangle$$
$$+\frac{4\pi^2}{81}(1+6L)\left(\frac{\alpha_s}{\pi}\right)^2(\nu)\sum_{k,l}\langle\bar\psi_k\gamma_\mu T_a\psi_k\bar\psi_l\gamma^\mu T_a\psi_l(\nu)\rangle\ . \tag{33.14}$$

where $L \equiv \log(Q^2/\nu^2)$. The upper component of $\binom{1}{\gamma_5}$ or $\binom{\gamma_5}{1}$ is for the vector(V) correlator, while the lower one is for the axial-vector (A).

33.6 Vector spectral function to higher order

33.6.1 Complete two-loop perturbative expression of the spectral function

In the case where one of the quark mass is zero, the spectral function of the vector current reads:

$$\begin{aligned}\text{Im}\Pi^{(1)}(t) = {} & \frac{(2+x)}{3m^2t}\text{Im}\Psi_5(t)\\ & -\frac{1}{6\pi}\left(\frac{\alpha_s}{\pi}\right)\left[(3+x)(1-x)^3\log\frac{x}{1-x}\right.\\ & \left.+2x\log x+(3-x^2)(1-x)\right],\end{aligned} \tag{33.15}$$

where $x \equiv m^2/t$, and becomes:

$$\begin{aligned}\frac{1}{\pi}\text{Im}\Pi^{(1)}(t) = \frac{1}{8\pi^2}&\left[(2+x)\left[1+\frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left[\frac{13}{4}+2\log x+\log x\log(1-x)\right.\right.\right.\\ & \left.\left.+\frac{3}{2}\log\frac{x}{1-x}-\log(1-x)-x\log\frac{x}{1-x}-\frac{x}{1-x}\log x\right]\right]\\ & +\frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left[-(3+x)(1-x)\log\frac{x}{1-x}-\frac{2x}{(1-x)^2}\log x\right.\\ & \left.\left.-5-2x-\frac{2x}{1-x}\right]\right]\theta(t-m^2)\ .\end{aligned} \tag{33.16}$$

For the case of the electromagnetic current, one has the well-known QED result, which is accurately reproduced by the Schwinger interpolating formula:

$$\frac{1}{\pi}\text{Im}\Pi^{(1)}(t) = \frac{1}{4\pi}v\left(\frac{3-v^2}{2}\right)\left[1+\frac{4}{3}\alpha_s f(v)\right]\theta(t-4m^2)\ , \tag{33.17}$$

where:

$$\begin{aligned}v &\equiv \sqrt{1-4x}\ ,\\ f(v) &\equiv \frac{\pi}{2v}-\frac{(3+v)}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\ .\end{aligned} \tag{33.18}$$

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33.6.2 Four-loop perturbative expression of the spectral function

The neutral vector spectral function can be related to the $e^+e^- \to$ hadrons total cross-section as:

$$R_{e^+e^-}(t) \equiv \frac{\sigma(e^+e^- \to \text{ hadrons}(\gamma))}{\sigma(e^+e^- \to \mu^+\mu^-(\gamma))} = 12\pi \,\text{Im}\Pi_{\text{em}}(t+i\epsilon) , \qquad (33.19)$$

where $\text{Im}\Pi_{\text{em}}$ is associated to the conserved electromagnetic current $J^\mu_{\text{em}} \equiv \sum_i Q_i \bar\psi_i \gamma^\mu \psi_i$ $(i = u, d, s, \ldots)$. However, the perturbative calculation has been done in the Euclidian region and corresponds to the D-function:

$$D(Q^2) \equiv -Q^2 \frac{d}{dQ^2} \Pi_{\text{em}}(Q^2) , \qquad (33.20)$$

which can be related to $R_{e^+e^-}$ through:

$$R(s) = \frac{1}{2i\pi} \int_{-s-i\epsilon}^{-s+i\epsilon} \frac{dQ^2}{Q^2} D(Q^2) , \qquad (33.21)$$

where it is necessary to transform the result into the physical region by taking into account the effects due to the analytic continuation of the terms of the type:

$$\log^n(-q^2/\nu^2) \to (\log(t/\nu^2) + i\pi)^n . \qquad (33.22)$$

The asymptotic four-loop expression reads:

$$(16\pi^2)\frac{1}{\pi}\text{Im}\Pi_{\text{em}}(t) = 3\left(\sum_i Q_i^2\right)\left[1 + \frac{\bar\alpha_s}{\pi} + F_3\left(\frac{\bar\alpha_s}{\pi}\right)^2 + F_4\left(\frac{\bar\alpha_s}{\pi}\right)^3\right] + \left(\sum_i Q_i\right)^2 F_4'\left(\frac{\bar\alpha_s}{\pi}\right)^3 , \qquad (33.23)$$

where $\bar\alpha_s$ is the running coupling evaluated at the scale t and:

$$\begin{aligned} F_3 &= 1.9857 - 0.1153n , \\ F_4 &= -6.6368 - 1.2001n - 0.0052n^2 , \\ F_4' &= -1.2395 . \end{aligned} \qquad (33.24)$$

The last term comes from the light-by-light diagrams specific for the neutral electromagnetic current. The expression of the D-function reads:

$$\begin{aligned}(16\pi^2)D(-Q^2) = 3\left(\sum_i Q_i^2\right)\Bigg[1 + \left(\frac{\alpha_s}{\pi}\right) + \left[F_3 + \frac{b_1}{2}L\right]\left(\frac{\alpha_s}{\pi}\right)^2 \\ + \left[F_4 + \left(F_3\beta_1 + \frac{\beta_2}{2}\right)L + \frac{\beta_1^2}{4}\left(L^2 + \frac{\pi^2}{3}\right)\right]\left(\frac{\alpha_s}{\pi}\right)^3\Bigg] \\ + \left(\sum_i Q_i\right)^2 F_4'\left(\frac{\alpha_s}{\pi}\right)^3 , \end{aligned} \qquad (33.25)$$

where the values of the β-function have been given in Table 11.1 and $L \equiv \ln(Q^2/\nu^2)$.

33.7 Heavy-light correlator

In the following, we give useful lowest order expressions in α_s when $m \equiv m_i \ll M \equiv m_j$ for the (axial-)vector current. The notations are the same as in previous section but differs from $\Pi_{V/A}$ given in [488]. They are related as:

$$\Pi^{(1)}_{V/A} = \Pi_{V/A} - \frac{(M \mp m)^2}{q^2}\Psi_{\mp} - \frac{(M \mp m)}{q^2}\Big[\langle \bar{Q}Q \mp \bar{q}q\rangle\Big], \tag{33.26a}$$

where $\Pi^{(1)}_{V/A}$ is the $g_{\mu\nu}$ coefficients in [488], $\Psi_{\mp}$ has been given in Section (32.8), and:

$$\Pi_{V/A} = \Pi_{V/A}\Big|_{pert} + \Pi_{V/A}\Big|_{\bar{\psi}\psi}\langle\bar{\psi}\psi\rangle + \Pi_{V/A}\Big|_{\bar{Q}Q}\langle\bar{Q}Q\rangle + \Pi_{V/A}\Big|_{G^2}\langle\alpha_s G^2\rangle +$$
$$\Pi_{V/A}\Big|_{\bar{\psi}G\psi}\langle\bar{\psi}\frac{\lambda_a}{2}\sigma^{\mu\nu}G^a_{\mu\nu}\psi\rangle + \Pi^{(1)}_{V/A}\Big|_{\bar{Q}GQ}\langle\bar{Q}\frac{\lambda_a}{2}\sigma^{\mu\nu}G^a_{\mu\nu}Q\rangle. \tag{33.26b}$$

The different contributions read:

$$\begin{aligned}
\Pi_{V/A}\Big|_{pert} &= \frac{3}{24\pi^2}\Bigg[\frac{10}{3}q^2 + 4M^2 - 4\frac{M^4}{q^2} + 2(2M^2 - 3q^2)\frac{M^4}{q^4}\ln\frac{M^2}{W} + 2q^2\ln\frac{\mu^2}{W} \\
&\quad +6m^2\left(1 + 2\frac{M^2}{q^2} - 2\frac{M^4}{q^4}\ln\frac{M^2}{W}\right) \\
&\quad -3\frac{m^4}{W^2}\Big[\frac{(2M^2-q^2)^2}{q^2} - 2(2M^2 - 3q^2)\frac{M^4}{q^4}\ln\frac{M^2}{W} + 2q^2\ln\frac{m^2}{W}\Big]\Bigg] \\
\Pi_{V/A}\Big|_{\bar{\psi}\psi} &= \left[\frac{mq^4}{W^2} + \frac{2m^3q^4(4M^2-q^2)}{3W^4}\right] \\
\Pi_{V/A}\Big|_{\bar{Q}Q} &= \left[M - \frac{2M^3}{3q^2} + \frac{2m^2M}{q^2}\right] \quad \text{for } -q^2 > M^2 \\
\Pi_{V/A}\Big|_{G^2} &= -\frac{q^4}{12\pi W^2}\left[1 + \frac{2m^2}{W^2}\left(q^2 + 7M^2 + 6M^2\ln\frac{mM}{W}\right)\right] \\
\Pi_{V/A}\Big|_{\bar{\psi}G\psi} &= -\frac{mM^2q^6}{W^4} \\
\Pi_{V/A}\Big|_{\bar{Q}GQ} &= \mp\frac{mM^2}{3W} \quad \text{for } -q^2 > M^2\,.
\end{aligned} \tag{33.27a}$$

with $W \equiv M^2 - q^2$. Introducing the renormalized condensates defined in Eq. (27.56), and using relations similar to Eq. (32.26), one can deduce:

$$\begin{aligned}
\bar{\Pi}_{V/A}\Big|_{pert} &= \frac{3}{24\pi^2}\Bigg[10q^2 + 18(M^2+m^2) - 9(3M^4 - 4M^2m^2 + 3m^4)\frac{1}{q^2} \\
&\quad -6\Big[q^2 - 3(M^4+m^4)\frac{1}{q^2}\Big]\log\frac{-q^2}{\mu^2}\Bigg],
\end{aligned}$$

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$$\bar{\Pi}_{V/A}\Big|_{\bar{\psi}\psi} = \Pi_{V/A}\Big|_{\bar{\psi}\psi} ,$$
$$\bar{\Pi}_{V/A}\Big|_{\bar{Q}Q} = \Pi_{V/A}\Big|_{\bar{Q}Q} ,$$
$$\bar{\Pi}_{V/A}\Big|_{G^2} = \frac{1}{12} - \frac{(M^2+m^2)}{18q^2} ,$$
$$\bar{\Pi}_{V/A}\Big|_{\bar{\psi}G\psi} = \Pi_{V/A}\Big|_{\bar{\psi}G\psi} , \tag{33.27b}$$

from which the expressions for $\Pi^{(1)}_{V/A}$ can be easily derived.

33.8 Beyond the SVZ expansion: tachyonic gluon contributions to the (axial-)vector and (pseudo)scalar correlators

Here, we shall give contributions coming from the dimension $D = 2$ operators induced by a tachyonic gluon mass. This contribution has been introduced in [161], where one expects that the gluon mass phenomenologically mimics the resummation of the QCD perturbative series due to renormalons.

33.8.1 Vector correlator

This effect can be systematically obtained from the Feynman diagram given in Fig. 30.1. The derivation of the results is explictly given in [161]. Here, we only quote these results which are consistent if one uses normal non-ordered condensates for the $D = 4$ contribution. To first order in α_s and expanding in λ^2, $m^2_{1,2}$ we obtain:

$$\begin{aligned}(16\pi^2)\Pi^{(1)}_V = &\left[\frac{20}{3} + 6\frac{m_-^2}{Q^2} - 6\frac{m_+^2}{Q^2} + 4l_{\mu Q} + 6\frac{m_-^2}{Q^2}l_{\mu Q}\right]\\
&+\frac{\alpha_s}{\pi}\left[\frac{55}{3} - 16\,\zeta(3) + \frac{107}{2}\frac{m_-^2}{Q^2} - 24\,\zeta(3)\frac{m_-^2}{Q^2} - 16\frac{m_+^2}{Q^2}\right.\\
&\left.+4l_{\mu Q} + 22\frac{m_-^2}{Q^2}l_{\mu Q} - 12\frac{m_+^2}{Q^2}l_{\mu Q} + 6\frac{m_-^2}{Q^2}l^2_{\mu Q}\right]\\
&+\frac{\alpha_s}{\pi}\frac{\lambda^2}{Q^2}\left[-\frac{128}{3} + 32\,\zeta(3) - \frac{76}{3}\frac{m_-^2}{Q^2} + 16\,\zeta(3)\frac{m_-^2}{Q^2}\right.\\
&\left.-8\frac{m_+^2}{Q^2} + 12\frac{m_-^2}{Q^2}l_{\mu Q} - 12\frac{m_+^2}{Q^2}l_{\mu Q}\right],\end{aligned} \tag{33.28}$$

The above result is in the $\overline{MS}$ scheme and the notations are as follows: $m_\pm = m_1 \pm m_2$ and $l_{\mu Q} = \log(\mu^2/Q^2)$. Note that the terms of order λ^2/Q^2 in Eq. (33.28) are μ independent and, thus, do not depend on the way the overall UV subtraction of the vector correlator is

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implemented. The quark mass-logs appearing in the $\lambda^2 m^2/Q^4$ terms have been absorbed after adding the contribution of the quark condensate to the correlators and its modified renormalization group invariant combination:

$$\langle \bar{\psi}_i \psi_i \rangle = \frac{3m_i^3}{4\pi^2}\left[1 + \ln\left(\frac{\mu^2}{m_i^2}\right) + 2\frac{\alpha_s}{\pi}\left(\ln^2\left(\frac{\mu^2}{m_i^2}\right) + \frac{5}{3}\ln\left(\frac{\mu^2}{m_i^2}\right) + \frac{5}{3}\right)\right] + \frac{m_i\lambda^2}{4\pi^2}\frac{\alpha_s}{\pi}\left(-5 + 6\ln\frac{\mu^2}{m_i^2}\right). \tag{33.29}$$

In the light-quark case relevant to the ρ-channels we can neglect the m^2 terms and $\Pi_\rho(M^2)$ simplifies greatly:

$$\frac{1}{\pi}\mathrm{Im}\,\Pi_\rho(s) = \frac{1}{4\pi^2}\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left[1 - 1.05\,\frac{\lambda^2}{s}\delta(s)\right]\right\}. \tag{33.30}$$

33.8.2 (Pseudo)scalar correlator

In the chiral limit ($m_u \simeq m_d = 0$), the QCD expression of the absorptive part of the (pseudo)scalar correlator reads:

$$\frac{1}{\pi}\mathrm{Im}\psi_{(5)}(s) \simeq (m_i + (-)m_j)^2 \frac{3}{8\pi^2} s\left[1 + \left(\frac{\alpha_s}{\pi}\right)\left(-2L + \frac{17}{3} - 4\frac{\lambda^2}{s}\right)\right], \tag{33.31}$$

where one should notice that the coefficient of the λ^2 term:

$$b_\pi \approx 4b_\rho\,. \tag{33.32}$$

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