LETTERS TO THE EDITOR

HEREDITY OF STATIONARY AND REVERSIBLE STOCHASTIC PROCESSES

RICHARD F. SERFOZO, *Georgia Institute of Technology*

Abstract

When a stochastic process (a random measure, set, field, etc. on a group) is stationary, ergodic, or reversible, then certain functions of this process inherit these properties. We present sufficient conditions for this inheritance.

Introduction

Associated with a queueing network process (see Chapters 2,3 of Kelly (1979)), there are a number of stochastic processes that describe various aspects of the queues. Examples are (i) the point process of times at which customers move from a certain set of queues to another set, and (ii) the process depicting the service station with the largest queue and the length of that queue over time. When the queueing network process is stationary or reversible, do these associated processes inherit these properties? I shall present general criteria for this inheritance. The basic issue is: if a stochastic process is stationary or reversible, then what types of functions of it are also stationary or reversible?

Preliminaries

Let $X = \{X(t) : t \in R\}$ be a stochastic process with state space S. Here R denotes the real line or, more generally, a group with a σ-field $\mathcal{R}$ on it that renders the group operation (addition) measurable. Assume that X has sample paths in a subspace $\mathcal{X}$ of the space $F(R, S)$ of all measurable functions from R to S. The process X is *stationary* if, for any $s_1, \cdots, s_n$ in R,

$$X(s_1 + t), \cdots, X(s_n + t) \stackrel{\mathrm{d}}{=} X(s_1), \cdots, X(s_n), \ t \in R,$$

(here $\stackrel{\mathrm{d}}{=}$ means equality in distribution). A compact way of expressing this is $X \circ T_t \stackrel{\mathrm{d}}{=} X$, $t \in R$, where $\circ$ denotes the convolution operator and T_t is the time-shift transformation on $R(T_t s = s + t)$. A stationary process X is *ergodic* if each of its invariant events has probability 0 or 1 (an event A is invariant for X if there is a $B \in \mathcal{X}$ such that $\{X \circ T_t \in A\} = B$, $t \in R$). The process X is *reversible* if it is stationary and $X \circ T^- \stackrel{\mathrm{d}}{=} X$, where T^- is the time-reversal transformation on $R(T^- t = -t)$. Note that X is reversible if and only if $X \circ T_t \circ T^- \stackrel{\mathrm{d}}{=} X$, $t \in R$ (this is sometimes used as the definition).

The preceding terminology also applies to point processes and other random

---

elements. Let $N = \{N(B): B \in \mathcal{R}\}$ be a point process on R, where $N(B)$ represents the number of points (possibly infinite) in the set B. Stationarity, ergodicity, and reversibility of N are defined as above, where T_t and T^- are the time-shift and time-reversal transformations on $\mathcal{R}$ (instead of R: $T_t B = B + t, T^- B = -B$. For instance, N is reversible if it is stationary ($N \circ T_t \overset{\text{d}}{=} N$, $t \in R$) and $N \circ T^- \overset{\text{d}}{=} N$. There are similar definitions of stationarity, ergodicity, and reversibility for random measures, sets, etc., and vectors of these elements.

It is well known that if the process X is stationary and $g: S \to S'$ and $h: \mathcal{X} \to F(R, S')$ are measurable functions, then the processes $Y(t) = g(X(t))$ and $Z(t) = h(X \circ T_t)$ are also stationary. (The g is a special case of h.) Moreover, Y and Z are ergodic when X is. See for instance Chapter 6 of Breiman (1968). In addition, an easy check shows that Y and Z are reversible when X is. I shall now show how these results extend to a larger class of functions and to point processes and other random elements.

Results

A key observation is that stationarity and reversibility are special cases of the following notion of invariance of a process under a family of 'time' transformations. Let Y be a random element and let Φ be a family of measurable transformations from the domain of Y onto itself. The domain of Y is R, $\mathcal{R}$ or cartesian products of these. The random element Y is *invariant under the set of transformations* Φ if $Y \circ \phi \overset{\text{d}}{=} Y$ for each $\phi \in \Phi$. For example, the vector $Y = (X, N)$ is reversible if it is invariant under the transformations $\{T^-, T_t : t \in R\}$, where T^- and T_t are the time-reversal and time-shift transformations on $R \times \mathcal{R}$.

For the following results, let Y and Z denote random elements (possibly vectors of processes, measures, sets, etc.) with sample paths or realizations in the respective spaces $\mathcal{Y}$ and $\mathcal{Z}$. Assume that $Z = f(Y)$, where $f: \mathcal{Y} \to \mathcal{Z}$ is a measurable function.

Lemma. If Y is invariant under the set of transformations Φ, and f satisfies $f(y) \circ \phi = f(y \circ \phi)$, for each $\phi \in \Phi$, $y \in \mathcal{Y}$, then Z is invariant under the set of transformations Φ.

Proof. This follows since, for each $\phi \in \Phi$,

$$Z \circ \phi = f(Y) \circ \phi = f(Y \circ \phi) \overset{\text{d}}{=} f(Y) = Z.$$

Proposition. If Y is stationary and f satisfies

$$(1) \qquad f(y) \circ T_t = f(y \circ T_t), \text{ for each } t \in R, \ y \in \mathcal{Y},$$

then Z is stationary. If Y is reversible and f satisfies (1) and $f(y) \circ T^- = f(y \circ T^-)$, $y \in \mathcal{Y}$, then Z is reversible. If Y is stationary and ergodic and f satisfies (1), then Z is stationary and ergodic.

Proof. The first two assertions are special cases of the preceding lemma. The third assertion follows since $\{Z \circ T_t \in A\} = \{Y \circ T_t \in f^{-1}(A)\}$ implies that the invariant events of Z are the same as those for Y.

Example. Suppose $Y = \{(Y_1(t), \cdots, Y_m(t)): t \in R\}$ is a queueing network process, where $Y_j(t)$ denotes the number of customers at queue j at time t. Assume that Y cannot have an infinite number of transitions in a finite time interval. Specifics on the operation of the queues are not needed for this discussion. Let $Y_\mathcal{J}(t) = \sum_{j \in \mathcal{J}} Y_j(t)$, the number of customers in the set of queues $\mathcal{J} \supset \{1, \cdots, m\}$. Let N denote the point process of times at which customers move from $\mathcal{J}$ to another set of queues $\mathcal{K}$, and let

$X(t)$ denote the largest number of customers in $\mathcal{J}$ since the last time the number was 0. Clearly $(N, X) = f(Y) = (f_1(Y),\, f_2(Y))$, where

$$f_1(y) = \sum_{t \in B} I(Y_{\mathcal{J}}(t) = Y_{\mathcal{J}}(t-) - 1,\ Y_{\mathcal{K}}(t) = Y_{\mathcal{K}}(t-) + 1),$$

I is the indicator function, and

$$f_2(y)(t) = \max \{ y_{\mathcal{J}}(s) : \tau_t < s \leqq t \},\ t \in R,$$

and $\tau_t = \sup \{ s \leqq t : y_{\mathcal{J}}(s) = 0 \}$, assuming that τ_t exists for each t. Then the preceding proposition establishes that the vector (N, X) is stationary, ergodic or reversible when Y has these respective properties.

Extensions

The lemma extends to state-space transformations as well as time transformations, as follows. Define the random element Y to be (h, Φ)-*invariant* if $h(Y \circ \phi) \stackrel{\mathrm{d}}{=} h(Y)\, \phi$, $\phi \in \Phi$, where $h : \mathcal{Y} \to F(R, S')$ is measurable. For example, if the process X is $(h, \{ T_t : t \in R \})$-invariant, where $h(x)(t) = g(x(t))$ and $g : S \to S$ satisfies $g(g(s)) = s$, then X is *dynamically reversible* (see p. 31 of Kelly (1979)). The lemma for (h, Φ)-invariance is:

if Y is (h, Φ)-invariant and f satisfies $h(f(y) \circ \phi) = h(f(y \circ \phi))$, $\phi \in \Phi$, $Y \in \mathcal{Y}$, then $Z = f(Y)$ is also (h, Φ)–invariant.

The preceding results readily extend to time and space transformations h, Φ that are random.

References

BREIMAN, L. (1968) *Probability*. Addison-Wesley, Reading, Mass.
KELLY, F. P. (1979) *Reversibility and Stochastic Networks*. Wiley, New York.