ON LAPLACE TRANSFORM

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1. Introduction

In this paper we suppose that the function $s(x)$ is integrable in the Lebesgue sense for every finite interval of $x \geqq 0$. If

$$\lim_{A\to\infty}\int_0^A f(x)dx$$

exists, we say that the integral

$$\int_0^\infty f(x)dx$$

exists.

The function $s(x)$ is said to be summable by the Laplace method (L, γ) to s for some $\gamma > -1$ if

$$L(y) = L^{(\gamma)}(y) = \frac{y^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty s(u)u^\gamma e^{-u/y}du \tag{1}$$

exists for all $y > 0$ and $\lim_{y\to\infty} L^{(\gamma)}(y) = s$.

Let $\phi(u)$ and $\psi(u)$ be defined and continuous for $u \geqq 0$, positive for $u > 0$ and bounded in each finite interval $[0, R]$, $R > 0$. If

$$g(y) = \int_0^\infty s(yu)\phi(u)du \tag{2}$$

exists for $y > 0$ and $\lim_{y\to\infty} g(y) = s$, we say that $s(u)$ is summable (ϕ) to s. Suppose that $s(u)$ is bounded in every finite interval of $u \geqq 0$. Then from Theorem 6 in [1] a sufficient condition for the regularity of (ϕ) is

$$\int_0^\infty \phi(u)du = 1. \tag{3}$$

If we assume that, for each $\lambda > 0$,

$$\lim_{u\to\infty}\frac{\phi(\lambda u)}{\psi(u)} \tag{4}$$

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exists and is not zero,

(5) $$\int_1^\infty \left| \frac{d}{du}\left(\frac{\phi(\lambda u)}{\psi(u)}\right) \right| du < \infty,$$

and

(6) $$\int_1^\infty \left| \frac{d}{du}\left(\frac{\psi(u)}{\phi(\lambda u)}\right) \right| du < \infty,$$

then, by partial integration, a necessary and sufficient condition for the convergence of (2) is that

(7) $$\int_1^\infty s(u)\psi(u)du$$

should converge.

We shall prove

THEOREM 1. *Suppose that* $s(u)$ *is summable* (L, γ) $(\gamma > -1)$ *and bounded in every finite interval of* $u \geqq 0$, *and that, in addition to* (3), (4), (5) *and* (6), *the functions* $\phi(u)$ *and* $\psi(u)$ *satisfy the conditions*

(8) $$u^{\gamma-1}e^{-u/\omega}\int_1^\infty \left| \frac{d}{dx}\left(\frac{\phi(x)}{\psi(ux)}\right) \right| dx \in L_1(0, \infty),$$

(9) $$\frac{u^{\gamma-1}e^{-u/\omega}\phi(A)}{\psi(Au)} \leqq F(u) \in L_1(0, \infty),$$

for $A \geqq A_0 > 0$ *then*

$$\frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}g(u)du = \int_0^\infty L(\omega x)\phi(x)dx,$$

where $L(x)$ *and* $g(u)$ *are defined by* (1) *and* (2) *and further* $g(u)$ *is summable* (L, γ) *to* s.

By taking

$$\phi(u) = \frac{\Gamma(\alpha+\beta+1)u^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+1)(1+u)^{\alpha+\beta+1}},$$

$$\psi(u) = u^{\beta+2},$$

summability (ϕ) reduces to summability (C_t, α, β) defined by taking in (2)

(10) $$g(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^\infty \frac{u^{\alpha-1}s(u)}{(u+y)^{\alpha+\beta+1}} du,$$

and Theorem 1 reduces to

THEOREM 2. *Let* $\alpha > 0$, $\beta > -1$, $\gamma > -1$. *Suppose that* $s(u)$ *is summable* (L, γ) *to* s *and bounded in every finite interval of* $u \geqq 0$, *and that*

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$$\int_1^\infty \frac{s(u)}{u^{\beta+2}}\,du$$

converges. Then

$$\frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega} g(u)du = \frac{\Gamma(\alpha+\beta+1)\omega^{\beta+1}}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^\infty \frac{x^{\alpha-1}L(x)}{(x+\omega)^{\alpha+\beta+1}}\,dx,$$

where $L(x)$ and $g(u)$ are defined by (1) *and* (10), *and further $g(u)$ is summable (L, γ) to s.*

Summability $(C_t, \alpha, 0)$ has been considered by Kuttner [3] and $(C_t, 1, \beta+1)$ by me [4].

2. A lemma

LEMMA. *Let $s(u) = 0$ for $0 \leqq u \leqq 1$, bounded in every finite interval of $u \geqq 0$, and summable (L, γ) to s for some $\gamma > -1$, Then, for every fixed $\omega > 0$,*

$$(\omega x)^{-\gamma-1}\int_{Bx}^\infty s(u)u^\gamma e^{-u/\omega x}du$$

tends to zero as $B \to \infty$ uniformly in $0 < u \leqq 1$.

This lemma has been proved in [2].

PROOF OF THEOREM 1. Let

$$s_1(u) = s(u), \quad s_2(u) = 0 \quad \text{for } 0 \leqq u \leqq 1,$$
$$s_1(u) = 0, \quad s_2(u) = s(u) \quad \text{for } u > 1,$$

and let $g_1(x)$ and $M_1(x)$ be the (ϕ) and (L, γ) transformations of $s_1(u)$ and let $g_2(x)$ and $M_2(x)$ be those of $s_2(u)$.

By Fubini's theorem, since $s_1(u)$ is bounded,

$$\begin{aligned}\int_0^\infty M_1(\omega x)\phi(x)dx &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty x^{-\gamma-1}\phi(x)\int_0^\infty u^\gamma e^{-u/\omega x}s_1(u)du\,dx\\ &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty \phi(x)\int_0^\infty u^\gamma e^{-u/\omega}s_1(ux)du\,dx\\ &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}\int_0^\infty s_1(ux)\phi(x)\,dx\,du\\ &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}g_1(u)du.\end{aligned} \tag{11}$$

Let $A > 1$, and write

$$\int_0^\infty M_2(\omega x)\phi(x)dx = \frac{1}{\omega}\int_0^\infty M_2(x)\phi(x/\omega)dx$$
$$= \frac{1}{\Gamma(\gamma+1)\omega}\int_0^{A\omega} x^{-\gamma-1}\phi(x/\omega)\int_1^{Bx/\omega} u^\gamma e^{-u/x}s_2(u)du\,dx$$
$$+ \frac{1}{\Gamma(\gamma+1)\omega}\int_{A\omega}^\infty x^{-\gamma-1}\phi(x/\omega)\int_1^\infty u^\gamma e^{-u/x}s_2(u)du\,dx$$
$$+ \frac{1}{\Gamma(\gamma+1)\omega}\int_0^{A\omega} x^{-\gamma-1}\phi(x/\omega)\int_{Bx/\omega}^\infty u^\gamma e^{-u/x}s_2(u)du\,dx$$
$$= I_1+I_2+I_3.$$

Since $s(u)$ is summable (L, γ) and (ϕ) is regular,

$$\lim_{A\to\infty} I_2 = 0.$$

By changing the variable,

$$I_3 = \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^1 x^{-\gamma-1}\phi(x)\int_{Bx}^\infty u^\gamma e^{-u/\omega x}s_2(u)du\,dx$$
$$+ \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_1^A x^{-\gamma-1}\phi(x)\int_{Bx}^\infty u^\gamma e^{-u/\omega x}s_2(u)du\,dx$$
$$= J_1+J_2.$$

By Lemma 1,

$$\lim_{B\to\infty} J_1 = 0.$$

We have, by changing the variable,

$$J_2 = \frac{A^{-\gamma}\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_{1/A}^1 x^{-\gamma-1}\phi(Ax)\int_{ABx}^\infty u^\gamma e^{-u/A\omega x}s_2(u)du\,dx.$$

Hence, for any fixed $\omega > 0$, $A > 1$, all sufficiently large B, and $1/A \leqq x \leqq 1$, we have, by Lemma 1,

$$\left|(A\omega x)^{-\gamma-1}\int_{ABx}^\infty u^\gamma e^{-u/A\omega x}s_2(u)du\right| < \varepsilon.$$

It follows from (3) that

$$|J_2| < \varepsilon A\int_{1/A}^1 \phi(Ax)dx$$
$$< \varepsilon\int_0^\infty \phi(x)dx$$
$$= \varepsilon$$

for $A > 1$ and all sufficiently large B. Therefore

$$\lim_{B\to\infty} J_2 = 0.$$

Hence

$$\begin{aligned}\int_0^\infty M_2(\omega x)\phi(x)dx &= \lim_{B\to\infty}(I_1+I_2+I_3)\\ &= \lim_{B\to\infty}\frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^A \phi(x)\int_0^B u^\gamma e^{-u/\omega}s_2(xu)du\,dx+I_2\\ &= \lim_{B\to\infty}\frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^B u^\gamma e^{-u/\omega}\int_0^A s_2(xu)\phi(x)dx\,du+I_2 \qquad (12)\\ &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}\int_0^A s_2(xu)\phi(x)dx\,du+I_2\\ &= \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}g_2(u)du+\frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}\\ &\quad\times\int_A^\infty s_2(xu)\phi(x)dx\,du+I_2.\end{aligned}$$

By an obvious change of variable, the inner integral of the second integral is

$$H = \frac{1}{u}\int_{Au}^\infty s_2(x)\phi(x/u)dx.$$

Let

$$G(t) = \int_t^\infty s_2(v)\psi(v)dv$$

for $t \geqq 1$. Then, since (7) converges, $G(t)\to 0$ as $t\to\infty$. By partial integration,

$$\begin{aligned}H &= -\frac{1}{u}\int_{Au}^\infty \frac{\phi(x/u)}{\psi(x)}dG(x)\\ &= \frac{\phi(A)G(Au)}{u\psi(Au)}+\frac{1}{u}\int_{Au}^\infty G(x)d\left(\frac{\phi(x/u)}{\psi(x)}\right).\end{aligned}$$

Hence

$$|H| \leqq K\left\{\left|\frac{\phi(A)}{u\psi(Au)}\right|+\frac{1}{u}\int_1^\infty\left|d\left(\frac{\phi(x)}{\psi(ux)}\right)\right|dx\right\},$$

where K is independent of A as well as u. From (8) and (9) and dominated convergence, the second term of (12) tends to 0 as $A\to\infty$. Now let $A\to\infty$. We get

$$\int_0^\infty M_2(\omega x)\phi(x)dx = \frac{\omega^{-\gamma-1}}{\Gamma(\gamma+1)}\int_0^\infty u^\gamma e^{-u/\omega}g_2(u)du. \qquad (13)$$

The theorem follows from (11) and (13).

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References

[1] G. H. Hardy, *Divergent series* (Oxford).
[2] A. Jakimovski, 'Some remarks on Tauberian theorems', *Quart. J. Math.* (*Oxford*) (2) 9 (1958), 114–131.
[3] B. Kuttner, On translated quasi-Cesáro summability', *Proc. Camb. Phil. Soc.* 62 (1966), 705–712.
[4] B. Kwee, On Perron's method of summation', *Proc. Camb. Phil. Soc.* 63 (1967), 1033–1040.

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