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COMMUTATIVITY OF RINGS SATISFYING CERTAIN POLYNOMIAL IDENTITIES

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It is shown that an n-torsion-free ring R with identity such that, for all x, y in R, $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1}y^{n+1}$ is central, must be commutative. It is also shown that a periodic n-torsion-free ring (not necessarily with identity) for which $(xy)^n - (yx)^n$ is always in the centre is commutative provided that the nilpotents of R form a commutative set. Further, examples are given which show that all the hypotheses of both theorems are essential.

R is called periodic if for every x in R, there exist distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. By a theorem of Chacron (see [**6**, Theorem 1]), R is periodic if and only if for each $x \in R$, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$.

Throughout, R is an associative ring, N denotes the set of nilpotent elements of R, Z denotes the centre of R, $C(R)$ denotes the commutator ideal of R, and $[x, y]$ denotes the commutator $xy - yx$. We start with our first theorem:

Theorem 1. *Let R be a ring with identity and let n be a fixed positive integer. Suppose that R is n-torsion-free, and that for all x, y in R, $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1}y^{n+1}$ is in the centre Z of R. Then R is commutative.*

In preparation for the proof of Theorem 1, we state the following known lemmas [2,11,5].

Lemma 1. *If $[x, y]$ commutes with x, then $[x^k, y] = kx^{k-1}[x, y]$ for all positive integers k.*

Lemma 2. *Suppose that R is a ring with identity 1. If $x^m[x, y] = 0$ and $(x+1)^m[x, y] = 0$ for some x, y in R and some integer $m > 0$, then $[x, y] = 0$. A similar statement holds if we assume $[x, y]x^m = 0$ and $[x, y](x+1)^m = 0$ instead.*

Lemma 3. *Let R be an n-torsion-free ring with identity 1 such that $[x^n, y^n] = 0$ for all x, y in R. Let N denote the set of nilpotent elements of R. Then*

(i) *$a \in N$, $x \in R$ imply $[a, x^n] = 0$.*
(ii) *$a \in N$, $b \in N$ imply $[a, b] = 0$.*

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PROOF OF THEOREM 1: By hypothesis, $[x^n, y^n] = 0$ for all x, y in R and hence, by [8], the commutator ideal is nil. This implies that the set of nilpotent elements N forms an ideal. Hence, by Lemma 3(ii), N is a commutative ideal. This implies that

(1) $$N^2 \subseteq Z.$$

Let $a \in N$, $b \in R$. Then by hypothesis,

(2) $$((a+1)b)^{n+1} - (a+1)^{n+1}b^{n+1} \in Z, \text{ and}$$

(3) $$(b(a+1))^{n+1} - b^{n+1}(a+1)^{n+1} \in Z.$$

Subtracting (3) from (2), and using the fact that $N^2 \subseteq Z$ we get

$$ab^{n+1} - b^{n+1}a - (n+1)ab^{n+1} + (n+1)b^{n+1}a \in Z,$$

and thus $n[a, b^{n+1}] \in Z$. Hence, since R is n-torsion-free, we get

(4) $$[a, b^{n+1}] \in Z, \ (a \in N, \ b \in R).$$

Therefore,

(5) $$[a, b^{n+1}] = [a, b]b^n + b[a, b^n] \in Z, \text{ by (4)}.$$

But, by Lemma 3(i), $[a, b^n] = 0$, and hence by (5),

(6) $$[a, b]b^n \in Z, \ (a \in N, \ b \in R).$$

Thus, $$[[a, b]b^n, b] = 0 = [[a, b], b]\, b^n.$$

Replacing b by $b+1$ in the above argument and using Lemma 2, we see that

(7) $$[[a, b], b] = 0, \ (a \in N, \ b \in R).$$

Using Lemma 3(i), (7), and Lemma 1 we get

$$0 = [a, b^n] = nb^{n-1}[a, b].$$

Since R is n-torsion-free, we conclude that $b^{n-1}[a, b] = 0$. Putting $b+1$ instead of b, and using Lemma 2, we get

$$[a, b] = 0, \ (a \in N, \ b \in R).$$

Thus, the nilpotent elements are central and hence (since $C(R)$ is nil)

$$[x,y] \in Z, \text{ for all } x,y \text{ in } R. \tag{8}$$

Using (8) and Lemma 1, we have $0 = [x^n, y^n] = nx^{n-1}[x,y^n]$. Now, using the fact that R is n-torsion-free and Lemma 2 we get $[x,y^n] = 0$ for all x,y in R. Similarly, $0 = [x,y^n] = ny^{n-1}[x,y]$ yields $[x,y] = 0$ for all x,y in R. This completes the proof of Theorem 1. ∎

In preparation for the proof of the next theorem, we state the following lemma which is proved in [4].

LEMMA 4. *Let R be a periodic ring such that N is commutative. Then the commutator ideal of R is nil, and N forms an ideal of R.*

THEOREM 2. *Let n be a fixed positive integer and let R be an n-torsion-free periodic ring (not necessarily with identity) such that $(xy)^n - (yx)^n \in Z$. If N is commutative, then R is commutative.*

PROOF: Consider first the case that R has an identity 1. By Lemma 4, N is an ideal of R. Also, since N is commutative,

$$N^2 \subseteq Z.$$

Let $a \in N$, $b \in R$. Taking $x = (1+a)b, y = (1+a)^{-1}$, the hypothesis $(xy)^n - (yx)^n \in Z$ yields

$$(1+a)b^n(1+a)^{-1} - b^n \in Z, \tag{9}$$

and hence

$$[(1+a)b^n(1+a)^{-1} - b^n](1+a) = (1+a)[(1+a)b^n(1+a)^{-1} - b^n].$$

Therefore $\quad (1+a)b^n - b^n(1+a) = (1+a)[(1+a)b^n(1+a)^{-1} - b^n],$

$$ab^n - b^n a = (1+a)[(1+a)b^n(1+a)^{-1} - b^n]. \tag{10}$$

Since N is a *commutative ideal*, $(1+a)(ab^n - b^n a) = ab^n - b^n a$, and hence by (10),

$$(1+a)(ab^n - b^n a) = (1+a)[(1+a)b^n(1+a)^{-1} - b^n].$$

Further, since $a \in N$, $1+a$ is a unit in R, and thus

$$ab^n - b^n a = (1+a)b^n(1+a)^{-1} - b^n \in Z, \text{ by (9)}.$$

Thus,

$$(11) \qquad [a, b^n] \in Z, \ (a \in N, \ b \in R).$$

Now, suppose $x_1, \ldots, x_k \in R$. Since $R/C(R)$ is commutative,

$$(x_1 \ldots x_k)^n - x_1^n \ldots x_k^n \in C(R) \subseteq N, \text{ by Lemma 4.}$$

But N is commutative, and hence

$$(12) \qquad [a, (x_1 \ldots x_k)^n] = [a, x_1^n \ldots x_k^n], \ (a \in N).$$

Combining (11) and (12), we conclude that

$$(13) \qquad [a, x_1^n \ldots x_k^n] \in Z, \ (a \in N; \ x_1, \ldots, x_k \in R; \text{ any } k \geqslant 1).$$

Let S be the subring of R generated by the n-th powers of elements of R. Then, by (13),

$$(14) \qquad [a, x] \in Z(S) \text{ for all } a \in N(S), \ x \in S,$$

(here $Z(S)$ and $N(S)$ denote the centre of S and the set of nilpotents of S, respectively). Combining the facts that S is periodic, $N(S)$ is commutative, and (14), a theorem of [**3**] shows that S is commutative, and hence

$$(15) \qquad [x^n, y^n] = 0 \text{ for all } x, y \in R.$$

Note that R is an n-torsion-free ring with identity satisfying (15) and the hypothesis "$(xy)^n - (yx)^n$ is always central", and hence by Theorem 1 of [**1**], R is commutative (in the event R happens to have an identity).

We now consider the general case. We begin with the following two claims.

Claim 1. The idempotents of R are central.

Let $e^2 = e \in R$, $r \in R$. By hypothesis,

$$[e(e + er - ere)]^n - [(e + er - ere)e]^n \in Z,$$

and hence $er - ere \in Z$. Therefore,

$$er - ere = e(er - ere) = (er - ere)e = 0,$$

and thus $er = ere$. Similarly, $re = ere$, and the claim follows.

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CLAIM 2. If $\sigma : R \to S$ is a homomorphism of R onto S, then the nilpotents of S coincide with $\sigma(N)$, where N is the set of nilpotents of R.

This claim was essentially proved in [9].

To complete the proof of the theorem, first recall that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i $(i \in \Gamma)$. Suppose that

$$\sigma_i : R \to R_i$$

is the natural homomorphism of R onto R_i. Let $x_i \in R_i$ and let $\sigma_i(x) = x_i$, $x \in R$. Since R is periodic,

$$x^s = x^r \text{ for some integers } s > r > 0,$$

and hence

$$e = x^{(s-r)r} \text{ is idempotent.} \tag{16}$$

By Claim 1, e is central in R, and hence $\sigma_i(e)$ is a *central* idempotent of R_i. Since R_i is subdirectly irreducible, $\sigma_i(e) = 0$ or $\sigma_i(e) = 1_i$ (if $1_i \in R_i$).

CASE 1. R_i does not have an identity.

In this case, $\sigma_i(e) = 0$ and hence (see (16)), $x_i^{(s-r)r} = 0$. Thus R_i is nil and hence, by Claim 2,

$$R_i = \sigma_i(N).$$

By hypothesis, N is commutative; therefore R_i is commutative.

CASE 2. R_i has an identity 1_i.

Note that R_i need *not* be n-torsion-free. So let $\sigma_i(e_0) = 1_i$, $e_0 \in R$, and choose integers $s > r > 0$ such that $e_0^s = e_0^r$. Let

$$e = e_0^{(s-r)r}.$$

Then e is idempotent and, moreover, $\sigma_i(e) = 1_i^{(s-r)r} = 1_i$. Also, e is central (Claim 1), and hence e is a nonzero central idempotent element of R. Thus, eR is a ring *with identity* e. Because eR inherits all the hypotheses of the ground ring R (including n-torsion-free property), it follows by the first part of the proof that eR is commutative, and hence

$$[ex, ey] = 0 \text{ for all } x,\ y \in R.$$

This implies (since $\sigma_i(e) = 1_i$)

$$[\sigma_i(x),\ \sigma_i(y)] = 0 \text{ for all } x,\ y \in R,$$

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and thus $R_i = \sigma_i(R)$ is again commutative. Hence the ground ring R is commutative, and the theorem is proved.

We conclude by giving examples which show that all the hypotheses of Theorems 1 and 2 are essential.

EXAMPLE 1: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c \in GF(4) \right\},$$

and let $n = 6$. Then R satisfies all the hypotheses of Theorem 2 except that R is *not* n-torsion-free. Note that R is *not* commutative, and hence the hypothesis "R is n-torsion-free" *cannot* be omitted in Theorem 2.

EXMAPLE 2: Let R be as in Example 1, and let $n = 7$. Then R satisfies all the hypotheses of Theorem 2 *except* the hypothesis "$(xy)^n - (yx)^n \in Z$", and hence this hypothesis *cannot* be omitted in Theorem 2.

EXAMPLE 3: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c, d \in GF(3) \right\},$$

and let $n = 7$. Then R satisfies all the hypotheses of Theorem 2 except the hypothesis "N is commutative", and hence this hypothesis *cannot* be omitted in Theorem 2 (note that R is *not* commutative).

EXAMPLE 4: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c, d \in GF(2) \right\},$$

and let $n = 2$. This ring shows that the condition "n-torsion-free" *cannot* be omitted in Theorem 1.

EXAMPLE 5: Let R be as in Example 4 but with entries in $GF(3)$, and let $n = 2$. This ring shows that the condition "$[x^n, y^n] = 0$" *cannot* be omitted in Theorem 1.

EXAMPLE 6: Let R be as in Example 1 with $n = 3$. This ring shows that the condition "$(xy)^{n+1} - x^{n+1}y^{n+1} \in Z$" *cannot* be omitted in Theorem 1.

EXAMPLE 7:

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b, c \in GF(3) \right\},$$

and let $n = 4$. This ring shows that the condition "$1 \in R$" *cannot* be omitted in Theorem 1.

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