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ON FUNCTIONAL REPRESENTATIONS OF A RING WITHOUT NILPOTENT ELEMENTS

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1. In [3, p. 149], J. Lambek gives a proof of a theorem, essentially due to Grothendieck and Dieudonne, that if R is a commutative ring with 1 then R is isomorphic to the ring of global sections of a sheaf over the prime ideal space of R where a stalk of the sheaf is of the form $R/0_P$, for each prime ideal P, and $0_P=\{r \in R \mid ra=0, \exists a \notin P\}$. In this note we will show, this type of representation of a noncommutative ring is possible if the ring contains no nonzero nilpotent elements. If R is a ring with 1, let $X(R)$ be the set of prime ideals of R. For each ideal A of R define the support of A to be $\{P \in X(R) \mid A \nsubseteq P\}$ and let us write this set as supp (A). Let $\tau=\{\text{supp}(A) \mid A$ is an ideal of $R\}$. Then $(X(R), \tau)$ is a topological space which is compact (refer [1, p. 143]). If R is a ring without nilpotent elements then for any prime ideal P, 0_P is an ideal of R which is contained in P. Moreover $0_P=P$ if and only if P is a minimal prime ideal as it is in the case of a commutative ring and, furthermore, any minimal prime ideal P is a completely prime ideal in the sense that R/P is an integral domain. A principal result of this note is as follows:

Let R be a ring with 1 without nilpotent elements. Then R is isomorphic to the ring of global sections of a sheaf of rings $\bigcup_{P \in X(R)} R/0_P$ over $X(R)$ where $R/0_P$ is a ring without nilpotent elements and $R/0_P$ is an integral domain if and only if P is a minimal prime ideal.

2. Let R be a ring and P be an ideal in R. Then P is called a prime ideal provided that R/P is a prime ring and P is called a *completely prime ideal* provided that R/P is an integral domain. If R is a commutative ring then a prime ideal is a completely prime ideal, however, if R is not commutative, then a prime ideal may fail to be a completely prime ideal. If S is a nonempty subset of R, let $S^r=\{r \in R \mid sr=0$ for every $s \in S\}$, $S^l=\{r \in R \mid rs=0$ for every $s \in S\}$ and if $S^r=S^l$ then let $S^\perp=S^r$.

2.1. PROPOSITION. *Let R be a ring without nilpotent elements and let x be a nonzero element of R. Then $\{x\}^r$ is a two-sided ideal of R, $\{x\}^r=\{x\}^l$, $x \notin \{x\}^l$, $R/\{x\}^\perp$ has no nilpotent elements and if $r \in R$ and $rx \in \{x\}^l$ then $r \in \{x\}^l$.*

Proof. See [5].

2.2. PROPOSITION. (Stewart). *Let R be a ring without nilpotent elements and for each $x \neq 0$ in R, let $Z(x)=\{I \mid I$ is an ideal of R, $x \notin I$, if $rx \in I$ then $r \in I$, and*

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R/I has no nilpotent elements}. Then any maximal member of $Z(x)$ is a completely prime ideal. In particular, $\{x\}^{\perp}$ is contained in a completely prime ideal.

Proof. see [5].

2.3. PROPOSITION. *If P is a prime ideal of a ring R without nilpotent elements then $0_P=\{r \in R \mid ra=0, \exists a \notin P\}$ is an ideal, $0_P \subseteq P$ and $R/0_P$ is a ring without nilpotent elements.*

Proof. If $r_1, r_2 \in 0_P$ then there exist a_1, a_2 in $R \backslash P$ such that $r_1a_1=0=r_2a_2$. Hence by 2.1, $r_1Ra_1=0=r_2Ra_2$. Let $a_1ra_2 \notin P$ for some r in R. Then $(r_1-r_2)a_1ra_2=0$. Therefore, $r_1-r_2 \in 0_P$. Clearly if $r \in 0_P$ and $x \in R$ then rx and xr are elements of 0_P. It is also clear that $0_P \subseteq P$. If $a \in R$ such that $a^n \in 0_P$ for some integer n, then $a^ny=0$ for some $y \notin P$. Therefore, by 2.1, $(ay)^n=0$ and $a \in 0_P$.

2.4. THEOREM. *Let R be a ring without nilpotent elements. Then P is a minimal prime ideal if and only if $P=0_P$ and in this case, P is a completely prime ideal.*

Proof. Let P be a minimal prime ideal. Suppose $P \neq 0_P$ Then there is $a \in P$ such that $a \notin 0_P$ since $0_P \subseteq P$ by 2.3. Let $M=R \backslash P$. Then M is an m-system, that is for any $x, y \in M$ there is $r \in R$ such that $xry \in M$. Let $S=\{a, a^2, a^3, \ldots\}$ and let $T=\{r \in R \mid r \neq 0, r=a^{i_0}x_0a^{i_1}x \ldots a^{i_n}x_na^{i_{n+1}}$ for some nonnegative integer n where $x_j \in M$ for $j=0, 1, 2, \ldots, n$ and i_0, i_{n+1} are nonnegative integers and $i_1, \ldots, i_n$ are positive integers}. We let $ra^0=r=a^0r$ for any $r \in R$. We will prove that $\Gamma=M \cup S \cup T$ is an m-system. It is clear that $0 \notin \Gamma$. Let x, y be two elements in Γ. Let $x \in M$. If $y \in M$ or $y \in S$ then clearly there is $r \in R$ such that $xry \in \Gamma$ since M is an m-system and $\{a^n\}^{\perp} \subseteq P$ for any n. For if $xa^n=0$ for some n then $(xa)(xa)\ldots(xa)$ is zero by 2.1 and this in turn implies $xa=0={}^nax$ and $a \in 0_P$. This is impossible. Now let $y \in T$. Then $y=a^{i_0}x_0a^{i_1}2_1 \ldots a^{i_n}x_na^{i_{n+1}}$ for some $x_1 \in M$, $i=0, \ldots, n$. Since M is an m-system, there exist $r_0, r_1, r_2, \ldots, r_n$ such that $xr_0x_0r_1x_1r_2x_2 \ldots r_nx_n \in M$. Let $w=xr_0x_0r_1x_1r_2x_2 \ldots r_nx_n$. Let $i=i_0+i_1+\cdots+i_{n+1}$. If $wy=0$ then $wy \in T$ and therefore, $xRy \cap \Gamma \neq \varnothing$. So suppose $wy=0$. Then by 2.1, $0=[(a^iw)(a^iw)\ldots(a^iw)]/(n+2)$ and $a^iw=0$ and $\{a^i\}^{\perp} \nsubseteq P$. This is impossible. A similar argument shows that when $x \in S \cup T$, $xRy \cap \Gamma \neq \varnothing$. Let A be an ideal of R which is maximal with respect to the property that $\Gamma \cap A=\varnothing$. Then A is a prime ideal and $A \subseteq P$ and $A \neq P$. This contradicts the minimality of P. Thus, $0_P=P$. Conversely, suppose $0_P=P$ and P' is a prime ideal contained in P. Then for any $x \in P$ there exists $a \notin P$ such that $xa=0 \in P'$. This means that $xRa=0 \subseteq P'$ by 2.1, and $x \in P'$. Thus $P'=P$ is a minimal prime ideal of R. By 2.3 $R/0_P$ is a ring without nilpotent elements. Thus, $R/0_P$ is an integral domain.

2.5. COROLLARY. *Let Π be the subspace of $X(R)$ which consists of all minimal prime ideals of R. Then Π is a Hausdorff space with a base of open and closed sets.*

Proof. Let $P_1, P_2 \in \Pi$ such that $P_1 \neq P_2$. Then by 2.4, $P_i=0_{P_i}$ for $i=1, 2$. Hence,

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$0_{P_1} \nsubseteq P_2$. Let $x \in 0_{P_1}$ such that $x \notin P_2$ and let $s \notin P_1$ such that $xs=0$. Then $RxRsR=0$ by 2.1. Hence supp $(RxR) \cap$ supp $(RsR)=\varnothing$ and $P_1 \in$ supp (RsR) and $P_2 \in$ supp (RxR). Now for any $a \in R$ supp $(RaR)=\Pi \backslash$supp $(\{a\}^\perp)$. Thus, the assertion holds true.

2.6. Proposition. *If R is a ring without nilpotent elements then the right singular ideal of R is zero and the left singular ideal of R is zero.*

Proof. If not, there is $x \neq 0$ in R such that $\{x\}^\perp \cap I \neq 0$ for each nonzero right (or left in case the left singular ideal is not zero) ideal I. In particular $x^\perp \cap xR \neq 0$. Hence, there is $r \in R$ such that $xr \neq 0$ but $x(xr)=0=(xr)(xr)$ which is impossible.

2.7. Example. A maximal right (or left) ring of quotients of a ring without nilpotent elements may not be a ring without nilpotent elements. For example, let $R=Z/(2)[x, y]$, the polynomial ring in two variables x, y over the field of integers modulo 2 such that $xy \neq yx$. Then R is an integral domain such that $xR \cap yR=0$. By [2] $Q_r(R)$, the maximal right quotients of R, is a simple ring which is regular. Hence, if $Q_r(R)$ is a ring without nilpotent elements then $Q_r(R)$ would be a strongly regular ring and since it is simple, $Q_r(R)$ would be a division ring and $xR \cap yR \neq 0$.

2.8. Proposition. *Let R be a ring without nilpotent elements and assume $1 \in R$. If P is a prime ideal of R, let $\widehat{R/P}$ be the injective hull of the right R-module R/P. Then $0_P=(\widehat{R/P})^\perp$.*

Proof. If $(\widehat{R/P})^\perp \nsubseteq 0_P$ then there is $r_0 \in (\widehat{R/P})$ such that $r_0 \notin 0_P$ and $r_0 y \neq 0$ for any $y \in R\backslash P$. That is $(r_0 y)^\perp \subseteq P$ for any $y \in R\backslash P$. Let $T=\{x \in R/P \mid x(\{r_0\}^\perp)=0\}$. For each $y \in T$, define $f_y: r_0 x \to yx$ for all $x \in R$. Then f_y is an R-homomorphism from $r_0 R$ into R/P. Let $\bar{f}_y$ be an extension of f_y to R. Then $y=\bar{f}_y(r_0)=\bar{f}_y(1)r_0=0$ since $\bar{f}_y(1) \in R/P$. Therefore, $T=\{0\}$. Let $b \in R/P$. Then $b(\{r_0\}) \subseteq P$ since $\{r_0\}^\perp \subseteq P$ and $\bar{b}=b+P \in T=\{0\}$. This is impossible. Conversely, suppose $0_P \nsubseteq (\widehat{R/P})^\perp$. Then $(\widehat{R/P})0_P \neq 0$ and there exist $x \in (\widehat{R/P})$, $a \in 0_P$ such that $xa \neq 0$. Since $(\widehat{R/P})$ is an essential extension of R/P, $xaR \cap R/P=N$ is a nonzero submodule of R/P. Hence, there is a right ideal J in R such that $J \neq P$ and $J/P=N$. Since $a \in 0_P$, there is $b \notin P$ such that $ab=0$ and by 2.1, $aRb=0$. Therefore $Nb=0$. This is impossible since P is a prime ideal. Thus $(\widehat{R/P})^\perp \subseteq 0_P \subseteq (\widehat{R/P})^\perp$.

2.9. Proposition. *Let $S=\bigcup_{P \in X(R)} R/0_P$. For each $r \in R$, define $\hat{r}$ to be the function from $X(R)$ into S such that $\hat{r}(P)=r+0_P$. Let U be any open set in $X(R)$ and let $\hat{r}(U)=\{\hat{r}(P) \mid P \in U\}$. Let ρ be the topology on S generated by $\{\hat{r}(U) \mid r \in R$, U is open in $X(R)\}$. Then (S, ρ) forms a topological space and each point $\hat{r}(P_0)$ of S is contained in an open set which is homeomorphic to its image in $X(R)$ under the canonical projection $\hat{r}(P) \to P$, i.e. S is a sheaf of rings over $X(R)$.* (Refer to [4].)

Proof. Straightforward.

3.0. Proposition. *If R is a ring without nilpotent elements and $s \in R$, then $r \in 0_P$ for all $P \in$ supp $((s))$ if and only if $(s) \in \{r\}^\perp$ where (s) is an ideal generated by s.*

Proof. By 2.1, $R/\{r\}^{\perp}$ is a ring without nilpotent elements. Now the condition that $r \in 0_P$ for all $P \in \operatorname{supp}((s))$ is equivalent to that of $s \in \bigcap_{\{r\}^{\perp} \subseteq P_\alpha} P_\alpha$ where $P_\alpha \in X(R)$. Hence, $s+\{r\}^{\perp}$ is an element of rad $(R/\{r\}^{\perp})$, the intersection of prime ideals of $R/\{r\}^{\perp}$. Hence $s \in \{r\}^{\perp}$.

3.1. THEOREM. *If R is a ring without nilpotent elements then every section of the sheaf S of rings over $X(R)$ described in Proposition* 2.9 *has the form $\hat{r}$ for some $r \in R$.*

Proof. Let $f: X(R) \to S$ be any section. For each $P \in X(R)$, there exists $r_1 \in R$ such that $f(P)=\hat{r}_1(P)$. Hence, by Lemma 3.2 of [4, p. 11] there exists an open set, say supp $((s))$ for some $s \in R$ such that $P \in \operatorname{supp}((s))$ and $f(P')=\hat{r}_1(P')$ for all $P' \in \operatorname{supp}((s))$. Since $X(R)$ is compact, there exist $s_1, s_2, \ldots, s_m$ and $r_1, r_2, \ldots, r_m$ in R such that $\bigcup_{i=1}^m \operatorname{supp}((s_i))=X(R)$ and $f(P')=\hat{r}_i(P')$ for any $P' \in \operatorname{supp}((s_i))$. Hence, for any $P \in \operatorname{supp}((s_i)) \cap \operatorname{supp}((s_j))=\operatorname{supp}((s_i)(s_j))$, $i, j=1, 2, \ldots, m$, $r_i-r_j \in 0_P$. Therefore, by 3.0, $s_is_j(r_i-r_j)=0$. This means that $s_i(r_i-r_j)s_j(r_i-r_j)=0$ since $\{s_j(r_i-r_j)\}^{\perp}$ is an ideal and this, in turn, implies that $s_i(r_i-r_j)s_js_i(r_i-r_j)s_j=0$ and $s_i(r_i-r_j)s_j=0$ since R is a ring without nilpotent elements. Since $X(R)=\bigcup_{i=1}^n$ supp $((s_i))$, $1 \in \sum_{i=1}^m Rs_iR$ and $1=\sum_{i=1}^m b_is_it_i$ for some b_i, t_i in R.

Define $a=\sum_{i=1}^m r_ib_is_it_i$. Since $s_i(r_i-r_j)s_j=0$, by 2.1 $s_i(r_i-r_j)b_ls_j=0$ for any b_l, $l=1, 2, 3, \ldots, m$. Therefore, $s_ir_ib_ls_j=s_ir_jb_ls_j$. For any s_j,

$$\begin{aligned} s_ja &= s_jr_1b_1s_1t_1+s_jr_2b_2s_2t_2+\cdots+s_jr_mb_ms_mt_m \\ &= s_jr_jb_1s_1t_1+s_jr_jb_2s_2t_2+\cdots+s_jr_jb_ms_mt_m \\ &= s_jr_j\left(\sum_{i=1}^m b_is_it_i\right) = s_jr_j \quad \text{and} \quad s_j(a-r_j) = 0. \end{aligned}$$

Thus,

$$(a-r_j)s_j = 0.$$

Recall that $s_j \notin P_j$. It follows that $a-r_j \in 0_{P_j}$ and $\hat{a}=\hat{r}_j$ for all $j=1, 2, \ldots, m$. Thus, $f=\hat{a}$.

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