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Inverse scattering transforms for non-local reverse-space matrix non-linear Schrödinger equations

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The aim of the paper is to explore non-local reverse-space matrix non-linear Schrödinger equations and their inverse scattering transforms. Riemann–Hilbert problems are formulated to analyse the inverse scattering problems, and the Sokhotski–Plemelj formula is used to determine Gelfand–Levitan–Marchenko-type integral equations for generalised matrix Jost solutions. Soliton solutions are constructed through the reflectionless transforms associated with poles of the Riemann–Hilbert problems.

Keywords: Matrix spectral problem, non-local reduction, inverse scattering, Riemann–Hilbert problem, soliton solution

2020 Mathematics Subject Classification: 37K15 (Primary), 35Q55, 37K40 (Secondary)

1 Introduction

Non-local integrable non-linear Schrödinger (NLS) equations are generated from matrix spectral problems under specific symmetric reductions on potentials [3]. The corresponding inverse scattering transforms have been recently presented, under zero or non-zero boundary conditions, and there still exist N-soliton solutions in the non-local cases [2, 4, 15]. Such soliton solutions can be constructed more generally from the Riemann–Hilbert problems with the identity jump matrix [51] and by the Hirota bilinear method [16]. Some vector or matrix generalisations [5, 10, 32] and other interesting non-local integrable equations [20, 44] were also presented. We would like to propose a class of general non-local reverse-space matrix NLS equations and analyse their inverse scattering transforms and soliton solutions through formulating and solving associated Riemann–Hilbert problems.

The Riemann–Hilbert approach is one of the most powerful approaches for investigating integrable equations and particularly constructing soliton solutions [42]. Many integrable equations, such as the multiple wave interaction equations [42], the general coupled non-linear Schrödinger equations [46], the generalised Sasa–Satsuma equation [13], the Harry Dym equation [47] and the AKNS soliton hierarchies [27], have been studied by formulating and analysing their Riemann–Hilbert problems associated with matrix spectral problems.

A general procedure for formulating Riemann–Hilbert problems can be described as follows. We start from a pair of matrix spectral problems, say,

$$-i\phi_x = U\phi,\ -i\phi_t = V\phi,\ U = A(\lambda) + P(u, \lambda),\ V = B(\lambda) + Q(u, \lambda), \tag{1.1}$$

where i is the unit imaginary number, λ is a spectral parameter, u is a potential and ϕ is an $m \times m$ matrix eigenfunction. The compatibility condition of the above two matrix spectral problems, that is, the zero curvature equation:

$$U_t - V_x + i[U, V] = 0, \tag{1.2}$$

where $[\cdot, \cdot]$ is the matrix commutator, presents an integrable equation. To establish an associated Riemann–Hilbert problem for the above integrable equation, we adopt the following equivalent pair of matrix spectral problems:

$$\psi_x = i[A(\lambda), \psi] + \check{P}(u, \lambda)\psi,\ \psi_t = i[B(\lambda), \psi] + \check{Q}(u, \lambda)\psi,\ \check{P} = iP,\ \check{Q} = iQ, \tag{1.3}$$

where ψ is also an $m \times m$ matrix eigenfunction. We often assume that A and B are constant commuting $m \times m$ matrices, and P and Q are trace-less $m \times m$ matrices. The equivalence between (1.1) and (1.3) follows from the commutativity of A and B. The properties $(\det \psi)_x = (\det \psi)_t = 0$ are two consequences of $\mathrm{tr}P = \mathrm{tr}Q = 0$. There exists a direct connection between (1.1) and (1.3):

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{iA(\lambda)x + iB(\lambda)t}. \tag{1.4}$$

It is important to note that for the pair of matrix spectral problems in (1.3), we can impose the asymptotic conditions:

$$\psi^{\pm} \to I_m,\ \text{when } x \text{ or } t \to \pm\infty, \tag{1.5}$$

where I_m stands for the identity matrix of size m. From these two matrix eigenfunctions $\psi^{\pm}$, we need to pick the entries and build two generalised matrix Jost solutions $T^{\pm}(x, t, \lambda)$, which are analytic in the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in the closed upper and lower half-planes $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$, respectively, to formulate a Riemann–Hilbert problem on the real line:

$$G^+(x, t, \lambda) = G^-(x, t, \lambda)G_0(x, t, \lambda),\ \lambda \in \mathbb{R}, \tag{1.6}$$

where two unimodular generalised matrix Jost solutions G^+ and G^- and the jump matrix G_0 are generated from T^+ and T^-. The jump matrix G_0 carries all basic scattering data from the scattering matrix $S_g(\lambda)$ of the matrix spectral problems, defined through

$$\psi^- E_g = \psi^+ E_g S_g(\lambda). \tag{1.7}$$

Solutions to the associated Riemann–Hilbert problems (1.6) provide the required generalised matrix Jost solutions in recovering the potential of the matrix spectral problems, which solves the corresponding integrable equation. Such solutions G^+ and G^- can be presented by applying the Sokhotski–Plemelj formula to the difference of G^+ and G^-. A recovery of the potential comes from observing asymptotic behaviours of the generalised matrix Jost solutions $G^{\pm}$ at infinity of λ. This then completes the corresponding inverse scattering transforms. Soliton solutions can be worked out from the reflectionless transforms, which correspond to the Riemann–Hilbert problems with the identity jump matrix G_0.

In thispaper, we first present a class of non-local reverse-space matrix NLS equations by making a specific group of non-local reductions and then analyse their inverse scattering transforms and soliton solutions, based on associated Riemann–Hilbert problems. One example with two components is

$$\begin{cases} ip_{1,t}(x,t) = p_{1,xx}(x,t) - 2[\gamma_1 p_1(x,t)p_1^*(-x,t) + \gamma_2 p_2(x,t)p_2^*(-x,t)]p_1(x,t), \\ ip_{2,t}(x,t) = p_{2,xx}(x,t) - 2[\gamma_1 p_1(x,t)p_1^*(-x,t) + \gamma_2 p_2(x,t)p_2^*(-x,t)]p_2(x,t), \end{cases} \tag{1.8}$$

where γ_1 and γ_2 are arbitrary non-zero real constants.

The rest of the paper is organised as follows. In Section 2, within the zero curvature formulation, we recall the Ablowitz–Kaup–Newell–Segur (AKNS) integrable hierarchy with matrix potentials, based on an arbitrary-order matrix spectral problem suited for the Riemann–Hilbert theory, and conduct a group of non-local reductions to generate non-local reverse-space matrix NLS equations. In Section 3, we build the inverse scattering transforms by formulating Riemann–Hilbert problems associated with a kind of arbitrary-order matrix spectral problems. In Section 4, we compute soliton solutions to the obtained non-local reverse-space matrix NLS equations from the reflectionless transforms, that is, the special associated Riemann–Hilbert problems on the real axis where an identity jump matrix is taken. The conclusion is given in the last section, together with a few concluding remarks.

2 Non-local reverse-space matrix NLS equations

2.1 Matrix AKNS hierarchy

Let m and $n \geq 1$ be two arbitrary integers, and α_1 and α_2 are different arbitrary real constants. We focus on the following matrix spectral problem:

$$-i\phi_x = U\phi = U(p,q;\lambda)\phi, \; U = \begin{bmatrix} \alpha_1\lambda I_m & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.1}$$

where λ is a spectral parameter, and p and q are two matrix potentials:

$$p = (p_{jl})_{m\times n}, \; q = (q_{lj})_{n\times m}. \tag{2.2}$$

When $m = 1$, that is, p and q are vectors, (2.1) gives a matrix spectral problem with vector potentials [37]. When there is only one pair of non-zero potentials p_{jl}, q_{lj}, (2.1) becomes the standard AKNS spectral problem [1]. On account of these, we call (2.1) a matrix AKNS matrix spectral problem, and its associated hierarchy, a matrix AKNS integrable hierarchy. Because of the existence of a multiple eigenvalue of $\frac{\partial U}{\partial \lambda}$, we have a degenerate matrix spectral problem in (2.1).

To construct an associated matrix AKNS integrable hierarchy, as usual, we begin with the stationary zero curvature equation:

$$W_x = i[U, W], \tag{2.3}$$

corresponding to (2.1). We search for a solution W of the form:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.4}$$

where a, b, c and d are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices, respectively. Obviously, the stationary zero curvature equation (2.3) equivalently presents

$$\begin{cases} a_x = i(pc - bq), \\ b_x = i(\alpha\lambda b + pd - ap), \\ c_x = i(-\alpha\lambda c + qa - dq), \\ d_x = i(qb - cp), \end{cases} \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$. We take W as a formal series:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{s=0}^{\infty} W_s \lambda^{-s}, \; W_s = W_s(p, q) = \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \; s \geq 0, \tag{2.6}$$

and then, the system (2.5) exactly engenders the following recursion relations:

$$b^{[0]} = 0, \; c^{[0]} = 0, \; a_x^{[0]} = 0, \; d_x^{[0]} = 0, \tag{2.7a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p), \; s \geq 0, \tag{2.7b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q), \; s \geq 0, \tag{2.7c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q), \; d_x^{[s]} = i(qb^{[s]} - c^{[s]}p), \; s \geq 1. \tag{2.7d}$$

Let us now fix the initial values:

$$a^{[0]} = \beta_1 I_m, \; d^{[0]} = \beta_2 I_n, \tag{2.8}$$

where β_1 and β_2 are arbitrary but different real constants, and take zero constants of integration in (2.7d), which says that we require

$$W_s|_{p,q=0} = 0, \; s \geq 1. \tag{2.9}$$

In this way, with $a^{[0]}$ and $d^{[0]}$ given by (2.8), all matrices W_s, $s \geq 1$, defined recursively, are uniquely determined. For example, a direct computation, based on (2.1), generates that

$$b^{[1]} = \frac{\beta}{\alpha}p, \; c^{[1]} = \frac{\beta}{\alpha}q, \; a^{[1]} = 0, \; d^{[1]} = 0; \tag{2.10a}$$

$$b^{[2]} = -\frac{\beta}{\alpha^2}ip_x, \; c^{[2]} = \frac{\beta}{\alpha^2}iq_x, \; a^{[2]} = -\frac{\beta}{\alpha^2}pq, \; d^{[2]} = \frac{\beta}{\alpha^2}qp; \tag{2.10b}$$

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$$\begin{cases} b^{[3]} = -\frac{\beta}{\alpha^3}(p_{xx} + 2pqp), \ c^{[3]} = -\frac{\beta}{\alpha^3}(q_{xx} + 2qpq), \\ a^{[3]} = -\frac{\beta}{\alpha^3}i(pq_x - p_xq), \ d^{[3]} = -\frac{\beta}{\alpha^3}i(qp_x - q_xp); \end{cases} \tag{2.10c}$$

$$\begin{cases} b^{[4]} = \frac{\beta}{\alpha^4}i(p_{xxx} + 3pqp_x + 3p_xqp), \\ c^{[4]} = -\frac{\beta}{\alpha^4}i(q_{xxx} + 3q_xpq + 3qpq_x), \\ a^{[4]} = \frac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_xq_x + p_{xx}q], \\ d^{[4]} = -\frac{\beta}{\alpha^4}[3(qp)^2 + qp_{xx} - q_xp_x + q_{xx}p]; \end{cases} \tag{2.10d}$$

where $\beta = \beta_1 - \beta_2$. Using (2.7d), we can derive, from (2.7b) and (2.7c), a recursion relation for $b^{[s]}$ and $c^{[s]}$:

$$\begin{bmatrix} c^{[s+1]} \\ b^{[s+1]} \end{bmatrix} = \Psi \begin{bmatrix} c^{[s]} \\ b^{[s]} \end{bmatrix}, \ s \geq 1, \tag{2.11}$$

where Ψ is a matrix operator:

$$\Psi = \frac{i}{\alpha}\begin{bmatrix} (\partial_x + q\partial_x^{-1}(p\,\cdot) + [\partial_x^{-1}(\cdot\, p)]q) & -q\partial_x^{-1}(\cdot\, q) - [\partial_x^{-1}(q\,\cdot)]q \\ p\partial_x^{-1}(\cdot\, p) + [\partial_x^{-1}(p\,\cdot)]p & -\partial_x - p\partial_x^{-1}(q\,\cdot) - [\partial_x^{-1}(\cdot\, q)]p \end{bmatrix}. \tag{2.12}$$

The matrix AKNS integrable hierarchy is associated with the following temporal matrix spectral problems:

$$-i\phi_t = V^{[r]}\phi = V^{[r]}(p, q; \lambda)\phi, \ V^{[r]} = \sum_{s=0}^{r} W_m\lambda^{r-s}, \ r \geq 0. \tag{2.13}$$

The compatibility conditions of the two matrix spectral problems (2.1) and (2.13), that is, the zero curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \ r \geq 0, \tag{2.14}$$

yield the so-called matrix AKNS integrable hierarchy:

$$\begin{bmatrix} p \\ q \end{bmatrix}_t = i\begin{bmatrix} \alpha b^{[r+1]} \\ -\alpha c^{[r+1]} \end{bmatrix}, \ r \geq 0. \tag{2.15}$$

The first non-linear integrable system in this hierarchy gives us the standard matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2}i(p_{xx} + 2pqp), \ q_t = \frac{\beta}{\alpha^2}i(q_{xx} + 2qpq). \tag{2.16}$$

When $m=1$ and $n=2$, under a special kind of symmetric reductions, the matrix NLS equations (2.16) can be reduced to the Manokov system [39], for which a decomposition into finite-dimensional integrable Hamiltonian systems was made in [8].

2.2 Non-local reverse-space matrix NLS equations

Let us now take a specific group of non-local reductions for the spectral matrix:

$$U^{\dagger}(-x,t,-\lambda^{*}) = -CU(x,t,\lambda)C^{-1},\ C=\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix},\ \Sigma_i^{\dagger}=\Sigma_i,\ i=1,2, \tag{2.17}$$

which is equivalent to

$$P^{\dagger}(-x,t) = -CP(x,t)C^{-1}. \tag{2.18}$$

Henceforth, $\dagger$ stands for the Hermitian transpose, $*$ denotes the complex conjugate, $\Sigma_{1,2}$ are two constant invertible Hermitian matrices, and for brevity, we adopt

$$\begin{cases} A(x,t,\lambda) = A(u(x,t),\lambda),\\ A^{\dagger}(f(x,t,\lambda)) = (A(f(x,t,\lambda)))^{\dagger},\\ A^{-1}(f(x,t,\lambda)) = (A(f(x,t,\lambda)))^{-1}, \end{cases} \tag{2.19}$$

for a matrix A and a function f.

The matrix spectral problems of the matrix NLS equations (2.16) are given as follows:

$$-i\phi_x = U\phi = U(p,q;\lambda)\phi,\ -i\phi_t = V^{[2]}\phi = V^{[2]}(p,q;\lambda)\phi. \tag{2.20}$$

The involved Lax pair reads

$$U=\lambda\Lambda + P,\ V^{[2]} = \lambda^2\Omega + Q, \tag{2.21}$$

where $\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n)$, $\Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n)$, and

$$P=\begin{bmatrix}0 & p\\ q & 0\end{bmatrix},\ Q=\begin{bmatrix}a^{[1]}\lambda + a^{[2]} & b^{[1]}\lambda + b^{[2]}\\ c^{[1]}\lambda + c^{[2]} & d^{[1]}\lambda + d^{[2]}\end{bmatrix} = \frac{\beta}{\alpha}\lambda\begin{bmatrix}0 & p\\ q & 0\end{bmatrix} - \frac{\beta}{\alpha^2}\begin{bmatrix}pq & ip_x\\ -iq_x & -qp\end{bmatrix}. \tag{2.22}$$

In the above matrices P and Q, p and q are defined by (2.2), and $a^{[s]}, b^{[s]}, c^{[s]}, d^{[s]}$, $1\le s\le 2$, are determined in (2.10).

Based on (2.18), we arrive at

$$q(x,t) = -\Sigma_2^{-1}p^{\dagger}(-x,t)\Sigma_1. \tag{2.23}$$

The vector function c in (2.5) under such a non-local reduction could be taken as:

$$c(x,t,\lambda) = \Sigma_2^{-1}b^{\dagger}(-x,t,-\lambda^{*})\Sigma_1. \tag{2.24}$$

It is easy to see that those non-local reduction relations ensure that

$$a^{\dagger}(-x,t,-\lambda^{*}) = \Sigma_1 a(x,t,\lambda)\Sigma_1^{-1},\ d^{\dagger}(-x,t,-\lambda^{*}) = \Sigma_2 d(x,t,\lambda)\Sigma_2^{-1}, \tag{2.25}$$

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where a and d satisfy (2.5). For instance, under (2.23) and (2.24), we can compute that

$$\begin{aligned}
&(a^{\dagger}(-x,t,-\lambda^*))_x = -a_x^{\dagger}(-x,t,-\lambda^*)\\
&= i[c^{\dagger}(-x,t,-\lambda^*)p^{\dagger}(-x,t) - q^{\dagger}(-x,t)b^{\dagger}(-x,t,-\lambda^*)]\\
&= i\{[\Sigma_1 b(x,t,\lambda)\Sigma_2^{-1}][-\Sigma_2 q(x,t)\Sigma_1^{-1}] - [-\Sigma_1 p(x,t)\Sigma_2^{-1}][\Sigma_2 c(x,t,\lambda)\Sigma_1^{-1}]\}\\
&= -i\Sigma_1[b(x,t,\lambda)q(x,t) - p(x,t)c(x,t,\lambda)]\Sigma_1^{-1} = \Sigma_1 a_x(x,t,\lambda)\Sigma_1^{-1},
\end{aligned}$$

from which the first relation in (2.25) follows. Furthermore, by using the Laurent expansions for a,b,c and d, we can get

$$\begin{cases}
(a^{[s]})^{\dagger}(-x,t) = (-1)^s\Sigma_1 a^{[s]}(x,t)\Sigma_1^{-1},\\
(b^{[s]})^{\dagger}(-x,t) = (-1)^s\Sigma_2 c^{[s]}(x,t)\Sigma_1^{-1},\\
(d^{[s]})^{\dagger}(-x,t) = (-1)^s\Sigma_2 d^{[s]}(x,t)\Sigma_2^{-1},
\end{cases} \tag{2.26}$$

where $s \geq 0$. It then follows that

$$(V^{[2]})^{\dagger}(-x,t,-\lambda^*) = CV^{[2]}(x,t,\lambda)C^{-1},\ Q^{\dagger}(-x,t,-\lambda^*) = CQ(x,t,\lambda)C^{-1}, \tag{2.27}$$

where $V^{[2]}$ and Q are defined in (2.21) and (2.22), respectively.

The above analysis guarantees that the non-local reduction (2.18) does not require any new condition for the compatibility of the spatial and temporal matrix spectral problems in (2.20). Therefore, the standard matrix NLS equations (2.16) are reduced to the following nonlocal reverse-space matrix NLS equations:

$$ip_t(x,t) = \frac{\beta}{\alpha^2}[p_{xx}(x,t) - 2p(x,t)\Sigma_2^{-1}p^{\dagger}(-x,t)\Sigma_1 p(x,t)], \tag{2.28}$$

where Σ_1 and Σ_2 are two arbitrary invertible Hermitian matrices.

When $m = n = 1$, we can get two well-known scalar examples [3]:

$$ip_t(x,t) = p_{xx}(x,t) - 2\sigma p^2(x,t)p^*(-x,t),\ \sigma = \mp 1. \tag{2.29}$$

When $m = 1$ and $n = 2$, we can obtain a system of non-local reverse-space two-component NLS equations (1.8).

3 Inverse scattering transforms

3.1 Distribution of eigenvalues

We consider the non-local reduction case and so q is defined by (2.23). We are going to analyse the scattering and inverse scattering transforms for the non-local reverse-space matrix NLS equations (2.28) by the Riemann–Hilbert approach (see, e.g., [9, 14, 42]). The results will prepare the essential foundation for soliton solutions in the following section.

Assume that all the potentials sufficiently rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. For the matrix spectral problems in (2.20), we can impose the asymptotic behaviour: $\phi \sim e^{i\lambda\Lambda x + i\lambda^2\Omega t}$, when $x, t \to \pm\infty$. Therefore, if we take the variable transformation:

$$\phi = \psi E_g,\ E_g = e^{i\lambda\Lambda x + i\lambda^2\Omega t},$$

then we can have the canonical asymptotic conditions: $\psi \to I_{m+n}$, when $x, t \to \infty$ or $-\infty$. The equivalent pair of matrix spectral problems to (2.20) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \ \check{P} = iP, \tag{3.1}$$

$$\psi_t = i\lambda^2[\Omega, \psi] + \check{Q}\psi, \ \check{Q} = iQ. \tag{3.2}$$

Applying a generalised Liouville's formula [34], we can obtain

$$\det \psi = 1, \tag{3.3}$$

because $(\det \psi)_x = 0$ due to $\operatorname{tr} \check{P} = \operatorname{tr} \check{Q} = 0$.

Recall that the adjoint equation of the x-part of (2.20) and the adjoint equation of (3.1) are given by:

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{3.4}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \tag{3.5}$$

respectively. Obviously, there exist the links: $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$. Each pair of adjoint matrix spectral problems and equivalent adjoint matrix spectral problems do not create any new condition, either, except the non-local reverse-space matrix NLS equations (2.28).

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.1) associated with an eigenvalue λ. It is easy to see that $C\psi^{-1}(x, t, \lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Under the non-local reduction in (2.18), we have

$$\begin{aligned} &i[\psi^{\dagger}(-x, t, -\lambda^*)C]_x = i[-(\psi_x)^{\dagger}(-x, t, -\lambda^*)C] \\ &= -i\{(-i)(-\lambda)[\psi^{\dagger}(-x, t, -\lambda^*), \Lambda] + (-i)\psi^{\dagger}(-x, t, -\lambda^*)P^{\dagger}(-x, t)\}C \\ &= \lambda[\psi^{\dagger}(-x, t, -\lambda^*), \Lambda]C + \psi^{\dagger}(-x, t, -\lambda^*)C[-C^{-1}P^{\dagger}(-x, t)C] \\ &= \lambda[\psi^{\dagger}(-x, t, -\lambda^*)C, \Lambda] + \psi^{\dagger}(-x, t, -\lambda^*)CP(x, t). \end{aligned}$$

Thus, the matrix

$$\tilde{\psi}(x, t, \lambda) := \psi^{\dagger}(-x, t, -\lambda^*)C, \tag{3.6}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ. That is to say that $\psi^{\dagger}(-x, t, -\lambda^*)C$ solves the adjoint spectral problem (3.5).

Finally, we observe the asymptotic conditions for the matrix eigenfunction ψ, and see that by the uniqueness of solutions, we have

$$\psi^{\dagger}(-x, t, -\lambda^*) = C\psi^{-1}(x, t, \lambda)C^{-1}, \tag{3.7}$$

when $\psi \to I_{m+n}$, x or $t \to \infty$ or $-\infty$. This tells that if λ is an eigenvalue of (3.1) (or (3.5)), then $-\lambda^*$ will be another eigenvalue of (3.1) (or (3.5)), and there is the property (3.7) for the corresponding eigenfunction ψ.

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3.2 Riemann–Hilbert problems

Let us now start to formulate a class of associated Riemann–Hilbert problems with the variable x. In order to clearly state the problems, we also make the assumptions:

$$\alpha = \alpha_1 - \alpha_2 < 0, \ \beta = \beta_1 - \beta_2 < 0. \tag{3.8}$$

In the scattering problem, we first introduce the two matrix eigenfunctions $\psi^{\pm}(x, \lambda)$ of (3.1) with the asymptotic conditions:

$$\psi^{\pm} \to I_{m+n}, \text{ when } x \to \pm\infty, \tag{3.9}$$

respectively. It then follows from (3.3) that $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Because

$$\phi^{\pm} = \psi^{\pm} E, \ E = \mathrm{e}^{i\lambda\Lambda x}, \tag{3.10}$$

are both matrix eigenfunctions of (2.20), they must be linearly dependent, and as a result, one has

$$\psi^{-} E = \psi^{+} E S(\lambda), \ \lambda \in \mathbb{R}, \tag{3.11}$$

where $S(\lambda)$ is the corresponding scattering matrix. Note that $\det S(\lambda) = 1$, thanks to $\det \psi^{\pm} = 1$.

Through the method of variation in parameters, we can transform the x-part of (2.20) into the following Volterra integral equations for $\psi^{\pm}$ [42]:

$$\psi^{-}(\lambda, x) = I_{m+n} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y) \psi^{-}(\lambda, y) \mathrm{e}^{i\lambda\Lambda(y-x)} \, dy, \tag{3.12}$$

$$\psi^{+}(\lambda, x) = I_{m+n} - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y) \psi^{+}(\lambda, y) \mathrm{e}^{i\lambda\Lambda(y-x)} \, dy, \tag{3.13}$$

where the asymptotic conditions (3.9) have been imposed. Now, the theory of Volterra integral equations tells that by the Neumann series [18], we can show that the eigenfunctions $\psi^{\pm}$ exist and allow analytic continuations off the real axis $\lambda \in \mathbb{R}$ provided that the integrals on their right-hand sides converge (see, e.g., [6]). From the diagonal form of Λ and the first assumption in (3.8), we can see that the integral equation for the first m columns of ψ^{-} contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(x-y)}$, which decays because of $y < x$ in the integral, if λ takes values in the upper half-plane $\mathbb{C}^{+}$, and the integral equation for the last n columns of ψ^{+} contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(x-y)}$, which also decays because of $y > x$ in the integral, when λ takes values in the upper half-plane $\mathbb{C}^{+}$. Therefore, we see that these $m + n$ columns are analytic in the upper half-plane $\mathbb{C}^{+}$ and continuous in the closed upper half-plane $\bar{\mathbb{C}}^{+}$. In a similar manner, we can know that the last n columns of ψ^{-} and the first m columns of ψ^{+} are analytic in the lower half-plane $\mathbb{C}^{-}$ and continuous in the closed lower half-plane $\bar{\mathbb{C}}^{-}$.

In what follows, we give a detailed proof for the above statements. Let us express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \cdots, \psi_{m+n}^{\pm}), \tag{3.14}$$

that is, $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1 \le j \le m+n)$. We would like to prove that ψ_j^{-}, $1 \le j \le m$, and ψ_j^{+}, $m+1 \le j \le m+n$, are analytic at $\lambda \in \mathbb{C}^{+}$ and continuous at $\lambda \in \bar{\mathbb{C}}^{+}$; and ψ_j^{+}, $1 \le j \le m$, and ψ_j^{-}, $m+1 \le j \le m+n$, are analytic at $\lambda \in \mathbb{C}^{-}$ and continuous at $\lambda \in \bar{\mathbb{C}}^{-}$. We

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only to prove the result for ψ_j^-, $1 \le j \le m$, and the proofs for the other eigenfunctions follow analogously.

It is easy to obtain from the Volterra integral equation (3.12) that

$$\psi_j^-(\lambda, x) = \mathrm{e}_j + \int_{-\infty}^{x} R_1(\lambda, x, y)\psi_j^-(\lambda, y)\, dy, \;\; 1 \le j \le m, \tag{3.15}$$

and

$$\psi_j^-(\lambda, x) = \mathrm{e}_j + \int_{-\infty}^{x} R_2(\lambda, x, y)\psi_j^-(\lambda, y)\, dy, \;\; m+1 \le j \le m+n, \tag{3.16}$$

where the e_i are standard basis vectors of $\mathbb{R}^{m+n}$ and the matrices R_1 and R_2 are defined by:

$$R_1(\lambda, x, y) = i\begin{bmatrix} 0 & p(y) \\ \mathrm{e}^{-i\alpha\lambda(x-y)}q(y) & 0 \end{bmatrix}, \; R_2(\lambda, x, y) = i\begin{bmatrix} 0 & \mathrm{e}^{i\alpha\lambda(x-y)}p(y) \\ q(y) & 0 \end{bmatrix}. \tag{3.17}$$

Let us prove that for each $1 \le j \le m$, the solution to (3.15) is determined by the Neumann series:

$$\sum_{k=0}^{\infty} \phi_{j,k}^-(\lambda, x), \tag{3.18}$$

where

$$\phi_{j,0}^-(\lambda, x) = \mathrm{e}_j, \; phi_{j,k+1}^-(\lambda, x) = \int_{-\infty}^{x} R_1(\lambda, x, y)\phi_{j,k}^-(\lambda, y)\, dy, \;\; k \ge 1. \tag{3.19}$$

This will be true if we can prove that the Neumann series converges uniformly for $x \in \mathbb{R}$ and $\lambda \in \bar{\mathbb{C}}^+$. By the mathematical induction, we can have

$$|\phi_{j,k}^-(\lambda, x)| \le \frac{1}{k!}\left(\int_{-\infty}^{x} \|P(y)\| dy\right)^k, \;\; 1 \le j \le m, \;\; k \ge 0, \tag{3.20}$$

for $x \in \mathbb{R}$ and $\lambda \in \bar{\mathbb{C}}^+$, where $|\cdot|$ denotes the Euclidean norm for vectors and $\|\cdot\|$ stands for the Frobenius norm for square matrices. By the Weierstrass M-test, this estimation guarantees that

$$\phi_j^-(\lambda, x) = \sum_{k=0}^{\infty} \phi_{j,k}^-(\lambda, x), \;\; 1 \le j \le m, \tag{3.21}$$

uniformly converges for $\lambda \in \bar{\mathbb{C}}^+$ and $x \in \mathbb{R}$, and all $\phi_j^-(\lambda, x)$, $1 \le j \le m$, are continuous with respect to λ in $\bar{\mathbb{C}}^+$, since so are all $\phi_{j,k}^-(\lambda, x)$, $1 \le j \le m$, $k \ge 0$.

Let us now consider the differentiability of $\phi_j^-(\lambda, x)$, $1 \le j \le m$, with respect to λ in $\mathbb{C}^+$ (similarly, we can prove the differentiability with respect to x in $\mathbb{R}$). Fix an integer $1 \le j \le m$ and a number μ in $\mathbb{C}^+$. Choose a disc $B_r(\mu) = \{\lambda \in \mathbb{C} \,|\, |\lambda - \mu| \le r\}$ with a radius $r > 0$ such that $B_r(\mu) \subseteq \mathbb{C}^+$, and then we can have a constant $C(r) > 0$ such that $|\alpha x \mathrm{e}^{-i\alpha\lambda x}| \le C(r)$ for $\lambda \in B_r(\mu)$ and $x \ge 0$. We define the following Neumann series:

$$\sum_{k=0}^{\infty} \phi_{j,\lambda,k}^-(\lambda, x), \tag{3.22}$$

where $\phi^-_{j,\lambda,0}=0$ and

$$\phi^-_{j,\lambda,k+1}(\lambda,x)=\int_{-\infty}^{x} R_{1,\lambda}(\lambda,x,y)\phi^-_{j,k}(\lambda,y)\,dy+\int_{-\infty}^{x} R_1(\lambda,x,y)\phi^-_{j,\lambda,k}(\lambda,y)\,dy,\ k\geq 0, \tag{3.23}$$

with the $\phi^-_{j,k}$ being given by (3.19) and $R_{1,\lambda}$ being defined by:

$$R_{1,\lambda}(\lambda,x,y)=\frac{\partial}{\partial\lambda}R_1(\lambda,x,y)=\begin{bmatrix} 0 & 0 \\ \alpha(x-y)\mathrm{e}^{-i\alpha\lambda(x-y)}q(y) & 0 \end{bmatrix}. \tag{3.24}$$

We can readily verify by the mathematical induction that

$$|\phi^-_{j,\lambda,k}(\lambda,x)|\leq \frac{1}{k!}\left\{\left[C(r)+1\right]\int_{-\infty}^{x}\|P(y)\|dy\right\}^k,\ k\geq 0, \tag{3.25}$$

for $x\in\mathbb{R}$ and $\lambda\in B_r(\mu)$. Therefore, by the Weierstrass M-test, the Neumann series defined by (3.22) converges uniformly for $x\in\mathbb{R}$ and $\lambda\in B_r(\mu)$; and by the term-by-term differentiability theorem, it converges to the derivative of ϕ^-_j with respect to λ, due to $\psi^-_{j,\lambda,k}=\frac{\partial}{\partial\lambda}\phi^-_{j,k}$, $k\geq 0$. Therefore, ϕ^-_j is analytic at an arbitrarily fixed point $\mu\in\mathbb{C}^+$. It then follows that all ϕ^-_j, $1\leq j\leq m$, are analytic with respect to λ in $\mathbb{C}^+$. The required proof is done.

Based on the above analysis, we can then form the generalised matrix Jost solution T^+ as follows:

$$T^+=T^+(x,\lambda)=(\psi^-_1,\cdots,\psi^-_m,\psi^+_{m+1},\cdots,\psi^+_{m+n})=\psi^-H_1+\psi^+H_2, \tag{3.26}$$

which is analytic with respect to λ in $\mathbb{C}^+$ and continuous with respect to λ in $\bar{\mathbb{C}}^+$. The generalised matrix Jost solution:

$$(\psi^+_1,\cdots,\psi^+_m,\psi^-_{m+1},\cdots,\psi^-_{m+n})=\psi^+H_1+\psi^-H_2 \tag{3.27}$$

is analytic with respect to λ in $\mathbb{C}^-$ and continuous with respect to λ in $\bar{\mathbb{C}}^-$. In the above definition, we have used

$$H_1=\mathrm{diag}(I_m,\underbrace{0,\cdots,0}_{n}),\ H_2=\mathrm{diag}(\underbrace{0,\cdots,0}_{m},I_n). \tag{3.28}$$

To construct the other generalised matrix Jost solution T^-, we adopt the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$, which can be generated from the adjoint counterparts of the matrix spectral problems. Note that the inverse matrices $\tilde{\phi}^{\pm}=(\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm}=(\psi^{\pm})^{-1}$ solve those two adjoint equations, respectively. Then, stating $\tilde{\psi}^{\pm}$ as:

$$\tilde{\psi}^{\pm}=(\tilde{\psi}^{\pm,1},\tilde{\psi}^{\pm,2},\cdots,\tilde{\psi}^{\pm,m+n})^T, \tag{3.29}$$

that is, $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^{\pm}$ $(1\leq j\leq m+n)$, we can verify by similar arguments that we can form the generalised matrix Jost solution T^- as the adjoint matrix solution of (3.5), that is,

$$T^-=(\tilde{\psi}^{-,1},\cdots,\tilde{\psi}^{-,m},\tilde{\psi}^{+,m+1},\cdots,\tilde{\psi}^{+,m+n})^T=H_1\tilde{\psi}^-+H_2\tilde{\psi}^+=H_1(\psi^-)^{-1}+H_2(\psi^+)^{-1}, \tag{3.30}$$

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which is analytic with respect to λ in $\mathbb{C}^-$ and continuous with respect to λ in $\bar{\mathbb{C}}^-$, and the other generalised matrix Jost solution of (3.5):

$$(\tilde{\psi}^{+,1},\cdots,\tilde{\psi}^{+,m},\tilde{\psi}^{-,m+1},\cdots,\tilde{\psi}^{-,m+n})^T=H_1\tilde{\psi}^{+}+H_2\tilde{\psi}^{-}=H_1(\psi^+)^{-1}+H_2(\psi^-)^{-1}, \quad (3.31)$$

is analytic with respect to λ in $\mathbb{C}^+$ and continuous with respect to λ in $\bar{\mathbb{C}}^+$.

Now we have finished the construction of the two generalised matrix Jost solutions, T^+ and T^-. Directly from $\det\psi^{\pm}=1$ and using the scattering relation (3.11) between ψ^+ and ψ^-, we arrive at

$$\lim_{x\to\infty}T^+(x,\lambda)=\begin{bmatrix} S_{11}(\lambda) & 0\\ 0 & I_n\end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^+,\ lim_{x\to-\infty}T^-(x,\lambda)=\begin{bmatrix} \hat{S}_{11}(\lambda) & 0\\ 0 & I_n\end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^-, \quad (3.32)$$

and

$$\det T^+(x,\lambda)=\det S_{11}(\lambda),\ \det T^-(x,\lambda)=\det\hat{S}_{11}(\lambda), \quad (3.33)$$

where we split $S(\lambda)$ and $S^{-1}(\lambda)$ as follows:

$$S(\lambda)=\begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda)\\ S_{21}(\lambda) & S_{22}(\lambda)\end{bmatrix},\ S^{-1}(\lambda)=(S(\lambda))^{-1}=\begin{bmatrix} \hat{S}_{11}(\lambda) & \hat{S}_{12}(\lambda)\\ \hat{S}_{21}(\lambda) & \hat{S}_{22}(\lambda)\end{bmatrix}, \quad (3.34)$$

$S_{11},\hat{S}_{11}$ being $m\times m$ matrices, $S_{12},\hat{S}_{12}$ being $m\times n$ matrices, $S_{21},\hat{S}_{21}$ being $n\times m$ matrices and $S_{22},\hat{S}_{22}$ being $n\times n$ matrices. Based on the uniform convergence of the previous Neumann series, we know that $S_{11}(\lambda)$ and $\hat{S}_{11}(\lambda)$ are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

In this way, we can introduce the following two unimodular generalised matrix Jost solutions:

$$\begin{cases} G^+(x,\lambda)=T^+(x,\lambda)\begin{bmatrix} S_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}, & \lambda\in\bar{\mathbb{C}}^+;\\ (G^-)^{-1}(x,\lambda)=\begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}T^-(x,\lambda), & \lambda\in\bar{\mathbb{C}}^-.\end{cases} \quad (3.35)$$

Those two generalised matrix Jost solutions establish the required matrix Riemann–Hilbert problems on the real line for the non-local reverse-space matrix NLS equations (2.28):

$$G^+(x,\lambda)=G^-(x,\lambda)G_0(x,\lambda),\ \lambda\in\mathbb{R}, \quad (3.36)$$

where the jump matrix G_0 is

$$G_0(x,\lambda)=E\begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}\tilde{S}(\lambda)\begin{bmatrix} S_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}E^{-1}, \quad (3.37)$$

based on (3.11). In the jump matrix G_0, the matrix $\tilde{S}(\lambda)$ has the factorisation:

$$\tilde{S}(\lambda)=(H_1+H_2S(\lambda))\,(H_1+S^{-1}(\lambda)H_2), \quad (3.38)$$

which can be shown to be

$$\tilde{S}(\lambda)=\begin{bmatrix} I_m & \hat{S}_{12}\\ S_{21} & I_n\end{bmatrix}. \quad (3.39)$$

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Following the Volterra integral equations (3.12) and (3.13), we can obtain the canonical normalisation conditions:

$$G^{\pm}(x,\lambda) \to I_{m+n}, \text{ when } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty, \tag{3.40}$$

for the presented Riemann–Hilbert problems. From the property (3.7), we can also observe that

$$(G^{+})^{\dagger}(-x,t,-\lambda^{*}) = C(G^{-})^{-1}(x,t,\lambda)C^{-1}, \tag{3.41}$$

and thus, the the jump matrix G_0 possesses the following involution property:

$$G_0^{\dagger}(-x,t,-\lambda^{*}) = CG_0(x,t,\lambda)C^{-1}. \tag{3.42}$$

3.3 Evolution of the scattering data

To complete the direct scattering transforms, let us take the derivative of (3.11) with time t and use the temporal matrix spectral problems:

$$\psi_t^{\pm} = i\lambda^2[\Omega,\psi^{\pm}] + \check{Q}\psi^{\pm}. \tag{3.43}$$

It then follows that the scattering matrix S satisfies the following evolution law:

$$S_t = i\lambda^2[\Omega, S]. \tag{3.44}$$

This tells the time evolution of the time-dependent scattering coefficients:

$$S_{12} = S_{12}(t,\lambda) = S_{12}(0,\lambda)\,\mathrm{e}^{i\beta\lambda^2 t},\ S_{21} = S_{21}(t,\lambda) = S_{21}(0,\lambda)\,\mathrm{e}^{-i\beta\lambda^2 t}, \tag{3.45}$$

and all other scattering coefficients are independent of the time variable t.

3.4 Gelfand–Levitan–Marchenko-type equations

To obtain Gelfand–Levitan–Marchenko-type integral equations to determine the generalised matrix Jost solutions, let us transform the associated Riemann–Hilbert problem (3.36) into

$$\begin{cases} G^{+} - G^{-} = G^{-}v,\ v = G_0 - I_{m+n},\ \text{on } \mathbb{R}, \\ G^{\pm} \to I_{m+n} \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty, \end{cases} \tag{3.46}$$

where the jump matrix G_0 is defined by (3.37) and (3.38).

Let $G(\lambda) = G^{\pm}(\lambda)$ if $\lambda \in \mathbb{C}^{\pm}$. Assume that G has simple poles off $\mathbb{R}$: $\{\mu_j\}_{j=1}^{R}$, where R is an arbitrary integer. Define

$$\tilde{G}^{\pm}(\lambda) = G^{\pm}(\lambda) - \sum_{j=1}^{R} \frac{G_j}{\lambda - \mu_j},\ \lambda \in \bar{\mathbb{C}}^{\pm};\ \tilde{G}(\lambda) = \tilde{G}^{\pm}(\lambda),\ \lambda \in \mathbb{C}^{\pm}, \tag{3.47}$$

where G_j is the residue of G at $\lambda = \mu_j$, that is,

$$G_j = \mathrm{res}(G(\lambda),\lambda_j) = \lim_{\lambda \to \mu_j} (\lambda - \mu_j)G(\lambda). \tag{3.48}$$

This tells that we have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v, \text{ on } \mathbb{R}, \\ \tilde{G}^\pm \to I_{m+n} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty. \end{cases} \tag{3.49}$$

By applying the Sokhotski–Plemelj formula [12], we get the solution of (3.49):

$$\tilde{G}(\lambda) = I_{m+n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda} \, d\xi. \tag{3.50}$$

Further taking the limit as $\lambda \to \mu_l$ yields

$$\text{LHS} = \lim_{\lambda \to \mu_l} \tilde{G} = F_l - \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j},$$

$$\text{RHS} = I_{m+n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l} \, d\xi,$$

where

$$F_l = \lim_{\lambda \to \mu_l} \frac{(\lambda - \mu_l) G(\lambda) - G_l}{\lambda - \mu_l}, \ 1 \leq l \leq R, \tag{3.51}$$

and consequently, we see that the required Gelfand–Levitan–Marchenko-type integral equations are as follows:

$$I_{m+n} - F_l + \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l} \, d\xi = 0, \ 1 \leq l \leq R. \tag{3.52}$$

All these equations are used to determine solutions to the associated Riemann–Hilbert problems and thus the generalised matrix Jost solutions. However, little was yet known about the existence and uniqueness of solutions. In the case of soliton solutions, a formulation of solutions, where eigenvalues could equal adjoint eigenvalues, will be presented for non-local integrable equations in the next section.

3.5 Recovery of the potential

To recover the potential matrix P from the generalised matrix Jost solutions, as usual, we make an asymptotic expansion:

$$G^+(x, t, \lambda) = I_{m+n} + \frac{1}{\lambda} G_1^+(x, t) + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \ \lambda \to \infty. \tag{3.53}$$

Then, plugging this asymptotic expansion into the matrix spectral problem (3.1) and comparing O(1) terms generates

$$P = \lim_{\lambda \to \infty} \lambda [G^+(\lambda), \Lambda] = -[\Lambda, G_1^+]. \tag{3.54}$$

This leads exactly to the potential matrix:

$$P = \begin{bmatrix} 0 & -\alpha G_{1,12}^+ \\ \alpha G_{1,21}^+ & 0 \end{bmatrix}, \tag{3.55}$$

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where we have similarly partitioned the matrix G_1^+ into four blocks as follows:

$$G_1^+ = \begin{bmatrix} G_{1,11}^+ & G_{1,12}^+ \\ G_{1,21}^+ & G_{1,22}^+ \end{bmatrix} = \begin{bmatrix} (G_{1,11}^+)_{n\times n} & (G_{1,12}^+)_{n\times m} \\ (G_{1,21}^+)_{m\times n} & (G_{1,22}^+)_{m\times m} \end{bmatrix}. \tag{3.56}$$

Therefore, the solutions to the standard matrix NLS equations (2.16) read

$$p = -\alpha G_{1,12}^+, \; q = \alpha G_{1,21}^+. \tag{3.57}$$

When the non-local reduction condition (2.18) is satisfied, the reduced matrix potential p solves the non-local reverse-space matrix NLS equations (2.28).

To conclude, this completes the inverse scattering procedure for computing solutions to the non-local reverse-space matrix NLS equations (2.28), from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^+(\lambda), G^-(\lambda)\}$ of the associated Riemann–Hilbert problems, to the potential matrix P.

4 Soliton solutions

4.1 Non-reduced local case

Let $N \geq 1$ be another arbitrary integer. Assume that $\det S_{11}(\lambda)$ has N zeros $\{\lambda_k \in \mathbb{C},\; 1 \leq k \leq N\}$, and $\det \hat{S}_{11}(\lambda)$ has N zeros $\{\hat{\lambda}_k \in \mathbb{C},\; 1 \leq k \leq N\}$.

In order to present soliton solutions explicitly, we also assume that all these zeros, λ_k and $\hat{\lambda}_k,\; 1 \leq k \leq N$, are geometrically simple. Then, each of $\ker T^+(\lambda_k),\; 1 \leq k \leq N$, contains only a single basis column vector, denoted by $v_k,\; 1 \leq k \leq N$; and each of $\ker T^-(\hat{\lambda}_k),\; 1 \leq k \leq N$, a single basis row vector, denoted by $\hat{v}_k,\; 1 \leq k \leq N$:

$$T^+(\lambda_k)v_k = 0, \; \hat{v}_k T^-(\hat{\lambda}_k) = 0, \; 1 \leq k \leq N. \tag{4.1}$$

Soliton solutions correspond to the situation where $G_0 = I_{m+n}$ is taken in each Riemann–Hilbert problem (3.36). This can be achieved if we assume that $S_{21} = \hat{S}_{12} = 0$, which means that the reflection coefficients are taken as zero in the scattering problem.

This kind of special Riemann–Hilbert problems with the canonical normalisation conditions in (3.40) and the zero structures given in (4.1) can be solved precisely, in the case of local integrable equations [21, 42], and consequently, we can exactly work out the potential matrix P. However, in the case of non-local integrable equations, we often do not have

$$\{\lambda_k | 1 \leq k \leq N\} \cap \{\hat{\lambda}_k | 1 \leq k \leq N\} = \emptyset. \tag{4.2}$$

Without this condition, the solutions to the special Riemann–Hilbert problem with the identity jump matrix can be presented as follows (see, e.g., [32]):

$$G^+(\lambda) = I_{m+n} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \; (G^-)^{-1}(\lambda) = I_{m+n} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{4.3}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix with its entries:

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \leq k, l \leq N, \tag{4.4}$$

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and we need an orthogonal condition:

$$\hat{v}_k v_l = 0, \text{ if } \lambda_l = \hat{\lambda}_k, \ 1 \leq k, l \leq N, \tag{4.5}$$

to guarantee that $G^+(\lambda)$ and $G^-(\lambda)$ solve

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{m+n}. \tag{4.6}$$

Note that the zeros λ_k and $\hat{\lambda}_k$ are constants, that is, space- and time-independent, and so we can easily determine the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \leq k \leq N$, in the kernels. For instance, let us compute the x-derivative of both sides of the first set of equations in (4.1). Applying (3.1) first and then again the first set of equations in (4.1), we arrive at

$$P^+(x,\lambda_k)\Big(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\Big) = 0, \ 1 \leq k \leq N. \tag{4.7}$$

This implies that for each $1 \leq k \leq N$, $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ is in the kernel of $P^+(x,\lambda_k)$, and thus, a constant multiple of v_k, since λ_k is geometrically simple. Without loss of generality, we can simply take

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k, \ 1 \leq k \leq N. \tag{4.8}$$

The time dependence of v_k:

$$\frac{dv_k}{dt} = i\lambda_k^2 \Omega v_k, \ 1 \leq k \leq N, \tag{4.9}$$

can be achieved similarly through an application of the t-part of the matrix spectral problem, (3.2). In consequence of these differential equations, we obtain

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k, \ 1 \leq k \leq N, \tag{4.10}$$

and completely similarly, we can have

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t}, \ 1 \leq k \leq N, \tag{4.11}$$

where w_k and $\hat{w}_k$, $1 \leq k \leq N$, are arbitrary constant column and row vectors, respectively, but need to satisfy an orthogonal condition:

$$\hat{w}_k w_l = 0, \text{ if } \lambda_l = \hat{\lambda}_k, \ 1 \leq k, l \leq N, \tag{4.12}$$

which is a consequence of (4.5).

Finally, from the solutions in (4.3), we get

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.13}$$

and thus, the presentations in (3.57) yield the following N-soliton solution to the standard matrix NLS equations (2.16):

$$p = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,2}, \ q = -\alpha \sum_{k,l=1}^{N} v_{k,2}(M^{-1})_{kl}\hat{v}_{l,1}. \tag{4.14}$$

Here for each $1 \le k \le N$, we split $v_k = ((v_{k,1})^T, (v_{k,2})^T)^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2})$, where $v_{k,1}$ and $\hat{v}_{k,1}$ are m-dimensional column and row vectors, respectively, and $v_{k,2}$ and $\hat{v}_{k,2}$ are n-dimensional column and row vectors, respectively.

4.2 Reduced non-local case

To compute N-soliton solutions for the non-local reverse-space matrix NLS equations (2.28), we need to check if G_1^+ defined by (4.13) satisfies an involution property:

$$(G_1^+(-x,t))^\dagger = CG_1^+(x,t)C^{-1}. \tag{4.15}$$

This equivalently requires that the potential matrix P determined by (3.55) satisfies the non-local reduction condition (2.18). Thus, the N-soliton solution to the standard matrix NLS equations (2.16) is reduced to the N-soliton solution:

$$p = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,2}, \tag{4.16}$$

for the non-local reverse-space matrix NLS equations (2.28), where we split $v_k = ((v_{k,1})^T, (v_{k,2})^T)^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2})$, $1 \le k \le N$, as before.

Let us now show how to realise the involution property (4.15). We first take N distinct zeros of $\det T^+(\lambda)$ (or eigenvalues of the spectral problems under the zero potential): $\lambda_k \in \mathbb{C},\ 1 \le k \le N$, and define

$$\hat{\lambda}_k = \begin{cases} -\lambda_k^*, \text{ if } \lambda_k \notin i\mathbb{R},\ 1 \le k \le N, \\ \text{any value } \in i\mathbb{R}, \text{ if } \lambda_k \in i\mathbb{R},\ 1 \le k \le N, \end{cases} \tag{4.17}$$

which are zeros of $\det T^-(\lambda)$. We recall that the $\ker T^+(\lambda_k)$, $1 \le k \le N$, are spanned by:

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k,\ 1 \le k \le N, \tag{4.18}$$

respectively, where $w_k,\ 1 \le k \le N$, are arbitrary column vectors. These column vectors in (4.18) are eigenfunctions of the spectral problems under the zero potential associated with $\lambda_k,\ 1 \le k \le N$. Furthermore, following the previous analysis in Subsection 3.1, the $\ker T^-(\lambda_k)$, $1 \le k \le N$, are spanned by:

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)C = w_k^\dagger \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t}C,\ 1 \le k \le N, \tag{4.19}$$

respectively. These row vectors are eigenfunctions of the adjoint spectral problems under the zero potential associated with $\hat{\lambda}_k,\ 1 \le k \le N$. To satisfy the orthogonal property (4.12), we require the following orthogonal condition:

$$w_k^\dagger C w_l = 0,\ \text{if } \lambda_l = \hat{\lambda}_k,\ 1 \le k,l \le N, \tag{4.20}$$

on the constant columns $\{w_k \mid 1 \le k \le N\}$. Interestingly, the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when $\lambda_k \in i\mathbb{R}$ and $\hat{\lambda}_k = -\lambda_k^*$.

Now, we can directly see that if the solutions to the specific Riemann–Hilbert problems, determined by (4.3) and (4.4), satisfy the property (3.41), then the corresponding matrix G_1^+ possesses the involution property (4.15) generated from each non-local reduction in (2.17). Accordingly, the formula (4.16), together with (4.3), (4.4), (4.18) and (4.19), presents the required N-soliton solutions to the non-local reverse-space matrix NLS equations (2.28).

When $m=n=N=1$, we choose $\lambda_1 = i\eta_1,\ \hat{\lambda}_1 = -i\eta_1,\ \eta_1 \in \mathbb{R}$ and denote $w_1 = (w_{1,1}, w_{1,2})^T$. Then, we can obtain the following one-soliton solution to the non-local reverse-space scalar NLS equations in (2.29):

$$p(x,t) = \frac{2\eta_1 i w_{1,1} w_{1,2}^*}{\varepsilon |w_{1,1}|^2 \mathrm{e}^{-\eta_1 x + i\eta_1^2 t} + |w_{1,2}|^2 \mathrm{e}^{\eta_1 x + i\eta_1^2 t}}, \tag{4.21}$$

where $\varepsilon = \pm 1$, η_1 is an arbitrary real number, and $w_{1,1}$ and $w_{1,2}$ are arbitrary complex numbers but satisfy $\sigma |w_{1,1}|^2 + |w_{1,2}|^2 = 0$, which comes from the involution property (4.15). The condition for w_1 implies that we need to take $\sigma = -1$. This solution has a singularity at $x = -\frac{\ln \varepsilon\sigma}{2\eta_1}$ when $\varepsilon\sigma > 0$, and the case of $\varepsilon = 1$ and $\sigma = -1$ can present the breather one-soliton in [4].

When $m=1,\ n=2$ and $N=1$, we take $C = \mathrm{diag}(1, \gamma_1, \gamma_2)$, where γ_1 and γ_2 are arbitrary non-zero real numbers. Then the non-local reverse-space matrix NLS equations (2.28) becomes

$$\begin{cases} ip_{1,t}(x,t) = \dfrac{\beta}{\alpha^2}\left[p_{1,xx}(x,t) - 2\left(\dfrac{1}{\gamma_1} p_1(x,t) p_1^*(-x,t) + \dfrac{1}{\gamma_2} p_2(x,t) p_2^*(-x,t)\right) p_1(x,t)\right], \\ ip_{2,t}(x,t) = \dfrac{\beta}{\alpha^2}\left[p_{2,xx}(x,t) - 2\left(\dfrac{1}{\gamma_1} p_1(x,t) p_1^*(-x,t) + \dfrac{1}{\gamma_2} p_2(x,t) p_2^*(-x,t)\right) p_2(x,t)\right]. \end{cases} \tag{4.22}$$

According to our formulation of solutions above, this system has the following one-soliton solution:

$$\begin{cases} p_1(x,t) = \dfrac{\alpha(\lambda_1 + \lambda_1^*) w_{1,1} w_{1,2}^*}{|w_{1,1}|^2 \mathrm{e}^{i[\alpha\lambda_1^* x - \beta(\lambda_1^*)^2 t]} + (\gamma_1 |w_{1,2}|^2 + \gamma_2 |w_{1,3}|^2) \mathrm{e}^{-i(\alpha\lambda_1 x + \beta\lambda_1^2 t)}}, \\ p_2(x,t) = \dfrac{\alpha(\lambda_1 + \lambda_1^*) w_{1,1} w_{1,3}^*}{|w_{1,1}|^2 \mathrm{e}^{i[\alpha\lambda_1^* x - \beta(\lambda_1^*)^2 t]} + (\gamma_1 |w_{1,2}|^2 + \gamma_2 |w_{1,3}|^2) \mathrm{e}^{-i(\alpha\lambda_1 x + \beta\lambda_1^2 t)}}, \end{cases} \tag{4.23}$$

provided that $w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^T$ satisfies the orthogonal conditions:

$$w_{1,1}^2 + \gamma_1 w_{1,2}^2 + \gamma_2 w_{1,3}^2 = 0,\ |w_{1,1}|^2 + \gamma_1 |w_{1,2}|^2 + \gamma_2 |w_{1,3}|^2 = 0.$$

Note that the product $\gamma_1\gamma_2$ could be either positive or negative.

5 Concluding remarks

The paper aims to present a class of non-local reverse-space integrable matrix non-linear Schrödinger (NLS) equations and their inverse scattering transforms. The main analysis is based on Riemann–Hilbert problems associated with a kind of arbitrary-order matrix spectral problems with matrix potentials. Through the the Sokhotski–Plemelj formula, the associated Riemann–Hilbert problems were transformed into Gelfand–Levitan–Marchenko-type integral equations, and the corresponding reflectionless problems were solved to generate soliton solutions for the non-local reverse-space matrix NLS equations.

The Riemann–Hilbert technique, which is very effective in generating soliton solutions (see also, e.g., [22, 48]), has been recently generalised to solve various initial-boundary value problems of continuous integrable equations on the half-line and the finite interval [11, 23]. There

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are many other approaches to soliton solutions that work well and are easy to use, among which are the Hirota direct method [19], the generalised bilinear technique [25], the Wronskian technique [17, 33] and the Darboux transformation [35, 40]. It would be significantly important to search for clear connections between different approaches to explore dynamical characteristics of soliton phenomena.

We also emphasise that it would be particularly interesting to construct various kinds of solutions other than solitons to integrable equations, such as positon and complexiton solutions [24, 41], lump and rogue wave solutions [38]–[31], Rossby wave solutions [50], solitonless solutions [28, 29, 43] and algebro-geometric solutions [7, 26], from a perspective of the Riemann–Hilbert technique. Another interesting topic for further study is to establish a general formulation of Riemann–Hilbert problems for solving generalised integrable equations, for example, integrable couplings, super integrable equations and fractional analogous equations.

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Conflict of interest

None.

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