Bull. Aust. Math. Soc. **81** (2010), 414–417
doi:10.1017/S0004972709001063

A NEW PROOF OF THE AMENABILITY OF $C(X)$

MORTAZA ABTAHI ✉ and YONG ZHANG

(Received 17 April 2009)

Abstract

In this paper, we present a constructive proof of the amenability of $C(X)$, where X is a compact space.

2000 *Mathematics subject classification*: primary 46J10.
Keywords and phrases: Banach algebra of continuous functions, amenability, approximate diagonal.

The concept of amenability for a Banach algebra A was introduced by Johnson in 1972 [5], and has proved to be of enormous importance in Banach algebra theory. Several modifications of this notion were introduced in [2, 3].

Let A be a Banach algebra, and let X be a Banach A-bimodule. A *derivation* is a continuous linear map $D : A \to X$ such that

$$D(ab) = aD(b) + D(a)b \quad (a, b \in A).$$

For $x \in X$, set $\mathrm{ad}_x : a \mapsto ax - xa$, $A \to X$. Then ad_x is the *inner derivation* induced by x.

The dual of a Banach space X is denoted by X^*; in the case where X is a Banach A-bimodule, X^* is also a Banach A-bimodule. For the standard dual module definitions, see [1].

According to Johnson's original definition, a Banach algebra A is *amenable* if, for every Banach A-bimodule X, every derivation from A into X^* is inner.

It is known that $C(X)$, for a compact space X, is an amenable Banach algebra, [4, Theorem 5.1.87]. Here, we give a constructive proof of this result. First, we prepare some preliminaries.

DEFINITION 1. Let A be a Banach algebra. An *approximate diagonal* for A is a net (u_α) in $A \widehat{\otimes} A$ such that, for each $a \in A$,

$$au_\alpha - u_\alpha a \to 0 \quad \text{and} \quad a\pi(u_\alpha) \to a.$$

Here, and in what follows, π always denotes the product morphism from $A\widehat{\otimes}A$ into A, specified by $\pi(a\otimes b)=ab$.

It proved very useful in the classical theory of amenability to have characterizations in terms of virtual diagonals or approximate diagonals. Later, Johnson [6] obtained the following theorem.

THEOREM 2. *A Banach algebra A is amenable if and only if it has a bounded approximate diagonal.*

LEMMA 3 [4, Proposition 0.3.30]. *For Banach algebras A and B, if* $u=\sum_1^n u_k\otimes v_k\in A\otimes B$, *then*

$$\|u\|_p\le\frac{1}{n}\sum_{k=1}^n\Big\|\sum_{\ell=1}^n u_\ell\zeta^{k\ell}\Big\|\,\Big\|\sum_{j=1}^n v_j\zeta^{-kj}\Big\|,$$

where $\zeta=e^{i\theta}$, $\theta=2\pi/n$ *and* $\|u\|_p$ *represents the projective tensor norm of* u.

LEMMA 4. *Let* $n\in\mathbb{N}$ *and let* z_k, w_k $(k=1,\ldots,n)$ *be complex numbers. Let* $\zeta=e^{i\theta}$, *where* $\theta=2\pi/n$. *Then*

$$\frac{1}{n}\sum_{k=1}^n\Big|\sum_{\ell=1}^n z_\ell\zeta^{k\ell}\Big|\Big|\sum_{j=1}^n w_j\zeta^{-kj}\Big|\le\frac{1}{2}\Big(\sum_{\ell=1}^n|z_\ell|^2+\sum_{j=1}^n|w_j|^2\Big).$$

PROOF. For $k\in\{1,\ldots,n\}$, let $R_k=|\sum_{\ell=1}^n z_\ell\zeta^{k\ell}|$ and $T_k=|\sum_{j=1}^n w_j\zeta^{-kj}|$. Then

$$A=\frac{1}{n}\sum_{k=1}^n R_kT_k\le\frac{1}{2n}\sum_{k=1}^n(R_k^2+T_k^2).$$

We have

$$R_k^2=\Big(\sum_{\ell=1}^n z_\ell\zeta^{k\ell}\Big)\Big(\sum_{j=1}^n\overline{z}_j\zeta^{-kj}\Big)=\sum_{\ell=1}^n|z_\ell|^2+2\,\mathrm{Re}\sum_{j<\ell}z_\ell\overline{z}_j\zeta^{k(\ell-j)}.$$

For $1\le j<\ell\le n$, $\zeta^{\ell-j}$ is an nth root of unity and $\zeta^{\ell-j}\ne1$, so $\sum_{k=1}^n\zeta^{k(\ell-j)}=0$. This implies that

$$\sum_{k=1}^n R_k^2=n\sum_{\ell=1}^n|z_\ell|^2+2\ \mathrm{Re}\sum_{j<\ell}z_\ell\overline{z}_j\sum_{k=1}^n\zeta^{k(\ell-j)}=n\sum_{\ell=1}^n|z_\ell|^2.$$

Similarly,

$$\sum_{k=1}^n T_k^2=n\sum_{j=1}^n|w_j|^2.$$

We, therefore, have

$$A\le\frac{1}{2}\Big(\sum_{\ell=1}^n|z_\ell|^2+\sum_{j=1}^n|w_j|^2\Big).\qquad\square$$

https://doi.org/10.1017/S0004972709001063 Published online by Cambridge University Press

COROLLARY 5. *Let X be a compact space. Let $u = \sum_{k=1}^{n} u_k \otimes v_k \in C(X) \otimes C(X)$. Then*

$$\|u\|_\ell \leq \frac{1}{2}\left(\left\|\sum_{\ell=1}^{n} |u_\ell|^2\right\| + \left\|\sum_{j=1}^{n} |v_j|^2\right\|\right),$$

where $\|\cdot\|$ is the uniform norm on $C(X)$.

MAIN THEOREM. *Let X be a compact Hausdorff space. Then $C(X)$ has a bounded approximate diagonal and so it is amenable.*

PROOF. Let F be a finite subset of $C(X)$ and $\varepsilon > 0$. For every $x \in X$, there exists a neighborhood V_x of x such that if $s \in V_x$ and $a \in F$, then $|a(s) - a(x)| < \varepsilon/2$. Since X is compact, there exist $x_1, \ldots, x_n \in X$ such that

$$X \subset V_1 \cup \cdots \cup V_n \quad (V_i = V_{x_i}).$$

There exist nonnegative continuous functions $h_1, \ldots, h_n$ such that $\text{Supp}(h_k) \subset V_k$ and $h_1 + \cdots + h_n = 1$ on X (see [7, Theorem 2.13]).

For $k = 1, \ldots, n$, let $u_k = \sqrt{h_k}$ and $u = \sum_{k=1}^{n} u_k \otimes u_k$. Clearly, $\pi(u) = \sum h_k = 1$. We prove that:

(1) $\|u\|_p \leq 1$;
(2) $\|au - ua\|_p < \varepsilon$ for all $a \in F$.

Claim (1) is clear from Corollary 5. Now we prove claim (2). For $a \in F$, let $a_k = a - a(x_k)$. Then, for any $s \in V_k$, $|a_k(s)| < \varepsilon/2$. We have

$$\begin{aligned} au - ua &= \sum_{k=1}^{n} (au_k \otimes u_k - u_k \otimes au_k) \\ &= \sum_{k=1}^{n} ((a - a(x_k))u_k \otimes u_k - u_k \otimes (a - a(x_k))u_k) \\ &= \sum_{k=1}^{n} a_k u_k \otimes u_k - \sum_{k=1}^{n} u_k \otimes a_k u_k. \end{aligned}$$

Therefore,

$$\|au - ua\|_p \leq \left\|\sum_{k=1}^{n} a_k u_k \otimes u_k\right\|_p + \left\|\sum_{k=1}^{n} u_k \otimes a_k u_k\right\|_p.$$

Denote $\eta = \varepsilon/2$ and write

$$\sum a_k u_k \otimes u_k = \sum \frac{1}{\sqrt{\eta}} a_k u_k \otimes \sqrt{\eta} u_k.$$

By Corollary 5,

$$\left\|\sum_{k=1}^{n} a_k u_k \otimes u_k\right\|_p \leq \frac{1}{2}\left(\left\|\sum_{k=1}^{n} \frac{|a_k|^2}{\eta} h_k\right\| + \eta\left\|\sum_{k=1}^{n} h_k\right\|\right) \leq \eta.$$

Similarly $\|\sum u_k \otimes a_k u_k\|_p < \varepsilon/2$. This completes the proof. □

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MORTAZA ABTAHI, Department of Mathematics, Damghan University of Basic Sciences, Damghan 3671641167, Iran
e-mail: abtahi@dubs.ac.ir

YONG ZHANG, Department of Mathematics, University of Manitoba, R3T 2N2, Canada
e-mail: zhangy@cc.umanitoba.ca

https://doi.org/10.1017/S0004972709001063 Published online by Cambridge University Press