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THE CLASS NUMBER FORMULA OF A REAL QUADRATIC FIELD AND AN ESTIMATE OF THE VALUE OF A UNIT

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ABSTRACT. Our aim is to give an arithmetical expression of the class number formula of real quadratic fields. Starting from the classical Dirichlet class number formula, our proof goes along arithmetical lines not depending on any analytical method such as an estimate for $L(1,\chi)$ [6].

1. **Introduction.** Let $\mathbf{Q}(\sqrt{d})$ be a real quadratic field with discriminant d over the rationals $\mathbf{Q}$. Let $\varepsilon > 1$ and h be the fundamental unit and the class number in the wide sense of $\mathbf{Q}(\sqrt{d})$ respectively. Let $\theta = \exp(2\pi i/d)$ be a d-th root of unity. As is well known [4],

DIRICHLET'S CLASS NUMBER FORMULA. It holds that

$$\varepsilon^{2h} = \frac{F_-(\theta)}{F_+(\theta)},$$

where

$$F_\pm(x) = \prod_{\substack{0<n<d \\ \chi(n)=\pm 1}} (x^n - 1)$$

and $\chi(n) = (\frac{d}{n})$ denotes the Kronecker symbol.

In this note we shall give a new arithmetical proof of our previous result [6], with including [3]. In the case of real quadratic fields of prime discriminants and of all the discriminants, Hasse, P. Chowla, T. Ono, the first author of this paper and H. Bergström gave the other methods for an arithmetical expression of the formula in [4], [2], [7], [5] and [1] respectively.

THEOREM. *The trace T of the unit ε^{2h} is the least positive integer* $\bmod \Phi_d(2)$ *of the rational number*

$$\frac{F_-(2)}{F_+(2)} + \frac{F_+(2)}{F_-(2)}$$

except for the field of discriminant 12, *where $\Phi_d(x)$ denotes the d-th cyclotomic polynomial.*

In [6], it is necessary for us to use an estimate for the upper bound of Dirichlet's L-function $L(s,\chi)$ at $s = 1$.

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2. **Lemma.** We shall prove the following key lemma.

LEMMA. *For any discriminant* $d \neq 5, 8, 12$ *and* 24 *there exists at least one representative n of a residue class modulo d such that*

$$0 < n < \frac{d}{4} \quad \textit{and} \quad \chi(n) = -1.$$

PROOF. In what follows we denote by χ_D the Kronecker symbol corresponding to the quadratic field of discriminant D.

First we assume $d \equiv 0 \bmod 4$. Let

$$n = \begin{cases} f-8 & \text{if } d = 4f \text{ with } f \equiv 3 \bmod 8, f > 3, \\ f-2 & \text{if } d = 4f \text{ with } f \equiv 7 \bmod 8, \\ f+8 & \text{if } d = 8f \text{ with } f \equiv 3 \bmod 4, f > 7, \\ f+2 & \text{if } d = 8f \text{ with } f \equiv 1 \bmod 4, f > 1. \end{cases}$$

Furthermore let $n = 3$ if $d = 8 \cdot 7$. It is easy to see that $0 < n < d/4$. We shall prove $\chi(n) = -1$. In the case that $d = 4f$ with $f \equiv 3 \bmod 4$, we compute $\chi(n) = \chi_{-4}(-1)\chi_{-f}(-2) = -1$ if $f \equiv 3 \bmod 8$, and $\chi(n) = \chi_{-4}(1)\chi_{-f}(-2) = -1$ if $f \equiv 7 \bmod 8$. When $d = 8f$ with $f \equiv 3 \bmod 4$, it follows that $\chi(n) = \chi_{-8}(f)\chi_{-f}(2) = -1$. If $d = 8f$ with $f \equiv 1 \bmod 4$ and $f > 1$, then $\chi(n) = \chi_8(f+2)\chi_f(2) = -1$. It holds $\chi(3) = -1$ when $d = 8 \cdot 7$.

We next consider the case that $d \equiv 1 \bmod 8$ and there is a prime $p \equiv 3 \bmod 4$ dividing d. Then $d = pq$ with $q \equiv p \bmod 8$. Putting $n = (p+q)/2$ we see that $\chi(n) = \chi(2)\chi_{-p}(q)\chi_{-q}(p) = -1$ and

$$n = \frac{1}{2}\left(\frac{d}{p} + \frac{d}{q}\right) \leq \frac{1}{2}\left(\frac{d}{3} + \frac{d}{11}\right) < \frac{d}{4}.$$

In the case that $d \equiv 1 \bmod 8$ and none of primes $p \equiv 3 \bmod 4$ divides d, it is impossible to give a desired integer n explicitly for every d. But our method is available for every $d \equiv 1 \bmod 8$ not divisible by 3. If $d \equiv 2 \bmod 3$ then $\chi(3) = -1$ and $3 < d/4$ because $d \geq 17$. Hence we suppose $d \equiv 1 \bmod 3$ and take an integer m such that $\chi(m) = -1$. Since χ is even and $\chi(2) = \chi(3) = 1$, we may assume that $0 < m < d/2$ and $(m, 6) = 1$. Put

$$n = \begin{cases} (d-m)/6 & \text{if } m \equiv 1 \bmod 6, \\ (d+m)/6 & \text{if } m \equiv 5 \bmod 6. \end{cases}$$

Then $\chi(n) = \chi(6)\chi(\pm m) = -1$, and $n < (d + d/2)/6 = d/4$.

Finally, when $d \equiv 5 \bmod 8$ and $d > 5$, the assertion is obvious because $\chi(2) = -1$. Thus we have finished the proof. ∎

REMARK 1. In the case that $d \equiv 1 \bmod 4$ or $d \not\equiv 0 \bmod 3$, we can give another proof of the Lemma. It is based on the following formula:

$$\varphi\left(d, \frac{d}{4}\right) = \begin{cases} \frac{\varphi(d)}{4} + 2^{r-2}, & \text{if for any prime } p \mid d, p \equiv 3 \bmod 4, \\ \frac{\varphi(d)}{4}, & \text{otherwise,} \end{cases}$$

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where $\varphi(d, d/4)$ denotes the number of rational integers n such that $0 < n \leq d/4$ and $(n, d) = 1$, and r is the number of prime factors of the discriminant d. This is shown from $\varphi(d, d/4) = \sum_{m|d} \mu(m)[d/4m]$, where $\mu(m)$ and $[x]$ are the Möbius function and the Gauss symbol, respectively.

3. **Proof of the Theorem.** For a number $T = \varepsilon^{2h} + \varepsilon^{-2h}$, let

$$p(x) = F_-(x)F_+(x)T - \left(F_-(x)^2 + F_+(x)^2\right).$$

Since it holds that $p(\theta) = F_-(\theta)F_+(\theta)(T - T) = 0$, we have $p(x) \equiv 0 \bmod \Phi_d(x)$. Then it follows that

$$F_-(2)F_+(2)T \equiv F_-(2)^2 + F_+(2)^2 \bmod \Phi_d(2).$$

namely

$$T \equiv \frac{F_-(2)}{F_+(2)} + \frac{F_+(2)}{F_-(2)} \bmod \Phi_d(2),$$

because $F_+(2)F_-(2)$ and $\Phi_d(2)$ are relatively prime.

We now claim that $T < \Phi_d(2)$. We know

$$\Phi_d(1) = \begin{cases} p, & \text{for } d \text{ a prime number } p, \\ 2, & \text{for } d = 8, \\ 1, & \text{otherwise.} \end{cases}$$

We first deal with the case that $r \geq 2$ and $d \neq 12, 24$. It is easy to see that

$$T = \varepsilon^{2h} + \varepsilon^{-2h} < \frac{F_-(\theta)^2}{F_+(\theta)F_-(\theta)} + 1 = F_-(\theta)^2 + 1,$$

i.e.

$$T - 1 < F_-(\theta)^2.$$

Since $|\theta^m - 1| \leq 2$ for any rational integer m and $|\theta^{\pm n} - 1| < \sqrt{2}$ for an integer n in the Lemma, it turns out that

$$F_-(\theta)^2 = \left\{(\theta^n - 1)(\theta^{-n} - 1) \prod_{\substack{0<m<d \\ \chi(m)=-1 \\ m \neq \pm n}} (\theta^m - 1)\right\}^2 < (2 \cdot 2^{\frac{\varphi(d)}{2} - 2})^2 = 2^{\varphi(d)-2}.$$

Put $t = \left(\Phi_d(2) - 1\right)/2^{\varphi(d)}$. By the well-known formulas $\Phi_d(x) = \prod_{m|d}(x^m - 1)^{\mu(\frac{d}{m})}$ and $\varphi(d) = \sum_{m|d} m\mu(\frac{d}{m})$, one has

$$t = \prod_{m|d}\left(1 - \frac{1}{2^m}\right)^{\mu(\frac{d}{m})} - \frac{1}{2^{\varphi(d)}}.$$

It is known that $1 + x \geq (1 + kx)^{1/k}$ for $x > -1$ and a rational integer $k > 0$ such that $1 + kx > 0$. We observe that $m \equiv m' \bmod 2$ if $m, m' \mid d$ and $\mu(d/m), \mu(d/m') \neq 0$. Then it follows that

$$\prod_{\substack{m|d \\ \mu(\frac{d}{m})=1}} \left(1 - \frac{1}{2^m}\right) > \left(1 - \frac{1}{2^u}\right)^{1+4^{-1}+4^{-2}+\cdots} = \left(1 - \frac{1}{2^u}\right)^{\frac{4}{3}} > 2^{-2},$$

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where

$$u = \begin{cases} 1, & \text{if } (d,2) = 1, \\ 2, & \text{if } 4 \mid d \text{ and } 8 \nmid d, \\ 4, & \text{if } 8 \mid d. \end{cases}$$

Using $(1 - \frac{1}{2^m})^{-1} > 1$, we have

$$t > \left(1 - \frac{1}{2^{\varphi(d)-2}}\right) \prod_{\substack{m \mid d \\ \mu(\frac{d}{m})=1}} \left(1 - \frac{1}{2^m}\right)$$
$$> \left(1 - \frac{1}{2^{4-2}}\right)\left(1 - \frac{1}{2}\right)^{\frac{4}{3}} > \left(\frac{1}{2}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{4}{3}} = 2^{-2}.$$

This implies that

$$F_-(\theta)^2 < 2^{\varphi(d)-2} < 2^{\varphi(d)}t = \Phi_d(2) - 1.$$

Therefore $T < F_-(\theta)^2 + 1 < \Phi_d(2)$.

When $r = 1$ the proof is simpler than the one of [3]. Let $d = p$ be a prime. Then

$$T = \frac{F_+(\theta)^2 + F_-(\theta)^2}{F_+(\theta)F_-(\theta)} < \frac{1}{p}(2^{p-1} + 2^{p-1}) = \frac{2^p}{p} < 2^p - 1 = \Phi_p(2).$$

For $d = 8$ we also have

$$T < \frac{1}{2}(2^4 + 2^4) = 2^4 < 2^4 + 1 = \Phi_8(2).$$

For $d = 24$ we compute

$$F_-(\theta)^2 = \{(\theta^7 - 1)(\theta^{11} - 1)(\theta^{13} - 1)(\theta^{17} - 1)\}^2$$
$$= -\{(\theta^7 - 1)(\theta^{11} - 1)\}^4$$

and $|\theta^7 - 1| < |\theta^8 - 1| = \sqrt{3}$. Hence $F_-(\theta)^2 < 16 \cdot 9 = 144$. On the other hand $\Phi_{24}(x) = x^8 - x^4 + 1$ and so $\Phi_{24}(2) = 241$. Therefore $T < F_-(\theta)^2 + 1 < \Phi_{24}(2)$. Hence the proof is complete.

REMARK 2. In the case that $d = p$ is a prime we again obtain

$$T < \frac{2^p}{p} < \frac{2^p + 1}{3} = \Phi_p(-2) < \Phi_p(2)$$

and

$$T \equiv \frac{F_-(-2)}{F_+(-2)} + \frac{F_+(-2)}{F_-(-2)} \bmod \Phi_p(-2).$$

Then we can improve a result of Chowla [3].

EXAMPLES. We examine our Theorem for several fields.

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The case of $d = 21 = 3 \cdot 7$. Let $n_\pm$ be the residue class modulo d such that $\chi(n_\pm) = \pm 1$ respectively. We obtain

$$n_+ = 1, 4, 5, 16, 17, 20,$$
$$n_- = 2, 8, 10, 11, 13, 19,$$
$$\varepsilon = \frac{5+\sqrt{21}}{2},$$

and

$$\Phi_d(2) = 2359.$$

Then we can see

$$F_+(2) = 4188247796043939375 \equiv 321 \bmod \Phi_d(2),$$
$$F_-(2) = 6879564182353905405 \equiv 2043 \bmod \Phi_d(2),$$

in the theorem respectively. Hence we get $T = 23$. Thus from the Pell's equation $T^2 - dU^2 = 4$, we get $U = 5$, *i.e.*

$$\varepsilon^{2h} = \frac{23+5\sqrt{21}}{2}.$$

Hence $h = 1$.

The case of $d = 101$. We obtain

$$n_+ = 1, 4, 5, 6, 9, 13, 14, 16, 17, 19, 20, 21, 22, 23, 24, 25, 30,$$
$$31, 33, 36, 37, 43, 45, 47, 49, 52, 54, 56, 58, 64, 65, 68, 70$$
$$71, 76, 77, 78, 79, 80, 81, 82, 84, 85, 87, 88, 92, 95, 96, 97, 100,$$
$$n_- = 2, 3, 7, 8, 10, 11, 12, 15, 18, 26, 27, 28, 29, 32, 34, 35, 38,$$
$$39, 40, 41, 42, 44, 46, 48, 50, 51, 53, 55, 57, 59, 60, 61, 62,$$
$$63, 66, 67, 69, 72, 73, 74, 75, 83, 86, 89, 90, 91, 93, 94, 98, 99,$$
$$\varepsilon = \frac{20+2\sqrt{101}}{2},$$
$$\Phi_d(2) = 2535301200456458802993406410751,$$

and

$$\Phi_d(-2) = 845100400152152934331135470251.$$

Then we can see

$$F_+(2) \equiv 1950259908319777068849563455334 \bmod \Phi_d(2),$$
$$F_-(2) \equiv 585041292136681734143842955619 \bmod \Phi_d(2),$$

and

$$F_+(-2) \equiv 584835916452220809277718941559 \bmod \Phi_d(-2),$$
$$F_-(-2) \equiv 260264483699932125053416528894 \bmod \Phi_d(-2),$$

in Theorem and Remark 1 respectively. Hence we get $T = 402$. Thus from the Pell's equation $T^2 - dU^2 = 4$, we get $U = 40$, *i.e*

$$\varepsilon^{2h} = \frac{402 + 40\sqrt{101}}{2}.$$

Hence $h = 1$.

REMARK 3. The inequality $\Phi_d(-2) < \Phi_d(2)$ is not always true for the case of a composite discriminant d (*e.g.* $\Phi_{21}(-2) = 5419 > \Phi_{21}(2)$). For the case of $d = 12$, see [6].

REMARK 4. We owe the numerical computation to the UBASIC86 Ver. 8.2.

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