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STABLE HOMEOMORPHISMS OF THE PSEUDO-ARC

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Noting that certain restrictions are placed on homeomorphisms of the pseudo-arc, since it is hereditarily indecomposable, in 1955 [**4**] R. H. Bing asked if the identity is the only stable homeomorphism of the pseudo-arc. In this paper we prove the following theorem.

THEOREM. *Let U be an open subset of the pseudo-arc P. Let p and q be distinct points of P such that the subcontinuum M irreducible between p and q does not intersect* $\mathrm{cl}(U)$. *Then there exists a homeomorphism $h : P \to P$ with $h(p) = q$ and $h|_U = 1_U$.*

1. Definitions. A *chain* C is a collection of open sets $C = \{C(i)\}_{i \leqq n}$ such that $C(i) \cap C(j) \neq \emptyset$ if and only if $|i - j| \leqq 1$, $\mathrm{cl}(C(i)) \cap \mathrm{cl}(C(j)) \neq \emptyset$ if and only if $|i - j| \leqq 1$, $C(0) - C(1) \neq \emptyset$, and $C(n) - C(n - 1) \neq \emptyset$. Each $C(i)$ is called a *link* of C. If H is a collection of open sets, and C is a subcollection of H which forms a chain, we shall by abuse of terminology call C a *subchain* of H and each element of H a *link* of H.

If $C = \{C(i)\}_{i \leqq n}$ is a chain, then $\mathrm{mesh}(C) = \max\{\mathrm{diam}(C(i)|i \leqq n)\}$.

Chain C_1 *closure refines* chain C_0 if for each i there exists j with $\mathrm{cl}(C_1(i)) \subset C_0(j)$.

If chain C_1 closure refines chain C_0, then chain C_1 is *crooked in* C_0 provided that for every p, s, i, j, where $j > i + 2$, $C_1(p) \cap C_0(i) \neq \emptyset$, and $C_1(s) \cap C_0(j) \neq \emptyset$, there exist q, r with $\mathrm{cl}(C_1(q)) \subset C_0(j - 1)$, $\mathrm{cl}(C_1(r)) \subset C_0(i + 1)$, and either $p < q < r < s$ or $p > q > r > s$.

A *pattern* is a surjection $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that for each $i < m$, $|f(i + 1) - f(i)| \leqq 1$.

Chain C_1 *follows the pattern f in chain* C_0 if $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ where $C_1 = \{C_1(i)\}_{i \leqq m}$, $C_0 = \{C_0(j)\}_{j \leqq n}$ and $\mathrm{cl}(C_1(i)) \subset C_0(f(i))$ for each $i \leqq m$. Note that in general there will be several patterns which C_1 follows in C_0.

Chain C_0 is an *amalgamation* of chain C_1 if every link of C_0 is the union of links of C_1.

Chain C_0 *properly covers* chain C_1 if C_1 closure refines C_0 and for every $C_0(j)$ there exists $C_1(k)$ with $\mathrm{cl}(C_1(k)) \subset C_0(j)$.

If A is a collection of sets, A^* will be used to denote the union of the elements of A.

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Throughout this paper **N** will be used to denote the set of non-negative integers.

A homeomorphism $h : X \to X$ is *stable* if $h = h_n \cdot h_{n-1} \cdots h_1 \cdot h_0$ where for each h_i there exists a non-empty open set U_i with $h_i|_{U_i} = 1_{U_i}$. If $n = 0$, then h will be called *primitively stable*. (Brown and Gluck [**6**] called such a homeomorphism *somewhere the identity*.)

If C is an open cover of the topological space X, and $x \in X$, then $\mathrm{St}(x, C)$, the star of x in C, is the collection of elements of C containing x.

2. Observations. In trying to construct stable homeomorphisms of the pseudo-arc P, there are a few observations one should make. If $h : P \to P$ is a homeomorphism of P such that, for the open subset U of P, $h|_U = 1_U$, then for each subcontinuum M of P which intersects $\mathrm{cl}(U)$, $h(M) = M$. Note also that for any distinct points p, q with $h(p) = q$, the subcontinuum M of P irreducible between p and q cannot intersect U, nor can M intersect $\mathrm{cl}(U)$ in a point in the same composant of M as either p or q. We do not know the answer to the following question.

Question 1. Can there exist a homeomorphism $h : P \to P$ of the pseudo-arc, with $h|_U = 1_U$ for the open set U and $h(p) = q$, where p and q are distinct points of P so that the subcontinuum M irreducible from p to q intersects $\mathrm{cl}(U)$ only in some of the composants of M not containing either p or q?

For any homeomorphism g of P and any point $p \in P$, the subcontinuum M irreducible between p and $g(p)$ must contain a point $r \in M$ with $g(r) = r$ [**8**]. So any non-identity stable homeomorphism h of P fixed on the open set U must contain a plethora of fixed points outside of U. However, this does not necessarily exclude the possibility that all other fixed points of h lie in $\mathrm{cl}(U)$, but this seems very unlikely if $h \neq 1_P$.

Question 2. Is there a homeomorphism $h : P \to P$ of the pseudo-arc such that the set of fixed points of h is the closure of an open set U with $\mathrm{cl}(U) \neq P$?

It was observations such as these which led to the question of whether there exist any stable homeomorphisms, other than the identity, of the pseudo-arc. As we shall see in Section 4, there are many.

3. Preliminary theorems. Essentially the only way to define homeomorphisms of the pseudo-arc is by the following theorem, the proof of which is left to the reader.

THEOREM 1. *Let $\{C_i\}_{i \in \mathbf{N}}$ and $\{D_i\}_{i \in \mathbf{N}}$ be two sequences of chains covering the chainable continuum X such that for each $i \in \mathbf{N}$:*

1) *C_i and D_i have the same number of links,*

2) $\mathrm{mesh}(C_i) < 1/i$ *and mesh* $(D_i) < 1/i$ *if* $i > 0$,

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3) *there exists a pattern* f_i *such that* C_{i+1} *follows* f_i *in* C_i *and* D_{i+1} *follows* f_i *in* D_i. *Then there exists a homeomorphism* $h : X \to X$ *such that for each* $p \in X$, *if* $\mathrm{St}(p, C_i)^* \subset C_i(j) \cup C_i(j+1)$ *then* $h(p) \in D_i(j) \cup D_i(j+1)$.

The theorems in this section are lengthy and technical, but they are often just what is needed. We shall assume that $\{C_i\}_{i \in \mathbf{N}}$ is a collection of chains covering the pseudo-arc P such that for each $i \in \mathbf{N}$:

1) $\mathrm{mesh}(C_i) < 1/(i+1)$
2) C_{i+1} is crooked in C_i
3) C_i is an amalgamation of C_{i+1}
4) it requires at least 3 links of C_{i+1} to span between two non-adjacent links of C_i.

Clearly, starting with any chain C_0, we can obtain such a sequence of chains. It should be clear how to make the appropriate modifications for pseudo-circles, pseudo-solenoids, etc.

THEOREM 2. *Let* $\{C_0(j), C_0(j+1), \ldots, C_0(k)\}$ *be a subchain of* C_0. *Let* A *and* B *be two disjoint closed subsets of* $C_0(j) - \mathrm{cl}(C_0(j+1))$. *Let* H *be a collection of links of* C_i, $i > 0$, *such that for each* $h \in H$, $\mathrm{cl}(h) \subseteq \bigcup_{j \leqq l \leqq k} C_0(l)$, *and any subchain of* H *which intersects both* A *and* B *is properly covered by* $\{C_0(j), C_0(j+1), \ldots, C_0(k)\}$ (*and there does exist at least one such subchain*). *Let* $f : \{0, 1, \ldots, m\} \to \{j, j+1, \ldots, k\}$ *be a pattern with* $f(0) = f(m) = j$. *Then there exists* $n \geqq i$ *and a chain* D *such that:*

1) *D is an amalgamation of the links of C_n each of whose closure is contained in H^*,*
2) *D follows the pattern f in $\{C_0(j), C_0(j+1), \ldots, C_0(k)\}$,*
3) $A \cap H^* \subseteq D(0) - \mathrm{cl}(D(1))$ *and* $B \cap H^* \subseteq D(m) - \mathrm{cl}(D(m-1))$.

Proof. There are basically three types of steps we will do (though some will be done several times). First we will assume that for each $\alpha < m, f(\alpha) \neq f(\alpha+1)$. Define a *bend* of f to be an integer $\beta, 0 < \beta < m$, such that $f(\beta - 1) = f(\beta + 1)$. Unless $j = k$ (in which case the theorem is trivial), the number η of bends of f is odd. We shall use induction on η. Step 1 will handle the case where $\eta = 1$, while step 2 will handle the general inductive step. Step 3 will then enable us to convert back to the case where there exists $\alpha < m$ with $f(\alpha) = f(\alpha+1)$.

Step 1. Let $\Gamma = \{\gamma | \gamma$ is a subchain of H maximal with respect to having no interior link intersect $A \cup B\}$. Γ consists of four disjoint subsets (some of which are possibly empty):

$\Gamma_1 = \{\gamma \in \Gamma |$ no link of γ intersects $A \cup B\}$

$\Gamma_2 = \{\gamma \in \Gamma |$ at least one link of γ intersects A, but no link of γ intersects $B\}$

$\Gamma_3 = \{\gamma \in \Gamma | \gamma$ has links intersecting each of A and $B\}$

$\Gamma_4 = \{\gamma \in \Gamma |$ at least one link of γ intersects B, but no link of γ intersects $A\}$.

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By hypothesis $\Gamma_3 \neq \emptyset$. Each $\gamma \in \Gamma_3$ has links γ_A intersecting A, γ_B intersecting B, and γ_k with $\mathrm{cl}(\gamma_k) \subseteq C_0(k)$. (If there are several links of γ with this last property, choose one and call it γ_k.) The links of a chain D are now formed as follows:

For $i < k - j$, $D(i)$ is the union of all links of $\Gamma_1^* \cup \Gamma_2^* \cup \{\delta \in \gamma \in \Gamma_3 | \delta$ is a link in the subchain of γ from γ_A to $\gamma_k\}$ whose closure is a subset of $C_0(j + i)$ but not a subset of $C_0(k)$.

$D(k - j)$ is the union of all links of H whose closure is a subset of $C_0(k)$.

For $k - j < i \leqq 2(k - j)$, $D(i)$ is the union of all links of $\Gamma_4^* \cup \{\zeta \in \gamma \in \Gamma_3 | \zeta$ is a link in the subchain of γ from γ_k to $\gamma_B\}$ whose closure is a subset of $C_0(2k - i - j)$ but not of $C_0(k)$.

Step 2. Suppose now that we can form a chain D satisfying the desired conditions whenever f has fewer than η bends ($\eta > 1$). We shall show how to then construct a chain D' satisfying the desired conditions whenever f has η bends. If f has more than one bend, we can associate with f a pattern g as follows: Let θ and κ be two distinct bends of f, $\theta < \kappa$, so that $\kappa - \theta$ is less than or equal to the corresponding value for any other pair of bends of f. Define $g : \{0, 1, \ldots, m - 2(\kappa - \theta)\} \rightarrow \{j, j + 1, \ldots, k\}$ by:

$$g(\alpha) = f(\alpha) \quad \text{for } \alpha < \theta$$
$$g(\alpha) = f(\alpha + 2(\kappa - \theta)) \quad \text{for } \alpha \geqq \theta.$$

Note that g has two fewer bends than f. Thus, by the inductive hypothesis we can construct a chain $\tilde{D}$ satisfying the desired conditions for the pattern g. We shall "split" $\tilde{D}$ along the subchain $\{\tilde{D}(2\theta - \kappa), \ldots, \tilde{D}(\theta)\}$ to obtain the desired chain D'.

Suppose $\tilde{D}$ is an amalgamation of C_t, where $\mathrm{mesh}(C_t) < \mathrm{dist}(A \cup B, C_0(j+1))$. Let $\Sigma = \{\sigma | \sigma$ is a subchain of C_{t+2} maximal with respect to σ^* being a subset of $\tilde{D}(2\theta - \kappa) \cup \ldots \cup \tilde{D}(\theta)$, and every interior link of σ having its closure in $\tilde{D}(2\theta - \kappa) \cup \ldots \cup \tilde{D}(\theta)\}$. Σ consists of three disjoint subsets:

$$\Sigma_1 = \{\sigma \in \Sigma | \text{both end links of } \sigma \text{ are in } \tilde{D}(2\theta - \kappa)\}$$
$$\Sigma_2 = \{\sigma \in \Sigma | \sigma \text{ has an end link in each of } \tilde{D}(2\theta - \kappa) \text{ and } \tilde{D}(\theta)\}$$
$$\Sigma_3 = \{\sigma \in \Sigma | \text{both end links of } \sigma \text{ are in } \tilde{D}(\theta)\}.$$

Note that Σ_2 is nonempty and by crookedness each $\sigma \in \Sigma_2$ consists of three subchains, $\sigma_1, \sigma_2, \sigma_3$ so that σ_1 contains the end link of σ in $\tilde{D}(2\theta - \kappa)$ and has its other end link in $\tilde{D}(\theta) - \tilde{D}(\theta - 1)$; σ_2 has one end link in $\tilde{D}(\theta) - \tilde{D}(\theta - 1)$ and the other end link in $\tilde{D}(2\theta - \kappa) - \tilde{D}(2\theta - \kappa + 1)$; σ_3 has one end link in $\tilde{D}(2\theta - \kappa) - \tilde{D}(2\theta - \kappa + 1)$ and contains the end link of σ in $\tilde{D}(\theta)$; σ_1 and σ_2 have one link in common; σ_2 and σ_3 have one link in common; and σ_1^* and σ_3^* are disjoint. We may also choose $\sigma_1, \sigma_2, \sigma_3$ so that if $2\theta - \kappa = 0$ then $(\sigma_2^* \cup \sigma_3^*) \cap A = \emptyset$, and if $\theta = m - 2(\kappa - \theta)$ then $(\sigma_1^* \cup \sigma_2^*) \cap B = \emptyset$.

D' is now defined as follows:

For $p < 2\theta - \kappa$, $D'(p)$ is the union of all links of C_{t+2} whose closure is a subset of $\tilde{D}(p)$ and which are not contained in $\Sigma_3^* \cup \{\sigma_2, \sigma_3 | \sigma \in \Sigma_2\}^*$.

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For $2\theta - \kappa \leqq p < \theta$, $D'(p)$ is the union of all links of $\Sigma_1^* \cup \{\sigma_1|\sigma \in \Sigma_2\}^*$ whose closure is a subset of $\tilde{D}(p)$.

$D'(\theta)$ is the union of all links of $\Sigma_1^* \cup \{\sigma_1, \sigma_2|\sigma \in \Sigma_2\}^*$ whose closure is a subset of $\tilde{D}(\theta)$.

For $\theta < p < \kappa$, $D'(p)$ is the union of all links of $\{\sigma_2|\sigma \in \Sigma_2\}^*$ whose closure is a subset of $\tilde{D}(2\theta - p)$.

$D'(\kappa)$ is the union of all links of $\{\sigma_2, \sigma_3|\sigma \in \Sigma_2\}^* \cup \Sigma_3^*$ whose closure is a subset of $\tilde{D}(2\theta - \kappa)$.

For $\kappa < p \leqq 2\kappa - \theta$, $D'(p)$ is the union of all links of $\{\sigma_3|\sigma \in \Sigma_2\}^* \cup \Sigma_3^*$ whose closure is a subset of $\tilde{D}(p - 2(\theta - \kappa))$. For $p > 2\kappa - \theta$, $D'(p)$ is the union of all links of C_{t+2} whose closure is a subset of $\tilde{D}(p - 2(\theta - \kappa))$ and which are not contained in $\Sigma_1^* \cup \{\sigma_1, \sigma_2|\sigma \in \Sigma_2\}^*$.

Step 3. We can now form a chain D' satisfying the desired conditions whenever $f(\alpha) \neq f(\alpha + 1)$ for each $\alpha < m$. Clearly, with any pattern there is an associated pattern of the same crookedness satisfying this condition. So we now need only split some of the links of the D' we have obtained by steps 1 and 2 into series of consecutive links. We will show how to split a link $D'(r)$ of D' into two consecutive links.

Suppose the chain D' is an amalgamation of C_s where $\text{mesh}(C_s) < \text{dist}(A \cup B, C_0(j + 1))$. Then, if $r \neq 0$, amalgamate C_{s+1} as follows:

For $q < r$, $D(q)$ is the union of all links of C_{s+1} whose closure is a subset of $D'(q)$.

$D(r)$ is the union of all links μ of C_{s+1} whose closure is a subset of $D'(r)$ such that some link adjacent to μ intersects $D'(r - 1)$.

$D(r + 1)$ is the union of all links of C_{s+1} whose closure is a subset of $D'(r)$ which are not contained in $D(r)$.

For $q > r + 1$, $D(q)$ is the union of all links of C_{s+1} whose closure is a subset of $D'(q - 1)$.

If $r = 0$, we split $D'(0)$ near $D'(1)$ instead.

If necessary, we can enlarge some of the links of D so as to satisfy condition 1) of the theorem.

The next two theorems are very similar, and their proof is a slight modification of the above. Therefore we state them without proof.

THEOREM 3. *Let* $\{C_0(j), C_0(j + 1), \ldots, C_0(k)\}$, $j < k$, *be a subchain of* C_0. *Let* A *be a closed subset of* $C_0(j) - \text{cl}(C_0(j + 1))$ *and* B *be a closed subset of* $C_0(k) - \text{cl}(C_0(k - 1))$. *Let* H *be a collection of links of* $C_i (i > 0)$ *so that for each* $h \in H$, $\text{cl}(h) \subseteq \bigcup_{j \leqq l \leqq k} C_0(l)$, *and some subchain of* H *intersects both* A *and* B. *Let* $f : \{0, 1, \ldots, m\} \rightarrow \{j, j + 1, \ldots, k\}$ *be a pattern with* $f(0) = j$ *and* $f(m) = k$. *Then there exists* $n \geqq i$ *and a chain* D *such that:*

1) *D is an amalgamation of the links of C_n contained in H^*.*

2) *D follows the pattern f in $\{C_0(j), \ldots, C_0(k)\}$.*

3) $A \cap H^* \subseteq D(0) - \text{cl}(D(1))$ *and*
$B \cap H^* \subseteq D(m) - \text{cl}(D(m - 1))$.

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THEOREM 4. *Let* $\{C_0(j), C_0(j+1), \ldots, C_0(k)\}$ *be a subchain of* C_0. *Let* A *be a closed subset of* $C_0(j) - \mathrm{cl}(C_0(j+1))$. *Let* H *be a collection of links of* C_i, $(i > 0)$, *such that for each* $h \in H$, $\mathrm{cl}(h) \subseteq \bigcup_{j \leqq l \leqq k} C_0(l)$, *and some subchain of* H *is properly covered by* $\{C_0(j), \ldots, C_0(k)\}$. *Let* $f : \{0, 1, \ldots, m\} \to \{j, j+1, \ldots, k\}$ *be a pattern with* $f(0) = j$. *Then there exists* $n \geqq i$ *and a chain* D *such that:*

1) D *is an amalgamation of the links of* C_n *contained in* H^*.
2) D *follows the pattern* f *in* $\{C_0(j), \ldots, C_0(k)\}$.
3) $A \cap H^* \subseteq D(0) - \mathrm{cl}(D(1))$.

The next theorem shows when we can amalgamate a chain to follow a pattern while including a specific point in a specific link.

THEOREM 5. *Let* $C = \{C(i)\}_{i \leqq n}$ *be a chain covering the pseudo-arc* P, $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ *a pattern,* $j \leqq m$, $f(j) = i_0$ *and* $p \in C(i_0)$. *Then there is a chain* D *covering* P *with* $p \in D(j)$ *and* D *following* f *in* C *if and only if for every subcontinuum* M *of* P *containing* p, *there exist* a, b *with* $0 \leqq a \leqq j \leqq b \leqq m$, *so that* $M \subseteq \bigcup_{a \leqq k \leqq b} C(f(k))$, *and* $M \cap C(f(k)) \neq \emptyset$ *for each* $a \leqq k \leqq b$.

Proof. This condition is clearly necessary. We will show that it is also sufficient.

Let $\{B_i\}_{i \in \mathbf{N}}$ be a sequence of chains covering P and refining C such that for each $i \in \mathbf{N}$:

1) $p \in B_i(0)$
2) B_{i+1} is crooked in B_i
3) $\mathrm{mesh}(B_i) < 1/(i+1)$
4) it takes 3 links of B_{i+1} to span the intersection of 2 links of B_i.

We will construct D inductively, at each step extending what we have done before. If $S = \{C(u), C(u+1), \ldots, C(v)\}$ is a subchain of C, containing $C(i_0)$, define $E(S)$ to be the subchain of C consisting of:

1) $S \cup \{C(v+1)\}$ if there is a subcontinuum M of P containing p such that $M \subseteq (S \cup \{C(v+1)\})^*$ and $M \cap (C(v+1) - S^*) \neq \emptyset$.

2) $S \cup \{C(u-1)\}$ if there is a subcontinuum M of P containing p such that $M \subseteq (S \cup \{C(u-1)\})^*$ and $M \cap (C(u-1) - S^*) \neq \emptyset$.

3) $S \cup \{C(u-1), C(v+1)\}$ otherwise. Note that if $u = 0$, 1) applies and if $v = n$, 2) applies, unless $S = C$ when we don't define $E(S)$.

Suppose for some subchain S of C, containing $C(i_0)$, and some $k \in \mathbf{N}$ we can amalgamate the maximal subchain of B_k which is properly covered by S and contains $B_k(0)$ to follow the maximal sub-pattern $\tilde{f}$ of f containing j with the image of $\tilde{f}$ in $\{l | C(l) \in S\}$. (A sub-pattern of f is a restriction of f to a set of consecutive integers.) Clearly we can do this for $S = \{C(i_0)\}$. Let $\tilde{B}_{k+1}$ be the maximal subchain of B_{k+1} contained in the amalgamated part of B_k and containing $B_{k+1}(0)$. Let B'_{k+1} be the rest of the subchain of B_{k+1} maximal with respect to being properly covered by $E(S)$ and containing $B_{k+1}(0)$. By Theorems 2–4, with $A = \mathrm{cl}(\tilde{B}_{k+1}{}^* \cap B'_{k+1}{}^*)$ and $H = B'_{k+1}$ we can obtain an

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amalgamation of the subchain of some B_r, $r \geqq k + 1$, maximal with respect to being properly covered by $E(S)$ and containing $B_r(0)$, so that the amalgamation follows the maximal subpattern of f, containing j, with image in $\{l | C(l) \in E(S)\}$.

Continuing in this manner we obtain the desired chain D.

Before we proceed to Theorem 6, we will illustrate some special types of chains which will be used. Let C_0 and D_0 be two chains covering the pseudo-arc P so that C_0 and D_0 have the same number of links. Let X be a collection of links of C_0 so that $C_0(i) = D_0(i)$ for each $C_0(i) \in X$. Let C_1 be a chain of very small mesh so that C_1 closure refines C_0, and each of C_0 and D_0 is an amalgamation of C_1.

We will be speaking about a pattern which C_1 follows in C_0. In doing so, we will wish to have a pattern f which "prefers" X. Thus if $C_1(2)$ is a closed subset of both $C_0(0)$ and $C_0(1)$, a pattern would assign it to either of these links. If X contained $C_0(0)$ but not $C_0(1)$ then we would assign it to $C_0(0)$. Thus whenever a link of C_1 could be assigned to a link of C_0 in X or to a link of C_0 not in X, we choose a pattern which assigns it to a link in X. These are all fairly easy conditions to satisfy. It is the next condition which requires more care to obtain, but it is what will enable us to actually construct stable homeomorphisms. Suppose $\tilde{C}_1$ is a subchain of C_1 so that the closure of some link of $\tilde{C}_1$ is in a link of X. By the pattern f, certain links of C_0 are assigned to the links of $\tilde{C}_1$. We will want the corresponding links of D_0 to properly cover $\tilde{C}_1$. This essentially says that, when C_1 is making progress away from a link of X, it must look similar in D_0 to what it does in C_0. But when it swings back toward X and fails to make progress, it can misbehave in D_0. Some condition like this last one will be necessary in the sequences of chains we will use to define our homeomorphism. In Theorem 6 we will hypothesize the existence of chains as described above.

All of the above conditions will be subsumed under the following definition: Let C_0, D_0 and C_1 be chains covering the pseudo-arc, X a collection of links of C_0, and f_1 a pattern. Then (C_1, f_1) is *compatible with* D_0 *relative to* (C_0, X) provided that:

1) C_0 and D_0 have the same number of links;

2) each of C_0 and D_0 is an amalgamation of C_1;

3) C_1 follows the pattern f_1 in C_0 with $C_0(f_1(j)) \in X$ for every $\mathrm{cl}(C_1(j)) \subseteq X^*$;

4) $C_0(i) = D_0(i)$ for each $C_0(i) \in X$;

5) for each subchain $\tilde{C}_1$ of C_1 with $\{C_0(f_1(k)) | C_1(k) \in \tilde{C}_1\} \cap X \neq \emptyset$, $\tilde{C}_1$ is properly covered by $\{D_0(f_1(k)) | C_1(k) \in \tilde{C}_1\}$.

In constructing stable homeomorphisms we shall obtain sequences of chains, patterns, and sets of links, as in this definition, of smaller and smaller mesh. In doing this, conditions 1 and 3 will allow us to define the final homeomorphism. Condition 5 at each stage will be necessary to achieve condition 4 at the next

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stage. Condition 4 at each stage will enable us to make sure that the homeomorphism defined is stable. Condition 2 is largely for convenience.

THEOREM 6. *Let C_0, D_0 and C_1 be chains covering the pseudo-arc, with X a non-empty collection of links of C_0 and f_1 a pattern, such that (C_1, f_1) is compatible with D_0 relative to (C_0, X). Then for each $\epsilon > 0$ there exists a chain E of mesh less than ϵ, a pattern f_2, and a chain D_1 covering P such that D_1 follows the pattern f_1 in D_0, and (E, f_2) is compatible with C_1 relative to $(D_1, \{D_1(j)|C_0(f_1(j)) \in X\})$.*

Proof. Let $\{A_n\}_{n\in\mathbf{N}}$ be a sequence of chains covering P such that for each $n \in \mathbf{N}$:

1) A_{n+1} is crooked in A_n;

2) $A_0 = C_1$, and mesh$(A_n) < 1/n$ for $n > 0$;

3) A_n is an amalgamation of A_{n+1};

4) It requires at least 4 links of A_{n+1} to span between any two non-adjacent links of A_n;

5) Any subchain of A_{n+1} spanning the intersection of two links of A_n contains at least two links whose closures are a subset of that intersection.

Let n be an integer so large that mesh$(A_n) < \epsilon$, and $n > 0$. Consider A_{n+1}. Amalgamate the links of A_{n+1} to form $D_1(l) = C_1(l)$ for each $\text{cl}(C_1(l)) \subseteq X^*$. Let $\Gamma = \{\gamma|\gamma$ is a subchain of A_{n+1} maximal with respect to having no link contained in $\{D_1(l)|\text{cl}(C_1(l)) \subseteq X^*\}\}$. We shall form the other links of D_1 from Γ^*. In doing so we shall restrict our attention to a single $\gamma \in \Gamma$, since a similar procedure applies to each. The amalgamation of the links of some A_m, $m \geqq n + 1$, in γ shall consist of several steps and the final D_1 and pattern f_2 will not be determined until the end of the proof. Some steps will modify what we have already amalgamated, but we will still refer to the result of each step as D_1 though it is not our final chain.

Suppose $\gamma \in \Gamma$ is minimally covered by $\{C_1(p), C_1(p + 1), \ldots, C_1(q)\}$. The amalgamated γ will follow the pattern $f_1|_{\{p,p+1,\ldots,q\}}$ in D_0 and match up on the links we have already amalgamated. For each α, $p \leqq \alpha \leqq q$, define:

> $F_\gamma(\alpha)$ is the first link of γ whose closure is contained in $C_1(\alpha)$ but not in $C_1(\alpha - 1)$;
> $L_\gamma(\alpha)$ is the first link of γ whose closure is contained in $C_1(\alpha) \cap C_1(\alpha + 1)$.

Let γ_1 be the closure of the first link of γ.

Step 1. Suppose that for some α and some $r \geqq 1$, the collection of all subchains δ of A_{n+1} contained in the part of γ from γ_1 through $L_\gamma(\alpha)$ has been amalgamated to follow the pattern $f_1|_{\{p,p+1,\ldots,\alpha\}}$ in D_0 with $\gamma_1 \cap \{\delta\}^{**} \subseteq D_1(p) - \text{cl}(D_1(p + 1))$, and condition 5 in the definition of compatible satisfied for $E = A_{n+r}$ and for the part of D_1 so far amalgamated. (This can certainly be done for $\alpha = p$, $r = 1$.) Consider the set of all subchains ζ of A_{n+r+2} maximal with respect to having both end links in γ_1 and having the closure of each link contained in the part of γ through $L_\gamma(\alpha)$. We will keep the amalgamation the same on these chains during this step.

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Consider the set of all subchains η of A_{n+r+2} contained in the part of γ to $F_\gamma(\alpha + 1)$ and intersecting the chains ζ at most in end links. (η contains no link whose closure is contained *only* in $F_\gamma(\alpha + 1)$.) Amalgamate into part of D_1 the links of some A_s, $s \geqq n + r + 2$, contained in the chains η to follow the pattern $f_1|_{\{p,\ldots,\alpha+1\}}$ in D_0 with $\gamma_1 \cap \{\eta\}^{**} \subseteq D_1(p) - \mathrm{cl}(D_1(p + 1))$.

Step 2. Consider the set of all subchains θ of A_{s+2} contained in the part of γ to $F_\gamma(\alpha + 1)$ and maximal with respect to having both end links in γ_1. We will keep the amalgamation into D_1 the same on these chains during this step.

Consider the set of all subchains κ of A_{s+2} contained in the part of γ through $L_\gamma(\alpha + 1)$ and intersecting the chains θ at most in end links. Amalgamate into part of D_1 the links of some A_t, $t \geqq s + 2$, contained in the chains κ to follow the pattern $f_1|_{\{p,\ldots,\alpha+1\}}$ in D_0 with $\gamma_1 \cap \{\kappa\}^{**} \subseteq D_1(p) - \mathrm{cl}(D_1(p + 1))$.

Continue in this manner. Do the analogous procedure from the opposite end of γ.

Step 3. When the part of A_u to $F_\gamma(q)$ has been amalgamated, and the analogous part from the other end of γ, consider the collection of all subchains λ of A_{u+2} contained in the part of γ to $F_\gamma(q)$ which are maximal with respect to having both end links in γ_1, and the analogous chains λ' from the other end of γ. We will keep the amalgamation into D_1 the same on these chains during this final step.

Consider the set of all subchains μ of A_{u+2} contained in γ and having only end links in common with chains λ and λ'. Amalgamate into D_1 the links of some A_v, $v \geqq u + 2$, contained in these chains μ to follow the pattern $f_1|_{\{p,\ldots,q\}}$ in D_0 with $y_1 \cap \{\mu\}^{**} \subseteq D_1(p) - \mathrm{cl}(D_1(p + 1))$ and analogously for the closure of the last link of γ. This is the final amalgamation and D_1 is the chain that results when this procedure has been done for each $\gamma \in \Gamma$. (The obvious modifications to this procedure can be made if only one end of γ intersects $\{D_1(l)|\mathrm{cl}(C_1(l)) \subseteq X^*\}^*$.) A_w is the desired chain E, where w is the largest v in the last step used for any γ. For the pattern f_2, by our construction any pattern preferring $\{D_1(l)|\mathrm{cl}(C_1(l)) \subseteq X^*\}$ which E follows in D_1 will do.

The following theorem, proved by Lehner [**9**] in 1959 will also be used.

THEOREM 7. *Suppose $H_0, H_1, \ldots, H_n$ are non-degenerate subcontinua of the pseudo-arc P such that H_i and H_j are in different composants of P if $i \neq j$. Suppose H_i is irreducible between the points P_i and Q_i $(i \leqq n)$. Then for each $\epsilon > 0$ there exists a chain D, with* mesh$(D) < \epsilon$, *covering P such that $D(P_i, Q_i)^*$ contains H_i, $D(P_i, Q_i)^* \cap D(P_j, Q_j)^* = \emptyset$ if $i \neq j$, $D(P_1, Q_n) = D$, and $D(Q_1) < D(P_2) < D(Q_2) < \cdots < D(P_n)$. (Here $D(A)$ is the first link of D containing the point A, and $D(A, B)$ is the subchain of D from $D(A)$ through $D(B)$.)*

4. Main result. We now have everything necessary to prove the following theorem:

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THEOREM 8. *Let U be an open subset of the pseudo-arc P. Let p and q be distinct points of P such that the subcontinuum M irreducible between p and q does not intersect* $\mathrm{cl}(U)$. *Then there exists a homeomorphism $h : P \to P$ with $h(p) = q$ and $h|_U = 1_U$.*

Proof. Let $\epsilon < \mathrm{dist}(M, \mathrm{cl}(U))$. By Theorem 3 there exists a chain C with $\mathrm{mesh}(C) < \epsilon/2$ such that $\mathrm{St}(p, C) = C(0)$, $\mathrm{St}(q, C) = C(k)$, k even, $k > 2$, and M is covered by $\{C(0), \ldots, C(k)\}$. There exists a chain D refining C with $\mathrm{St}(p, D) = D(0)$, $\mathrm{St}(q, D) = D(k)$, such that D follows the pattern f in C, where:

$$f(m) = m \quad \text{for } m \leqq k;$$
$$f(m) = 2k - m \quad \text{for } k \leqq m \leqq 2k;$$
$$f(m) = m - 2k \quad \text{for } m \geqq 2k.$$

Let $C_0 = C$ and let D_0 be an amalgamation of D where:

$$D_0(n) = D(k-n) \cup D(k+n) \cup D(2k-n) \cup D(2k+n) \quad \text{if } n \leqq k/2;$$
$$D_0(n) = D(k-n) \cup D(2k+n) \quad \text{if } k/2 < n \leqq k$$
$$D_0(n) = D(2k+n) \quad \text{if } n > k.$$

We can now inductively construct chains C_r and D_r, $r \in \mathbf{N}$, such that for each $r \in \mathbf{N}$:

1) C_r and D_r have the same number of links;

2) there exists a pattern f_r such that C_{r+1} follows f_r in C_r and D_{r+1} follows f_r in D_r;

3) $\mathrm{mesh}(C_r) < 1/r$ and $\mathrm{mesh}(D_r) < 1/r$, for $r > 0$;

4) $C_r(s) = D_r(s)$ for each r, s such that $(C_r(s) \cup D_r(s)) \cap U \neq \emptyset$;

5) $\mathrm{St}(p, C_r) = C_r(0)$ and $\mathrm{St}(q, D_r) = D_r(0)$.

The first four conditions follow directly from Theorem 6. Theorem 7 and the proof of Theorem 6 guarantee that condition 5 can be satisfied at each stage if it is satisfied at the previous stage. (We have constructed it into $r = 0$.) Then by Theorem 1 there is a homeomorphism $h : P \to P$ with $h(p) = q$ and $h|_U = 1_U$.

COROLLARY 9. *Let U be an open subset of the pseudo-arc P. Let $\{p_i\}_{i \leqq n}$ and $\{q_i\}_{i \leqq n}$ be sets of points of P such that for each $i \leqq n$ the subcontinuum M_i of P irreducible between p_i and q_i does not intersect* $\mathrm{cl}(U)$, *and $M_i \cap M_j = \emptyset$ if $i \neq j$. Then there exists a homeomorphism $h : P \to P$ with $h(p_i) = q_i$ for each $i \leqq n$ and $h|_U = 1_U$.*

5. Questions. It is clear that each stable homeomorphism of the pseudo-arc P must be set-wise fixed on each composant of P. One might wonder about the converse.

Question 3. Suppose h is a homeomorphism of the pseudo-arc P so that for each composant A of P, $h(A) = A$. Is h stable? Is h the composition of at most two primitively stable homeomorphisms? (cf. [**6**].)

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In [**5**] Brechner discusses a "standard" embedding of the pseudo-arc in the plane and shows that any stable homeomorphism of the pseudo-arc which, under this embedding, can be extended to a homeomorphism of the plane must be the identity. She also shows that for this embedding there is a homeomorphism of the pseudo-arc which, while not extendable to the plane, is conjugate to a homeomorphism which is extendable. She then asks the question, is every homeomorphism of the pseudo-arc conjugate to a homeomorphism which is extendable under this embedding? Since any conjugate of a stable homeomorphism is stable, we have provided a negative answer to this question.

In the same paper she defines a homeomorphism h of the pseudo-arc to be *essentially extendable* if there exists an embedding $\phi : P \rightarrow E^2$ so that $\phi\, h\, \phi^{-1}$ extends to a homeomorphism of E^2. She then conjectures that each essentially extendable, primitively stable homeomorphism of P is the identity. If this is true, the results of this paper would show that there are homeomorphisms of P which are not essentially extendable, a not altogether surprising result.

Question 4. Does there exist an essentially extendable stable homeomorphism of the pseudo-arc, other than the identity?

The following question concerning a possible strengthening of the results of this paper is also of interest.

Question 5. If M is a proper subcontinuum of the pseudo-arc P, U is an open subset of P with $M \cap \mathrm{cl}(U) = \emptyset$, and $h : M \rightarrow M$ is a homeomorphism, does there exist a homeomorphism $\bar{h} : P \rightarrow P$ with $\bar{h}|_M = h$ and $\bar{h}|_U = 1_U$? (cf. [**9**].)

A straightforward modification of these techniques can be used to construct non-trivial stable homeomorphisms of the pseudo-circle [**2**], [**7**], and of any pseudo-solenoid [**7**].

Since the considerations which lead one to question the existence of non-trivial stable homeomorphisms of the pseudo-arc arise mainly from the fact that P is hereditarily indecomposable, rather than from the fact that it is a pseudo-arc per se, the following question is also of interest.

Question 6. Does every hereditarily indecomposable continuum have a stable homeomorphism which is not the identity?

The answer to the following question is likely to be no, but we state it for what interest it has.

Question 7. Is the set of stable homeomorphisms dense in the space of homeomorphisms of the pseudo-arc?

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