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Smooth integers and de Bruijn's approximation Λ

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This paper is concerned with the relationship of y-smooth integers and de Bruijn's approximation $\Lambda(x, y)$. Under the Riemann hypothesis, Saias proved that the count of y-smooth integers up to x, $\Psi(x, y)$, is asymptotic to $\Lambda(x, y)$ when $y \geqslant (\log x)^{2+\varepsilon}$. We extend the range to $y \geqslant (\log x)^{3/2+\varepsilon}$ by introducing a correction factor that takes into account the contributions of zeta zeros and prime powers. We use this correction term to uncover a lower order term in the asymptotics of $\Psi(x, y)/\Lambda(x, y)$. The term relates to the error term in the prime number theorem, and implies that large positive (resp. negative) values of $\sum_{n\leqslant y} \Lambda(n) - y$ lead to large positive (resp. negative) values of $\Psi(x, y) - \Lambda(x, y)$, and vice versa. Under the Linear Independence hypothesis, we show a Chebyshev's bias in $\Psi(x, y) - \Lambda(x, y)$.

Keywords: smooth integers; smooth numbers; de Bruijn's approximation

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1. Introduction

A positive integer is called y-smooth if each of its prime factors does not exceed y. We denote the number of y-smooth integers not exceeding x by $\Psi(x, y)$. We assume throughout $x \geqslant y \geqslant 2$. Let $\rho\colon [0, \infty) \to (0, \infty)$ be the Dickman function, defined as $\rho(t) = 1$ for $t \in [0, 1]$ and via the delay differential equation $t\rho'(t) = -\rho(t-1)$ for $t > 1$. Dickman [7] showed that

$$\Psi(x, y) \sim x\rho(\log x/\log y) \quad (x \to \infty) \tag{1.1}$$

holds when $y \geqslant x^{\varepsilon}$. For this reason, it is useful to introduce

$$u := \log x/\log y.$$

De Bruijn [3, Eqs. (1.3), (4.6)] showed that

$$\Psi(x, y) - x\rho(u) \sim (1-\gamma)\frac{x\rho(u-1)}{\log x} > 0 \tag{1.2}$$

when $x \to \infty$ and $(\log x)/2 > \log y > (\log x)^{5/8}$. Here and later γ is the Euler–Mascheroni constant. As we see, there is no arithmetic information in the leading behaviour of the error term $\Psi(x, y) - x\rho(u)$, and in particular it does not oscillate. Moreover, the error term is large: the saving (1.2) gives over the main term is merely $\asymp \log(u+1)/\log y$ [3, p. 56].

This begs the question, what is the correct main term for $\Psi(x, y)$ that leads to a small and arithmetically rich error term? De Bruijn [3, Eq. (2.9)] introduced a refinement of ρ, often denoted λ_y:

$$\lambda_y(u) := \int_0^\infty \rho\left(u - \frac{\log t}{\log y}\right) d\left(\frac{\lfloor t \rfloor}{t}\right) = \int_{\mathbb{R}} \rho(u-v) d\left(\frac{\lfloor y^v \rfloor}{y^v}\right)$$

if $y^u \notin \mathbb{Z}$; otherwise $\lambda_y(u) = \lambda_y(u+)$ (one has $\lambda_y(u) = \lambda_y(u-) + O(1/x)$ if $y^u \in \mathbb{Z}$ [3, p. 54]). The count $\Psi(x, y)$ should be compared to

$$\Lambda(x, y) := x\lambda_y(u).$$

We refer the reader to de Bruijn's original paper for the motivation for this definition. In particular, Λ satisfies the following continuous variant of Buchstab's identity:

$$\Lambda(x, y) = \Lambda(x, z) - \int_y^z \Lambda\left(\frac{x}{t}, t\right) \frac{\mathrm{d}t}{\log t}$$

for $y \leqslant z$, to be compared with $\Psi(x, y) = \Psi(x, z) - \sum_{y<p\leqslant z} \Psi(x/p, p)$. De Bruijn proved [3, Eq. (1.4)]

$$\Lambda(x, y) = x\rho(u)\left(1 + O_\varepsilon\left(\frac{\log(u+1)}{\log y}\right)\right) \tag{1.3}$$

holds for $\log y > \sqrt{\log x}$. Saias [17, Lem. 4] improved the range to $y \geqslant (\log x)^{1+\varepsilon}$. De Bruijn and Saias also provided asymptotic series expansion for $\lambda_y(u)$ in (roughly) powers of $\log(u+1)/\log y$. Hildebrand and Tenenbaum [14, Lem. 3.1] showed that for $y \geqslant (\log x)^{1+\varepsilon}$,

$$\Lambda(x, y) \asymp_\varepsilon x\rho(u) \tag{1.4}$$

for $y \geqslant (\log x)^{1+\varepsilon}$. Implicit in the proof of proposition 4.1 of La Bretèche and Tenenbaum [5] is the estimate

$$\Lambda(x, y) = x\rho(u)K\left(-\frac{\xi(u)}{\log y}\right)\left(1 + O_\varepsilon\left(\frac{1}{\log x}\right)\right), \quad K(t) := \frac{t\zeta(t+1)}{t+1}, \tag{1.5}$$

for $y \geqslant (\log x)^{1+\varepsilon}$ where ζ is the Riemann zeta function and $\xi\colon [1, \infty) \to [0, \infty)$ is defined via

$$e^{\xi(u)} = 1 + u\xi(u).$$

We include as an appendix a proof in English of (1.5). The function K originates in de Bruijn's work [3, Eq. (2.8)]. Evidently, $K(0) = 1$ and $\lim_{t\to -1^+} K(t) = \infty$. Moreover, K is strictly decreasing in $(-1, 0]$ [9].

Suppose $\pi(x) = \mathrm{Li}(x)(1 + O(\exp(-(\log x)^a)))$ for some $a \in (0, 1)$. Saias [17, Thm.], improving on De Bruijn [3], proved that

$$\Psi(x, y) = \Lambda(x, y)(1 + O_\varepsilon(\exp(-(\log y)^{a-\varepsilon}))) \tag{1.6}$$

holds in the range $\log y \geqslant (\log\log x)^{1/a+\varepsilon}$. By the Vinogradov–Korobov zero-free region, we may take $a = 3/5$. Saias writes without proof [17, p. 81] that under the

Riemann hypothesis (RH) his methods give

$$\Psi(x,y) = \Lambda(x,y)(1 + O_\varepsilon(y^{\varepsilon-1/2}\log x)) \tag{1.7}$$

in the range $y \geqslant (\log x)^{2+\varepsilon}$, which recovers a conditional result of Hildebrand [11].

1.1. G

Define the entire function $I(s) = \int_0^s \frac{e^v-1}{v}\,\mathrm{d}v$. As shown in [14, Lem. 2.6], the Laplace transform of ρ is

$$\hat{\rho}(s) := \int_0^\infty e^{-sv}\rho(v)\ \mathrm{d}v = \exp(\gamma + I(-s)) \tag{1.8}$$

for all $s \in \mathbb{C}$. In [9] we studied in detail the ratio

$$G(s,y) := \zeta(s,y)/F(s,y)$$

where

$$\zeta(s,y) := \prod_{p\leqslant y}(1-p^{-s})^{-1} = \sum_{n \text{ is } y\text{-smooth}} n^{-s} \quad (\Re s > 0)$$

is the partial zeta function and

$$F(s,y) := \hat{\rho}((s-1)\log y)\zeta(s)(s-1)\log y. \tag{1.9}$$

The function $G(s,\,y)$ is defined for $\Re s > 0$ such that $\zeta(s) \neq 0$. Informally, G carries information about the ratio $\Psi(x,\,y)/\Lambda(x,\,y)$, since $s \mapsto \zeta(s,\,y)/s$ is the Mellin transform of $x \mapsto \Psi(x,\,y)$ while $s \mapsto F(s,\,y)/s$ is the Mellin transform of $x \mapsto \Lambda(x,\,y)$ [3, p. 54]. As in [9], it is essential to write G as G_1G_2 where

$$\log G_1(s,y) = \sum_{n\leqslant y}\frac{\Lambda(n)}{n^s\log n} - (\log(\zeta(s)(s-1)) + \log\log y + \gamma + I((1-s)\log y)),$$

$$\log G_2(s,y) = \sum_{k\geqslant 2}\sum_{y^{1/k}<p\leqslant y}\frac{p^{-ks}}{k}.$$

We assume $\log\zeta(s)$ is chosen to be real when $s > 1$.

1.2. Main results

Let $\psi(y) = \sum_{n\leqslant y}\Lambda(n)$ and

$$\beta := 1 - \frac{\xi(u)}{\log y}. \tag{1.10}$$

Theorem 1.1. *Assume RH. Fix $\varepsilon \in (0,\,1)$. Suppose that $x \geqslant C_\varepsilon$ and $x^{1-\varepsilon} \geqslant y \geqslant (\log x)^{2+\varepsilon}$. Then*

$$\Psi(x,y) = \Lambda(x,y)G(\beta,y)\left(1 + O_\varepsilon\left(\frac{\log(u+1)}{y\log y}\left(|\psi(y)-y| + y^{1/2}\right)\right)\right). \tag{1.11}$$

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The following theorem gives an asymptotic formula for $\Psi(x, y)$ for y smaller than $(\log x)^2$.

THEOREM 1.2. *Assume RH. Fix* $\varepsilon \in (0, 1/3)$. *Suppose that* $x \geqslant C_\varepsilon$ *and* $(\log x)^3 \geqslant y \geqslant (\log x)^{4/3+\varepsilon}$. *Then*

$$\Psi(x, y) = \Lambda(x, y)G(\beta, y)\left(1 + O_\varepsilon\left(\frac{(\log y)^3}{y^{1/2}} + \frac{(\log x)^3(\log y)^3}{y^2}\right)\right). \tag{1.12}$$

If $y \leqslant (\log x)^{2-\varepsilon}$ *then the error term can be improved to* $O_\varepsilon((\log x)^3/(y^2 \log y))$.

Theorems 1.1 and 1.2, proved in § 4, show that

$$\Psi(x, y) \sim \Lambda(x, y)G(\beta, y)$$

holds when $y/((\log x)^{3/2}(\log\log x)^{-1/2}) \to \infty$. This range is shown to be optimal in Theorem 2.14 of [**9**]. The same theorem also supplies an alternative proof of theorem 1.2 when $y \leqslant (\log x)^{2-\varepsilon}$ (the proof can be adapted to cover $(\log x)^{2-\varepsilon} \leqslant y \leqslant (\log x)^3$ as well).

Hildebrand showed that RH is equivalent to $\Psi(x, y) \asymp_\varepsilon x\rho(u)$ for $y \geqslant (\log x)^{2+\varepsilon}$ [**11**]. He conjectured that $\Psi(x, y)$ is not of size $\asymp x\rho(u)$ when $y \leqslant (\log x)^{2-\varepsilon}$ [**12**]. This was recently confirmed by the author [**9**]. This also follows (under RH) from theorem 1.2, since $\Lambda(x, y) \asymp_\varepsilon x\rho(u)$ for $y \geqslant (\log x)^{1+\varepsilon}$ while (under RH) $G(\beta, y) \to \infty$ when $y \leqslant (\log x)^{2-\varepsilon}$ and $x \to \infty$ (this follows from the estimates for G in [**9**], see § 2).

Theorems 1.1 and 1.2 and their proofs have their origin in our work in the polynomial setting [**10**], where $\Psi(x, y)$ corresponds to the number of m-smooth polynomials of degree n over a finite field, while $\Lambda(x, y)$ is analogous to the number of m-smooth permutations of S_n (multiplied by $q^n/n!$). In that setting, the analogue of $G_1(s, y)$ is identically 1 (the relevant zeta function has no zeros) which makes the analysis unconditional.

1.3. Applications: sign changes and biases

From theorem 1.1 we deduce in § 2.2 the following

COROLLARY 1.3. *Assume RH. Fix* $\varepsilon \in (0, 1)$. *Suppose that* $x \geqslant C_\varepsilon$ *and* $x^{1-\varepsilon} \geqslant y \geqslant (\log x)^{2+\varepsilon}$. *Then*

$$\begin{aligned}\Psi(x, y)/\Lambda(x, y) &= 1 + \frac{y^{-\beta}}{\log y}\Bigg(- \sum_{|\rho| \leqslant T} \frac{y^\rho}{\rho - \beta} \\ &\qquad + \frac{y^{1/2}}{2\beta - 1} + O_\varepsilon\big(\frac{y^{1/2}}{\log y} + \frac{y \log^2(yT)}{T} + \frac{|\psi(y) - y| + y^{1/2}}{u}\big)\Bigg) \\ &= 1 + \frac{y^{-\beta}}{\log y}((\psi(y) - y)(1 + O_\varepsilon(u^{-1})) + O_\varepsilon(y^{1/2})) \\ &= 1 + O_\varepsilon((\log(u + 1))(\log x)y^{-1/2})\end{aligned}$$

holds for $T \geqslant 4$, *where the sum is over zeros of* ζ.

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Corollary 1.3 implies that large positive (resp. negative) values of $\psi(y)-y$ lead to large positive (resp. negative) values of $\Psi(x, y)-\Lambda(x, y)$ and vice versa. Large and small values of $\psi(y)-y$ were exhibited by Littlewood [**15**, Thm. 15.11]. Note that corollary 1.3 sharpens (1.7) if $y \leqslant x^{1-\varepsilon}$.[1]

Let $\pi(x)$ be the count of primes up to x and $\mathrm{Li}(x)$ be the logarithmic integral. It is known that $\pi(x)-\mathrm{Li}(x)$ is biased towards positive values in the following sense. Assuming RH and the Linear Independence hypothesis (LI) for zeros of ζ, Rubinstein and Sarnak [**16**] showed that the set

$$\{x \geqslant 2 : \pi(x) > \mathrm{Li}(x)\}$$

has logarithmic density ≈ 0.999997. This is an Archimedean analogue of the classical Chebyshev's bias on primes in arithmetic progressions. We use corollary 1.3 to exhibit a similar bias for smooth integers. Let us fix the value of $\beta = 1-\xi(u)/\log y$ to be

$$\beta = \beta_0$$

where $\beta_0 \in (1/2, 1)$. This amounts to restricting x to be a function $x = x(y)$ of y defined by

$$x = \exp\left(\frac{y^{1-\beta_0}-1}{1-\beta_0}\right). \tag{1.13}$$

In particular, $y = (\log x)^{1/(1-\beta_0)+o(1)}$. Then corollary 1.3 shows

$$\frac{\Psi(x(y), y)-\Lambda(x(y), y)}{\Lambda(x(y), y)} y^{\beta_0-\frac{1}{2}} \log y = -\sum_{|\rho| \leqslant T} \frac{y^{\rho-1/2}}{\rho-\beta_0} + \frac{1}{2\beta_0-1} + O_{\beta_0}\left(\frac{y^{1/2}\log^2(yT)}{T} + \frac{1}{\log y}\right). \tag{1.14}$$

Applying the formalism of Akbary *et al.* [**1**] to the right-hand side of (1.14) we deduce immediately

Corollary 1.4. *Assume RH. Assume LI for ζ. Fix $\beta_0 \in (1/2, 1)$ and let x be a function of y defined as in (1.13). Then the set*

$$\{y \geqslant 2 : \Psi(x(y), y) > \Lambda(x(y), x)\}$$

has logarithmic density greater than $1/2$, and the left-hand side of (1.14) has a limiting distribution in logarithmic sense.

In the same way that Chebyshev's bias for primes relates to the contribution of prime squares, this is also the case for smooth integers. Writing G as G_1G_2 as in § 1.1, G_2 captures the contribution of proper powers of primes. When $\beta_0 \in (1/2, 1)$, the only significant term in $G_2(\beta_0, y)$ is $k = 2$, which corresponds to squares of

[1]For $x \geqslant y \geqslant x^{1-\varepsilon}$, de Bruijn proved $\Psi(x, y) = \Lambda(x, y)(1+O_{\varepsilon}((\log x)^2/y^{1/2}))$ under RH [**3**, Eq. (1.3)].

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primes. The squares lead to the term $y^{1/2}/(2\beta_0 - 1)$ in (1.14) which creates the bias.

Remark 1.5. Consider the arithmetic function $\alpha_y(n)$ defined implicitly via

$$\sum_{n \geqslant 1} \frac{\alpha_y(n)}{n^s} = \exp\left(\sum_{m \leqslant y} \frac{\Lambda(m)}{\log m} \frac{1}{m^s} \right).$$

This function is supported on y-smooth numbers and coincides with the indicator of y-smooth numbers on squarefree integers. Working with the summatory function of α_y instead of $\Psi(x, y)$, the bias discussed above disappears. This is because, modifying the proof of theorem 1.1, one finds that

$$\sum_{n \leqslant y} \alpha_y(n) = \Lambda(x,y) G_1(\beta, y) \left(1 + O_\varepsilon \left(\frac{\log(u+1)}{y \log y} (|\psi(y) - y| + y^{1/2}) \right) \right)$$

holds in $x^{1-\varepsilon} \geqslant y \geqslant (\log x)^{2+\varepsilon}$, meaning the bias-causing factor $G_2(\beta, y)$ does not arise. This is analogous to how the indicator function of primes is biased, while $\Lambda(n)/\log n$ is not.

Remark 1.6. It is interesting to see if one can formulate and prove variants of corollaries 1.3 and 1.4 in the range $y \leqslant (\log x)^{1-\varepsilon}$. In this range, an accurate main term for $\Psi(x, y)$ was established in [**6**].

1.4. Strategy behind theorems 1.1 and 1.2

We write $\Psi(x, y)$ as a Perron integral, at least for non-integer x:

$$\Psi(x,y) = \frac{1}{2\pi i} \int_{(\sigma)} \zeta(s,y) \frac{x^s}{s} \, \mathrm{d}s$$

where σ can be any positive real. For non-integer x we also have

$$\Lambda(x,y) = \frac{1}{2\pi i} \int_{(\sigma)} F(s,y) \frac{x^s}{s} \, \mathrm{d}s \tag{1.15}$$

whenever $\sigma > \varepsilon$ and $y \geqslant C_\varepsilon$. Indeed, the Laplace inversion formula expresses $\Lambda(x, y)$ as

$$\begin{aligned} \Lambda(x,y) = x\lambda_y(u) &= \frac{x}{2\pi i} \int_{(c)} \hat{\lambda}_y(s) e^{us} \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{(1+c/\log y)} (\hat{\lambda}_y((s-1)\log y) \log y) x^s \, \mathrm{d}s \end{aligned} \tag{1.16}$$

for any c such that

$$\hat{\lambda}_y(s) := \int_0^\infty e^{-sv} \lambda_y(v) \, \mathrm{d}v, \tag{1.17}$$

converges absolutely for $\Re s \geqslant c$. In particular, we may take $c > -(\log y)/(1+\varepsilon)$ if we assume $y \geqslant C_\varepsilon$, as Saias showed, see corollary A.2. As shown by de Bruijn

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[**3**, Eq. (2.6)] (cf. [**17**, Lem. 6]),

$$\hat{\lambda}_y(s) = \hat{\rho}(s)K(s/\log y).$$

By definition of F, (1.9), we can rewrite (1.16) as (1.15). As Saias does, we choose to work with $\sigma = \beta$, which is essentially a saddle point for $F(s,\, y)x^s$. If $x \geqslant y \geqslant (\log x)^{1+\varepsilon}$ and $x \geqslant C_\varepsilon$ then lemma 2.1 implies

$$\beta \geqslant c_\varepsilon > 0.$$

Saias proved (1.6) by showing that $\zeta(s,\, y)$ and $F(s,\, y)$ are close and so if we subtract

$$\Psi(x, y) - \Lambda(x, y) = \frac{1}{2\pi i}\int_{(\beta)} (\zeta(s, y) - F(s, y))\frac{x^s}{s}\,\mathrm{d}s$$

then we can bound the integral by using pointwise bounds for the integrand. Instead of subtracting $\Lambda(x,\, y)$, we subtract $\Lambda(x,\, y)$ times $G(\beta,\, y)$, which leads to

$$\Psi(x, y) = \Lambda(x, y)G(\beta, y)\left(1 + \frac{\Lambda(x, y)^{-1}}{2\pi i}\int_{(\beta)} \frac{G(s, y) - G(\beta, y)}{G(\beta, y)} F(s, y)\frac{x^s}{s}\,\mathrm{d}s\right). \tag{1.18}$$

We want to bound the integral in (1.18). The proof of theorem 1.1 considers separately the range

$$u \geqslant (\log y)(\log\log y)^3 \tag{1.19}$$

and its complement. When u satisfies (1.19), then in (1.18) one needs only small values of $\Re s$ to estimate the integral ($|\Re s| \leqslant 1/\log y$) with arbitrary power saving in y. This is an unconditional observation established in proposition 3.1. However, for smaller u, one needs $|\Re s|$ going up to a power of y if one desires power saving in y, which makes the proof more involved.

In our proofs, RH is only invoked at the very end to estimate G_1 and its derivatives. For instance, in the range where (1.19) and $y \geqslant (\log x)^{2+\varepsilon}$ hold, we prove in (4.12) the *unconditional* estimate

$$\Psi(x, y) = \Lambda(x, y)G(\beta, y)\left(1 + O_\varepsilon\left(\frac{\max_{|v|\leqslant 1}|G'(\beta + iv, y)|}{G(\beta, y)\log x} + \frac{\max_{|v|\leqslant 1}|G''(\beta + iv, y)|}{G(\beta, y)(\log x)(\log y)} + \frac{1}{y}\right)\right). \tag{1.20}$$

See (4.16) for a similar estimate for $u \leqslant (\log y)(\log\log y)^3$. In particular, our proofs are easily modified to recover (1.6).

Conventions

The letters C, c denote absolute positive constants that may change between different occurrences. We denote by C_ε, c_ε positive constants depending only on ε, which may also change between different occurrences. The notation $A \ll B$ means $|A| \leqslant CB$ for some absolute constant C, and $A \ll_\varepsilon B$ means $|A| \leqslant C_\varepsilon B$. We write $A \asymp B$ to mean $C_1 B \leqslant A \leqslant C_2 B$ for some absolute positive constants C_i, and

$A \asymp_\varepsilon B$ means C_i may depend on ε. The letter ρ will always indicate a non-trivial zero of ζ. When we differentiate a bivariate function, we always do so with respect to the first variable. We set

$$L(y) := \exp((\log y)^{3/5}(\log\log y)^{-1/5}).$$

2. Preliminaries

2.1. Standard lemmas

Recall β was defined in (1.10).

LEMMA 2.1. *[**13**, Lem. 1] For $u \geqslant 3$ we have $\xi(u) = \log u + \log\log u + O((\log\log u)/\log u)$. In particular,*

$$y^{1-\beta} \asymp u\log(u+1), \quad u \geqslant 1. \tag{2.1}$$

LEMMA 2.2 [**2**]. *For $u \geqslant 1$ we have $\rho(u) \asymp e^{-u\xi+I(\xi)}u^{-1/2} = x^{\beta-1}e^{I(\xi)}u^{-1/2}$.*

In the next lemmas we write $s \in \mathbb{C}$ as $s = \sigma + it$.

LEMMA 2.3. *[**15**, Cor. 10.5] For $|\sigma| \leqslant A$ and $|t| \geqslant 1$, $|\zeta(s)| \asymp_A (|t|+4)^{1/2-\sigma}|\zeta(1-s)|$.*

LEMMA 2.4. *[**15**, Cor. 1.17] Fix $\varepsilon > 0$. For $\sigma \in [\varepsilon, 2]$ and $|t| \geqslant 1$ we have*

$$\zeta(s) \ll_\varepsilon (1 + (|t|+4)^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log(|t|+4)\right\}.$$

LEMMA 2.5. *[**19**, Thm. 7.2(A)]*
We have, for $\sigma \in [1/2, 2]$ and $T \geqslant 2$,

$$\int_1^T |\zeta(\sigma+it)|^2 \,\mathrm{d}t \ll T\min\left\{\log T, \frac{1}{\sigma-\frac{1}{2}}\right\}.$$

LEMMA 2.6. *[**14**, Lem. 2.7] The following bounds hold for $s = -\xi(u) + it$:*

$$\hat{\rho}(s) = e^{\gamma + I(-s)} = \begin{cases} O\left(\exp\left(I(\xi) - \frac{t^2u}{2\pi^2}\right)\right) & \text{if } |t| \leqslant \pi, \\ O\left(\exp\left(I(\xi) - \frac{u}{\pi^2+\xi^2}\right)\right) & \text{if } |t| \geqslant \pi, \\ \frac{1}{s} + O\left(\frac{1+u\xi}{|s|^2}\right) & \text{if } 1+u\xi = O(|t|). \end{cases} \tag{2.2}$$

The third case of lemma 2.6 is usually stated in the range $1 + u\xi \leqslant |t|$, but the same proof works for $1 + u\xi = O(|t|)$. Since $1 + u\xi = e^\xi$, the third case can also be written as

$$s\hat{\rho}(s) = 1 + O(e^{-\sigma}/|t|) \tag{2.3}$$

for $s = \sigma + it$, assuming $\sigma < 0$ and $e^{-\sigma} = O(|t|)$. The following lemma is a variant of [**13**, Lem. 8], proved in the same way.

Lemma 2.7 [**13**]. *Fix* $\varepsilon > 0$. *Suppose* $x \geqslant y \geqslant (\log x)^{1+\varepsilon}$ *and* $x \geqslant C_\varepsilon$. *For* $|t| \leqslant 1/\log y$,

$$\left|\frac{\zeta(\beta+it,y)}{\zeta(\beta,y)}\right| \leqslant \exp(-ct^2(\log x)(\log y)).$$

For $1/\log y \leqslant |t| \leqslant \exp((\log y)^{3/2-\varepsilon})$,

$$\frac{\zeta(\beta+it,y)}{\zeta(\beta,y)} \ll_\varepsilon \exp\left(-\frac{cut^2}{(1-\beta)^2+t^2}\right). \tag{2.4}$$

2.2. More on G

Lemma 2.8 [**9**]. *Fix* $0 \leqslant i \leqslant 4$. *Let* $y \geqslant 4$. *Let* $s \in \mathbb{C}$ *with* $\Re s \in [0,\, 1]$ *and the property that*

$$\min_{\zeta(\rho)=0,\quad t\geqslant 0} |\rho - s - t| \gg 1. \tag{2.5}$$

Then for $T \geqslant 3 + |\Im s|$ *we have*

$$\begin{aligned}(\log G_1)^{(i)}(s,y) = &-\sum_{|\Im(\rho-s)|\leqslant T} \frac{\mathrm{d}^i}{\mathrm{d}s^i}\int_0^\infty \frac{y^{\rho-s-t}}{\rho-s-t}\,\mathrm{d}t \\ &+ O\left((\log y)^i y^{-\Re s} + \frac{\log^2(yT)(\log y)^{i-1}}{T} y^{1-\Re s}\right).\end{aligned} \tag{2.6}$$

Corollary 2.9. *Fix* $0 \leqslant i \leqslant 4$. *Let* $y \geqslant 4$. *Let* $s \in \mathbb{C}$ *with* $\Re s \in [0,\, 1]$. *If* $|\Im s| \leqslant 1$ *we have* $(\log G_1)^{(i)}(s,\, y) \ll L(y)^{-c} y^{1-\Re s}$ *unconditionally. Under RH, if* $T \geqslant 4$ *and* $|\Im s| \leqslant 1$ *then*

$$\begin{aligned}(\log G_1)^{(i)}(s,y) &= (-\log y)^{i-1} y^{-s}\left(\sum_{|\Im(\rho-s)|\leqslant T} \frac{y^\rho}{\rho-s} + O\left(\frac{y^{1/2}}{\log y} + \frac{y\log^2(yT)}{T}\right)\right) \\ &= (-1)^i(\log y)^{i-1} y^{-s}(\psi(y) - y + O(y^{1/2})) \ll y^{1/2-\Re s}(\log y)^{i+1}.\end{aligned} \tag{2.7}$$

Under RH, if $T \geqslant 4$, $\Re s \in [3/4,\, 1]$ *and* $|\Im s| \leqslant y^{9/10}$ *then*

$$\begin{aligned}(\log G_1)^{(i)}(s,y) &= (-1)^i(\log y)^{i-1} y^{-s}(\psi(y) - y + O(y^{1/2}\log^2(|\Im s|+2))) \\ &\ll y^{1/2-\Re s}(\log y)^{i+1}.\end{aligned} \tag{2.8}$$

Proof. If $|\Im s| \leqslant 1$ then (2.5) holds. It is easily seen that, for any zero ρ of ζ,

$$\frac{\mathrm{d}^i}{\mathrm{d}s^i}\int_0^\infty \frac{y^{\rho-s-t}}{\rho-s-t}\,\mathrm{d}t = -\frac{(-\log y)^{i-1}y^{\rho-s}}{\rho-s}\left(1 + O\left(\frac{1}{\min_{t\geqslant 0}|\rho-s-t|\log y}\right)\right) \tag{2.9}$$

if (2.5) holds. We apply lemma 2.8 with $T = L(y)^c$ and use the Vinogradov–Korobov zero-free region and (2.9) to simplify. Now assume RH, i.e. $|y^\rho| = y^{1/2}$. We demonstrate (2.7), and (2.8) is proved along similar lines. We apply lemma 2.8 with $T \geqslant 4$ and simplify it using (2.9). We bound the resulting error using the facts

$\min_{t\geqslant 0}|\rho - s - t| \asymp |\rho - s|$ and $\sum_{\rho} 1/|\rho - s|^2 \ll 1$ for $|s| \leqslant 2$, since there are $\ll \log T$ zeros of ζ between height T and $T+1$ [**15**, Thm. 10.13]. This gives the first equality in (2.7). The second equality in (2.7) follows by taking $T = y$, recalling the classical estimate

$$\psi(y) - y = -\sum_{|\rho|\leqslant y} \frac{y^{\rho}}{\rho} + O(\log^2 y) \tag{2.10}$$

given in [**15**, Thm. 12.5] (it also follows from lemma 2.8 with $(i, s, T) = (1, 0, y)$), and the bound $\sum_{\rho} 1/(|\rho - s||\rho|) \ll 1$. The last inequality in (2.7) is von Koch's bound $\psi(y) - y = O(y^{1/2}\log^2 y)$ [**20**]. □

We turn to G_2. By the non-negativity of the coefficients of $\log G_2$, for $i \geqslant 0$ and $\Re s > 0$ we have

$$|(\log G_2)^{(i)}(s, y)| \leqslant (-1)^i \log G_2^{(i)}(\Re s, y). \tag{2.11}$$

Lemma 2.10 [**9**]. *Fix $\varepsilon > 0$ and $0 \leqslant i \leqslant 4$. For $y \geqslant 2$ and $1 \geqslant s \geqslant \varepsilon$,*

$$\begin{aligned}(\log G_2)^{(i)}(s, y) &= (1 + O_{\varepsilon}(L(y)^{-c}))\frac{(-2)^i}{2}\int_{y^{1/2}}^{y} (\log t)^{i-1} t^{-2s}\,\mathrm{d}t \\ &\asymp_{\varepsilon} \frac{(-\log y)^i y^{\max\{1-2s, \frac{1}{2}-s\}}}{\max\{1, |s-1/2|\log y\}}.\end{aligned} \tag{2.12}$$

Corollary 2.9 and lemma 2.10, applied with $i = 0$, imply the following

Lemma 2.11. *Assume RH. Fix $\varepsilon > 0$. If $1 \geqslant s \geqslant 1/2 + \varepsilon$ and $T \geqslant 4$ then*

$$\begin{aligned}G(s, y) &= 1 + \frac{y^{-s}}{\log y}\left(-\sum_{|\rho|\leqslant T} \frac{y^{\rho}}{\rho - s} + \frac{y^{1/2}}{2s-1} + O_{\varepsilon}\left(\frac{y^{1/2}}{\log y} + \frac{y\log^2(yT)}{T}\right)\right) \\ &= 1 + \frac{y^{-s}}{\log y}(\psi(y) - y + O_{\varepsilon}(y^{1/2})) = 1 + O_{\varepsilon}(y^{1/2-s}\log y).\end{aligned}$$

Corollary 1.3 follows from theorem 1.1 by simplifying $G(\beta, y)$ using lemma 2.11 and (2.1).

3. Truncation estimates for Ψ and Λ

The purpose of this section is to prove the following two propositions.

Proposition 3.1 Medium u. *Suppose $x \geqslant y \geqslant 2$ satisfy*

$$u \geqslant (\log y)(\log\log y)^3.$$

Fix $\varepsilon > 0$. Suppose $y \geqslant (\log x)^{1+\varepsilon}$ and $x \geqslant C_{\varepsilon}$. Then

$$\begin{aligned}\Psi(x, y) &= \frac{1}{2\pi i}\int_{\beta - \frac{i}{\log y}}^{\beta + \frac{i}{\log y}} \zeta(s, y)\frac{x^s}{s}\,\mathrm{d}s \\ &\quad + O_{\varepsilon}\left(\frac{\Psi(x, y) + x\rho(u)G(\beta, y)}{\exp(c_{\varepsilon}\min\{u/\log^2(u+1), (\log y)^{4/3}\})}\right),\end{aligned} \tag{3.1}$$

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$$\Lambda(x,y) = \frac{1}{2\pi i}\int_{\beta-\frac{i}{\log y}}^{\beta+i/\log y} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O_\varepsilon\left(\frac{x\rho(u)}{\exp(cu/\log^2(u+1))}\right). \tag{3.2}$$

PROPOSITION 3.2 Small u. *Suppose $x \geqslant y \geqslant 2$ satisfy*

$$u \leqslant (\log y)(\log\log y)^3.$$

Suppose $x \geqslant C$ and let $T \in [(\log x)^5, x\rho(u)]$. Then

$$\Psi(x,y) = \frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT} \zeta(s,y)\frac{x^s}{s}\,\mathrm{d}s + O\left(\frac{\Psi(x,y)+x\rho(u)G(\beta,y)}{T^{4/5}}\right),$$
$$\Lambda(x,y) = \frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O\left(\frac{x\rho(u)}{T^{4/5}}\right).$$

3.1. Preparation

LEMMA 3.3. *Fix $\varepsilon \in (0, 1)$. For $\sigma \in [\varepsilon, 1]$ and $x \geqslant T \geqslant 2$ we have*

$$\frac{1}{2\pi i}\int_{\sigma+it\colon |t|>T} \zeta(s)\frac{x^s}{s}\,\mathrm{d}s \ll_\varepsilon \frac{x^\sigma}{T^\sigma}\log T + \log x. \tag{3.3}$$

The integral should be understood in principal value sense. Lemma 3.3 makes more precise a computation done in p. 96 of Saias' paper [**17**] (cf. [**18**, p. 537]), which is not stated for general T and σ but contains the same ideas.

Proof. By [**19**, Thm. 4.11], for every $r > 0$ we have

$$\zeta(s) = \sum_{n\leqslant r} n^{-s} - \frac{r^{1-s}}{1-s} + O_\varepsilon(r^{-\Re s})$$

as long as $s \neq 1$, $\Re s \geqslant \varepsilon$ and $|\Im s| \leqslant 2r$. Suppose $s = \sigma + it$ with $|t| \geqslant 1$. We apply this estimate with $r = |t|$, obtaining

$$\zeta(s) = \sum_{n\leqslant |t|} n^{-s} - \frac{|t|^{1-s}}{1-s} + O_\varepsilon(|t|^{-\sigma}) = \sum_{n\leqslant |t|} n^{-s} + O_\varepsilon(|t|^{-\sigma}). \tag{3.4}$$

We now plug (3.4) in the left-hand side of (3.3). The contribution of the error term to the integral is acceptable:

$$\int_{\sigma+it\colon |t|>T} O(|t|^{-\sigma})\frac{x^s}{s}\,\mathrm{d}s \ll x^\sigma \int_T^\infty |t|^{-\sigma-1}\,\mathrm{d}t \ll_\varepsilon \frac{x^\sigma}{T^\sigma}.$$

The contribution of $n^{-s}\mathbf{1}_{n\leqslant|t|}$ in (3.4) to the left-hand side of (3.3) is

$$\frac{1}{2\pi i}\int_{\sigma+it:|t|>\max\{n,T\}} n^{-s}\frac{x^s}{s}\,\mathrm{d}s. \tag{3.5}$$

Since

$$\frac{1}{2\pi i}\int_{\sigma+it\colon |t|\leqslant S} n^{-s}\frac{x^s}{s}\,\mathrm{d}s = \mathbf{1}_{x>n} + \frac{\mathbf{1}_{x=n}}{2} + O\left(\frac{(x/n)^\sigma}{1+S|\log(x/n)|}\right), \quad S \geqslant 1,$$

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by the truncated Perron's formula [**14**, p. 435], and

$$\frac{1}{2\pi i}\int_{(\sigma)} n^{-s}\frac{x^s}{s}\,\mathrm{d}s = \mathbf{1}_{x>n} + \frac{\mathbf{1}_{x=n}}{2}$$

by Perron's formula, it follows that the integral in (3.5) is bounded by

$$\ll \frac{(x/n)^{\sigma}}{1+\max\{n,T\}|\log(x/n)|}$$

and so the total contribution of the n-sum in (3.4) to the left-hand side of (3.3) is

$$\ll x^{\sigma}\sum_{n\geqslant 1}\frac{n^{-\sigma}}{1+\max\{n,T\}|\log(x/n)|}. \tag{3.6}$$

It remains to estimate (3.6), which we do according to the size of n. The contribution of $n \geqslant 2x$ is

$$\ll x^{\sigma}\sum_{n\geqslant 2x} n^{-\sigma-1} \ll_{\varepsilon} 1.$$

The contribution of $n \in (x/2,\, 2x)$ can be bounded by considering separately the n closest to x, and partitioning the rest of the ns according to the value of $k \geqslant 0$ for which $|\log(x/n)| \in [2^{-k},\, 2^{1-k})$:

$$\ll x^{\sigma}\sum_{n\in(x/2,2x)}\frac{n^{-\sigma}}{1+x|\log(x/n)|} \ll 1 + \sum_{k\geqslant 0:\, 2^k\leqslant 2x}\frac{x}{2^k}\frac{1}{1+x/2^k} \ll \log x.$$

The contribution of $n \leqslant T/2$ is

$$\ll \frac{x^{\sigma}}{T}\sum_{n\leqslant T/2} n^{-\sigma} \ll \frac{x^{\sigma}}{T^{\sigma}}\log T.$$

Finally, the contribution of $T/2 < n \leqslant x/2$ is

$$\ll x^{\sigma}\sum_{n>T/2} n^{-1-\sigma} \ll_{\varepsilon} \frac{x^{\sigma}}{T^{\sigma}},$$

acceptable as well. □

Corollary 3.4. *Fix $\varepsilon \in (0,\, 1)$. Suppose $x \geqslant y \geqslant C_{\varepsilon}$. For $\sigma \in [\varepsilon,\, 1]$ and $x \geqslant T \geqslant \max\{2,\, y^{1-\sigma}/\log y\}$ we have*

$$\Lambda(x,y) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O_{\varepsilon}\left(\frac{x^{\sigma}}{T^{\sigma}}\log T + \log x + x^{\sigma}\frac{y^{1-\sigma}}{\log y}\frac{\log^{1/2} T}{T^{\min\{1,1/2+\sigma\}}}\right).$$

Corollary 3.4 rests on lemma 3.3, and makes more precise Proposition 2 of Saias [**17**].

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Proof. Our starting point is the identity (1.15). (If $x \in \mathbb{Z}$ it still holds with an error term of $O(1)$, since the integral converges to the average $(\Lambda(x+,\, y) + \Lambda(x-,\, y))/2 = \Lambda(x,\, y) + O(1)$.) From that identity it follows that our task is equivalent to upper bounding

$$\left| \int_{\sigma+it\colon |t|>T} F(s,y) \frac{x^s}{s} \, \mathrm{d}s \right|.$$

Recall $F(s,\, y) = \hat{\rho}((s-1)\log y)\zeta(s)(s-1)\log y$. By (2.3) with $(s-1)\log y$ instead of s we find

$$F(s,y) = \zeta(s)\left(1 + O\left(\frac{y^{1-\sigma}}{|t|\log y}\right)\right)$$

if $y^{1-\sigma} = O(|t|\log y)$, which holds by our assumptions on T. By the triangle inequality,

$$\left| \int_{\sigma+it\colon |t|>T} F(s,y) \frac{x^s}{s} \, \mathrm{d}s \right| \ll \left| \int_{\sigma+it\colon |t|>T} \frac{\zeta(s)}{s} x^s \, \mathrm{d}s \right| + x^\sigma \frac{y^{1-\sigma}}{\log y} \int_{\sigma+it\colon |t|>T} \frac{|\zeta(s)|}{|t|^2} |\mathrm{d}s|. \tag{3.7}$$

The first integral in the right-hand side of (3.7) is estimated in lemma 3.3. To bound the second integral we apply the second moment estimate for ζ given in lemma 2.5. We first suppose that $\sigma \geqslant 1/2$. Using Cauchy–Schwarz, the second integral in the right-hand side of (3.7) is at most

$$\int_{\sigma+it\colon |t|>T} \frac{|\zeta(s)|}{|t|^2} |\mathrm{d}s| \ll \sum_{2^k \geqslant T/2} 4^{-k} \int_{2^k}^{2^{k+1}} |\zeta(\sigma+it)| \, \mathrm{d}t \ll \sum_{2^k \geqslant T/2} 2^{-k} k^{1/2} \ll \frac{\log^{1/2} T}{T}. \tag{3.8}$$

Multiplying this by the prefactor $x^\sigma y^{1-\sigma}/\log y$, we see that this is acceptable. If $\varepsilon \leqslant \sigma \leqslant 1/2$ we use lemma 2.3. We obtain that the second integral in the right-hand side of (3.7) is at most

$$\int_{\sigma+it\colon |t|>T} \frac{|\zeta(s)|}{|t|^2} |\mathrm{d}s| \ll \int_{1-\sigma+it\colon |t|>T} \frac{|\zeta(s)|}{|t|^{2+\sigma-1/2}} |\mathrm{d}s| \ll \sum_{2^k \geqslant T/2} 2^{-k(\sigma+1/2)} k^{1/2} \ll \frac{\log^{1/2} T}{T^{1/2+\sigma}}, \tag{3.9}$$

concluding the proof. □

Let $\alpha = \alpha(x,\, y)$ be the saddle point associated with y-smooth numbers up to x [**13**], that is, the minimizer of the convex function $s \mapsto x^s \zeta(s,\, y)$ $(s > 0)$.

Lemma 3.5. *For* $\sigma \in (0,\, 1]$, $x \geqslant y \geqslant C$ *and* $T \geqslant 2$ *we have*

$$\Psi(x, y) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \zeta(s, y)\frac{x^s}{s}\,\mathrm{d}s + O\left(\frac{x^\sigma\zeta(\sigma, y)}{T} + \frac{\Psi(x, y)\log T}{T^\alpha} + 1\right). \quad (3.10)$$

Our proof makes more precise a similar estimate appearing in Saias [**17**, p. 98], which does not allow general y and T but contains the main ideas.

Proof. The truncated Perron's formula [**14**, p. 435] bounds the error in (3.10) by

$$\ll x^\sigma \sum_{\substack{n\geqslant 1\\ n \text{ is } y\text{-smooth}}} \frac{1}{n^\sigma(1 + T|\log(x/n)|)}.$$

The contribution of the terms with $|\log(x/n)| \geqslant 1$ is

$$\ll \frac{x^\sigma}{T} \sum_{\substack{n\geqslant 1\\ n \text{ is } y\text{-smooth}}} \frac{1}{n^\sigma} = \frac{x^\sigma\zeta(\sigma, y)}{T}.$$

We now study the terms with $|\log(x/n)| < 1$. These contribute

$$\ll \sum_{\substack{e^{-1}x<n<ex\\ n \text{ is } y\text{-smooth}}} \frac{1}{1 + T|\log(x/n)|}. \quad (3.11)$$

The subset of terms with $|\log(x/n)| \leqslant 1/T$ contributes to (3.11)

$$\ll \sum_{\substack{|n-x|\leqslant Cx/T\\ n\, y\text{-smooth}}} 1 \ll \Psi\left(x + \frac{Cx}{T}, y\right) - \Psi\left(x - \frac{Cx}{T}, y\right). \quad (3.12)$$

The contribution of the rest of the terms to (3.11), namely, those terms with $1/T < |\log(x/n)| < 1$, can be dyadically dissected to terms with $|\log(x/n)| \in [2^{-k},\, 2^{1-k})$ for each integer $k \geqslant 1$ such that $2^k < 2T$ holds. Their total contribution is

$$\ll \frac{1}{T} \sum_{1\leqslant k\leqslant \log_2 T+1} 2^k \left(\Psi\left(x + \frac{Cx}{2^k}, y\right) - \Psi\left(x - \frac{Cx}{2^k}, y\right)\right), \quad (3.13)$$

where $\log_2$ is the base-2 logarithm. (We interpret $\Psi(a,\, y)$ for negative a as equal to 0.) Note that the sum in (3.13) dominates the right-hand side of (3.12). We shall make use of Hildebrand's inequality $\Psi(a + b,\, y) - \Psi(a,\, y) \leqslant \Psi(b,\, y)$, valid for

$y \geqslant C$ and $a,\, b \geqslant y$. It implies

$$\Psi(a+b,y) - \Psi(a,y) \leqslant \Psi(b,y) + 1 \tag{3.14}$$

for $y \geqslant C$ and all $a,\, b$. We apply (3.14) with $a = x - Cx/2^k$ and $b = 2Cx/2^k$ to find that (3.13) is bounded by

$$\ll \frac{1}{T} \sum_{1 \leqslant k \leqslant \log_2 T+1} 2^k \left(\Psi\left(\frac{Cx}{2^k}, y\right) + 1 \right) \ll \frac{1}{T} \sum_{1 \leqslant k \leqslant \log_2 T+1} 2^k \left(\Psi\left(\frac{x}{2^k}, y\right) + 1 \right) \tag{3.15}$$

where in the second inequality we replaced $\Psi(Cx,\, y)$ with $\Psi(x,\, y)$ using [**13**, Thm. 3]. To conclude, we recall Theorem 2.4 of [**5**] says $\Psi(x/d,\, y) \ll \Psi(x,\, y)/d^{\alpha}$ holds for $x \geqslant y \geqslant 2$ and $1 \leqslant d \leqslant x$. We apply this inequality with $d = 2^k$ and obtain

$$\begin{aligned} \frac{1}{T} \sum_{1 \leqslant k \leqslant \log_2 T+1} 2^k \left(\Psi\left(\frac{x}{2^k}, y\right) + 1 \right) &\ll 1 + \frac{\Psi(x,y)}{T} \sum_{1 \leqslant k \leqslant \log_2 T+1} 2^{(1-\alpha)k} \\ &\ll 1 + \frac{\Psi(x,y)\log T}{T^{\alpha}} \end{aligned} \tag{3.16}$$

as needed. □

3.2. Proof of proposition 3.1

We first truncate the Perron integral for $\Psi(x,\, y)$. We apply lemma 3.5 with $\sigma = \beta$ and $T = \exp((\log y)^{4/3})$. The assumption $y \geqslant (\log x)^{1+\varepsilon}$ implies $\beta \gg_{\varepsilon} 1$ and $\Psi(x,\, y) \geqslant x^{c_{\varepsilon}}$. Since $\alpha = \beta + O(1/\log y)$ [**13**, Lem. 2] it follows that $\alpha \gg_{\varepsilon} 1$ and so

$$\Psi(x,y) = \frac{1}{2\pi i} \int_{\beta - iT}^{\beta+iT} \zeta(s,y) \frac{x^s}{s}\, \mathrm{d}s + O_{\varepsilon}\left(\frac{x^{\beta}\zeta(\beta,y) + \Psi(x,y)}{T^{c_{\varepsilon}}} \right). \tag{3.17}$$

We use lemma 2.7 to bound the contribution of $1/\log y \leqslant |\Im s| \leqslant T$:

$$\begin{aligned} \int_{\beta+i/\log y}^{\beta+iT} \zeta(s,y)\frac{x^s}{s}\, \mathrm{d}s &\ll x^{\beta}\zeta(\beta,y) \int_{1/\log y}^{T} \left| \frac{\zeta(\beta+it,y)}{\zeta(\beta,y)} \right| \frac{\mathrm{d}t}{\beta+t} \\ &\ll x^{\beta}\zeta(\beta,y) \int_{1/\log y}^{T} \exp\left(- \frac{cut^2}{(1-\beta)^2+t^2} \right) \frac{\mathrm{d}t}{\beta+t} \\ &\ll x^{\beta}\zeta(\beta,y) \Bigg(\exp(-cu)\log T + \int_{1/\log y}^{\xi(u)/\log y} \\ &\qquad \exp\left(- \frac{c(\log x)(\log y)}{\log^2(u+1)} t^2 \right) \mathrm{d}t \Bigg) \\ &\ll x^{\beta}\zeta(\beta,y) \exp\left(- \frac{cu}{\log^2(u+1)} \right). \end{aligned}$$

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We estimate $x^{\beta}\zeta(\beta, y)$:

$$
\begin{aligned}
{}^{\beta}\zeta(\beta,y) &= \frac{x}{e^{u\xi(u)}}F(\beta,y)G(\beta,y)\\
&= \zeta(\beta)(\beta-1)\frac{xe^{I(\xi)+\gamma}\log y}{e^{u\xi(u)}}G(\beta,y) \ll_{\varepsilon} x\rho(u)\sqrt{(\log x)(\log y)}G(\beta,y)
\end{aligned}
\tag{3.18}
$$

using (1.9) and lemma 2.2. Finally, note that both T and $\exp(u/\log^2(u+1))$ grow faster than any power of $\log x$. We turn to $\Lambda(x, y)$. We apply corollary 3.4 with $\sigma = \beta$ and

$$
T = \frac{y^{1-\beta}}{\log y} = \frac{e^{\xi(u)}}{\log y} \asymp \frac{u\log(u+1)}{\log y} \gg (\log\log y)^4.
$$

We obtain

$$
\Lambda(x,y) = \frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O_{\varepsilon}\left(\frac{ux}{\exp(u\xi)}\right).
$$

We now treat the range $1/\log y \leqslant |\Im s| \leqslant T$. By the definition of F,

$$
\int_{\beta+\frac{i}{\log y}}^{\beta+iT} F(s,y)\frac{x^s}{s}\,\mathrm{d}s \ll_{\varepsilon} \frac{x\log y}{\exp(u\xi)}\int_{1/\log y}^{T} |\zeta(\beta+it)||\hat{\rho}(-\xi(u)+it\log y)|\,\mathrm{d}t.
\tag{3.19}
$$

First suppose $t \geqslant \pi/\log y$. By the second case of lemma 2.6, this range contributes

$$
\begin{aligned}
&\ll_{\varepsilon} \frac{x\exp(I(\xi))\log y}{\exp(u\xi)}\exp\left(-\frac{u}{\pi^2+\xi^2}\right)\int_{\pi/\log y}^{T}|\zeta(\beta+it)|\,\mathrm{d}t\\
&\ll x\rho(u)\sqrt{(\log x)(\log y)}\exp\left(-\frac{u}{\pi^2+\xi^2}\right)\int_{\pi/\log y}^{T}|\zeta(\beta+it)|\,\mathrm{d}t
\end{aligned}
\tag{3.20}
$$

using lemma 2.2 in the second inequality. Recall the second moment estimate for ζ given in lemma 2.5. It shows that right-hand side of (3.20) is bounded by

$$
\ll x\rho(u)\sqrt{(\log x)(\log y)}\exp\left(-\frac{u}{\pi^2+\xi^2}\right)T^{\max\{1,3/2-\beta\}}\sqrt{\log T}
$$

where we used the functional equation if $\beta < 1/2$ (lemma 2.3). The contribution of $1/\log y \leqslant t \leqslant \pi/\log y$ to the right-hand side of (3.19) is treated using the first part of lemma 2.6, and we find that it is at most

$$
\ll_{\varepsilon} \frac{x\exp(I(\xi))\log y}{\exp(u\xi)}\int_{1/\log y}^{\pi/\log y}\exp\left(-\frac{(\log x)(\log y)}{2\pi^2}t^2\right)\mathrm{d}t \ll_{\varepsilon} x\rho(u)\exp(-cu),
\tag{3.21}
$$

using lemma 2.2 in the second inequality. In conclusion,

$$
\Lambda(x,y) = \frac{1}{2\pi i}\int_{\beta-i/\log y}^{\beta+i/\log y} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + E
$$

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where

$$E \ll_\varepsilon \frac{ux}{\exp(u\xi)} + x\rho(u)\left(\sqrt{(\log x)(\log y)}\right.$$
$$\left.\exp\left(-\frac{u}{\pi^2+\xi^2}\right)T^{\max\{1,3/2-\beta\}}\sqrt{\log T} + \exp(-cu)\right).$$

By our choice of T and assumptions on u and y, this can be absorbed in the error term of (3.2).

3.3. Proof of proposition 3.2

We first truncate the Perron integral for $\Psi(x, y)$. We apply lemma 3.5 with $\sigma = \beta$ and our T, finding

$$\Psi(x,y) = \frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT} \zeta(s,y)\frac{x^s}{s}\,\mathrm{d}s + O\left(1 + \frac{\Psi(x,y)\log T}{T^\alpha} + \frac{x^\beta\zeta(\beta,y)}{T}\right). \quad (3.22)$$

In the considered range, $\Psi(x, y) \asymp x\rho(u)$. In particular, the error term $O(1)$ is acceptable since our T is $\ll x\rho(u) \ll \Psi(x, y)$ and so $1 \ll \Psi(x, y)/T^{4/5}$. Additionally, $\beta \sim 1$ as $x \to \infty$ by lemma 2.1 and $\alpha = \beta + O(1/\log y)$ [**13**, Lem. 2], so $\alpha \sim 1$. This implies that $(\log T)/T^\alpha \ll 1/T^{4/5}$ and the error term $O(\Psi(x, y)(\log T)/T^\alpha)$ is also acceptable. The estimate (3.18) treats the last error term and finishes the estimation. We turn to $\Lambda(x, y)$. We apply corollary 3.4 with our T, obtaining

$$\Lambda(x,y) = \frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT} F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O\left(\log x + x\exp(-u\xi)u\log(u+1)\frac{\log T}{T^\sigma}\right). \quad (3.23)$$

In our range $x\rho(u) \asymp x^{1+o(1)}$, so the term $\log x$ is acceptable. We have $\exp(-u\xi)u\log(u+1) \ll \rho(u)$ by lemma 2.2, so the second term in the error term of (3.23) is also acceptable.

4. Proofs of theorems 1.1 and 1.2

Proposition 4.1 Medium u. *Suppose $x \geqslant y \geqslant 2$ satisfy*

$$u \geqslant (\log y)(\log\log y)^3.$$

Fix $\varepsilon > 0$ and suppose $y \geqslant (\log x)^{1+\varepsilon}$ and $x \geqslant C_\varepsilon$. Let

$$t_0 := (\log x)^{-1/3}(\log y)^{-2/3}, \quad T := \exp(\min\{u/\log^2(u+1), (\log y)^{4/3}\}).$$

Then $\Psi(x, y) = \Lambda(x, y)G(\beta, y)(1 + E)$ for

$$E \ll_\varepsilon \frac{|G'(\beta,y)|}{G(\beta,y)\log x} + \frac{\max_{|v|\leqslant t_0}|G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)}$$
$$+ \frac{\max_{|v|\leqslant \frac{1}{\log y}}|G'(\beta+iv,y)|\exp(-u^{1/3}/20)}{G(\beta,y)\log x} + \frac{1}{T^{c_\varepsilon}}. \quad (4.1)$$

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Proof. Our strategy is to establish $\Psi(x,\,y) = \Lambda(x,\,y)G(\beta,\,y)(1+E_1+E_2)+E_3$ for

$$E_1 \ll_\varepsilon \frac{|G'(\beta,y)|}{G(\beta,y)\log x} + \frac{\max_{|v|\leqslant t_0}|G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)},$$

$$E_2 \ll_\varepsilon \frac{\max_{|v|\leqslant \frac{1}{\log y}}|G'(\beta+iv,y)|\exp(-u^{1/3}/20)}{G(\beta,y)\log x},$$

$$E_3 \ll_\varepsilon \frac{\Psi(x,y)+x\rho(u)G(\beta,y)}{T^{c_\varepsilon}}.$$

The theorem will then follow by rearranging, once we recall that $x\rho(u) \asymp_\varepsilon \Lambda(x,\,y)$. From proposition 3.1,

$$\Psi(x,y) - \Lambda(x,y)G(\beta,y)$$
$$= \frac{1}{2\pi i}\int_{\beta-\frac{i}{\log y}}^{\beta+\frac{i}{\log y}} (G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s + O_\varepsilon\left(\frac{\Psi(x,y)+x\rho(u)G(\beta,y)}{T^{c_\varepsilon}}\right), \tag{4.2}$$

which explains E_3. Let t_0 be as in the statement of the proposition. We upper bound the contribution of $t_0 \leqslant |\Im s| \leqslant 1/\log y$ to the integral in the right-hand side of (4.2). We have

$$|G(s,y)-G(\beta,y)| \leqslant |\Im s| \max_{|t|\leqslant|\Im s|}|G'(\beta+it,y)|.$$

The triangle inequality shows, by definition of F, that

$$\int_{\beta+it_0}^{\beta+\frac{i}{\log y}} (G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s$$
$$\ll_\varepsilon \max_{|t|\leqslant\frac{1}{\log y}}|G'(\beta+it,y)|x^\beta\log y\int_{t_0}^{1/\log y} t|e^{I(\xi-it\log y)}|\,\mathrm{d}t. \tag{4.3}$$

Since $-e^{-v^2/2}$ is the antiderivative of $e^{-v^2/2}v$, the first part of lemma 2.6 shows

$$\int_{t_0}^{1/\log y} t|e^{I(\xi-it\log y)}|\,\mathrm{d}t \ll \exp(I(\xi))\int_{t_0}^{1/\log y} t\exp(-(\log x)(\log y)t^2/(2\pi^2))\,\mathrm{d}t$$
$$\ll \exp(I(\xi))\frac{\exp(-u^{1/3}/(2\pi^2))}{(\log x)(\log y)}.$$

Hence, $t_0 \leqslant |\Im s| \leqslant 1/\log y$ contributes in total

$$\ll_\varepsilon \max_{|t|\leqslant 1/\log y}|G'(\beta+it,y)|x\rho(u)\exp(-u^{1/3}/20)/\log x$$

where we used lemma 2.2 to simplify. Once we divide this by $\Lambda(x,\,y)G(\beta,\,y) \asymp_\varepsilon x\rho(u)G(\beta,\,y)$ we obtain the error term E_2. It remains to study the contribution of $|\Im s| \leqslant t_0$ to the integral in the right-hand side of (4.2), which will yield E_1.

We Taylor-expand the integrand at $s=\beta$. We write $s=\beta+it$, $|t|\leqslant t_0$. We first simplify the integrand using the definition of F:

$$\begin{aligned}\frac{F(s,y)x^s}{s} &= (\log y)K(s-1)e^{\gamma+I(\xi)}x^{\beta+it}\exp(I(\xi-it\log y)-I(\xi))\\ &= (\log y)K(s-1)x^{\beta}e^{\gamma+I(\xi)}\exp(I(\xi-it\log y)-I(\xi)+it\log x).\end{aligned}$$

We Taylor-expand $\log K(s-1)$ and $G(s,\,y)-G(\beta,\,y)$:

$$\begin{aligned}K(s-1) &= K(\beta-1)(1+O_\varepsilon(t)),\\ G(s,y)-G(\beta,y) &= itG'(\beta,y)+O(t^2\max_{|v|\leqslant t}|G''(\beta+iv,y)|).\end{aligned}$$

We expand $I(\xi-it\log y)-I(\xi)+it\log x$:

$$I(\xi-it\log y)-I(\xi)+it\log x=-\frac{t^2}{2}I''(\xi)\log^2 y+O(|t|^3(\log x)(\log y)^2), \tag{4.4}$$

where we used $I'(\xi(u))=u$ and $I^{(3)}(\xi(u)+it)\ll e^{\xi(u)}/(1+\xi(u))\asymp u$. This implies

$$\begin{aligned}&\exp(I(\xi-it\log y)-I(\xi)-it\log y)\\ &\quad=\exp\left(-\frac{t^2}{2}I''(\xi)\log^2 y\right)(1+O(|t|^3(\log x)(\log y)^2))\end{aligned} \tag{4.5}$$

for $|t|\leqslant t_0$. By two basic properties of moments of the Gaussian,

$$\begin{aligned}&\int_{-t_0}^{t_0} t\exp\left(-\frac{t^2}{2}I''(\xi)\log^2 y\right)\,\mathrm{d}t=0,\\ &\int_{-t_0}^{t_0}|t|^k\exp\left(-\frac{t^2}{2}I''(\xi)\log^2 y\right)\,\mathrm{d}t\\ &\qquad\ll_k (I''(\xi)\log^2 y)^{-k+1/2}\ll_k((\log x)(\log y))^{-k+1/2},\end{aligned}$$

we find

$$\begin{aligned}&\int_{\beta-it_0}^{\beta+it_0}(G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s\\ &\qquad\ll_\varepsilon x^\beta e^{I(\xi)}\left(\frac{|G'(\beta,y)|\sqrt{\log y}}{(\log x)^{3/2}}+\frac{\max_{|v|\leqslant t_0}|G''(\beta+iv)|}{(\log x)^{3/2}(\log y)^{1/2}}\right).\end{aligned} \tag{4.6}$$

By lemma 2.2, we can replace $x^\beta e^{I(\xi)}$ with $x\rho(u)\sqrt{u}$, to obtain

$$\begin{aligned}&\int_{\beta-it_0}^{\beta+it_0}(G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s\\ &\qquad\ll_\varepsilon x\rho(u)\left(\frac{|G'(\beta,y)|}{\log x}+\frac{\max_{|v|\leqslant t_0}|G''(\beta+iv)|}{(\log x)(\log y)}\right).\end{aligned} \tag{4.7}$$

Dividing by $G(\beta,\,y)\Lambda(x,\,y)\asymp_\varepsilon G(\beta,\,y)x\rho(u)$ gives the error term E_1. □

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Proposition 4.2 Small u. *Suppose* $x \geqslant y \geqslant C$ *satisfy*

$$u \leqslant (\log y)(\log\log y)^3. \tag{4.8}$$

Let

$$t_0 := (\log x)^{-1/3}(\log y)^{-2/3}, \quad t_1 := \frac{u\log(u+1)}{\log y}, \quad t_2 \in [(\log x)^5, y^{4/5}]. \tag{4.9}$$

Then $\Psi(x, y) = \Lambda(x, y)G(\beta, y)(1+E)$ *for*

$$\begin{aligned} E \ll &\frac{|G'(\beta,y)|}{\log x} + \frac{\max_{|v|\leqslant t_0}|G''(\beta+iv,y)|}{(\log x)(\log y)} \\ &+ \frac{\max_{|v|\leqslant t_1}|G'(\beta+iv,y)|\exp(-u^{1/3}/20)}{\log x} + t_2^{-4/5} \\ &+ \exp(-u/2)\left(\max_{|t|\leqslant t_2}\left|\frac{G(\beta+it,y)}{G(\beta,y)}-1\right|\right. \\ &\left.+\left|\int_{t_1\leqslant|t|\leqslant t_2} K(\beta+it-1)x^{it}\frac{G(\beta+it,y)-G(\beta,y)}{G(\beta,y)}\frac{\mathrm{d}t}{t}\right|\right). \end{aligned}$$

Proof. Our strategy is to establish $\Psi(x, y) = \Lambda(x, y)G(\beta, y)(1 + E_1 + E_2 + E_3 + E_4) + E_5$ for

$$\begin{aligned} E_1 &\ll \frac{|G'(\beta,y)|}{G(\beta,y)\log x} + \frac{\max_{|v|\leqslant t_0}|G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)}, \\ E_2 &\ll \frac{\max_{|v|\leqslant t_1}|G'(\beta+iv,y)|\exp(-u^{1/3}/20)}{G(\beta,y)\log x}, \\ E_3 &\ll \frac{\exp(-u/2)}{\log y}\int_{t_1\leqslant|t|\leqslant t_2}\left|\frac{G(\beta+it)-G(\beta,y)}{G(\beta,y)}\right|\frac{\log(|t|+2)}{t^2}\,\mathrm{d}t, \\ E_4 &\ll \exp(-u/2)\left|\int_{t_1\leqslant|t|\leqslant t_2}K(\beta+it-1)x^{it}\frac{G(\beta+it,y)-G(\beta,y)}{G(\beta,y)}\frac{\mathrm{d}t}{t}\right|, \\ E_5 &\ll t_2^{-4/5}(\Psi(x,y)+x\rho(u)G(\beta,y)). \end{aligned} \tag{4.10}$$

The proposition will then follow by rearranging and the fact that $G(\beta, y) \asymp 1$ in the considered range, unconditionally, as follows from corollary 2.9 and lemma 2.10. From proposition 3.2 with $T = t_2$,

$$\begin{aligned} &\Psi(x,y) - \Lambda(x,y)G(\beta,y) \\ &\quad = \frac{1}{2\pi i}\int_{\beta-it_2}^{\beta+it_2}(G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s \\ &\qquad + O(t_2^{-4/5}(\Psi(x,y)+x\rho(u)G(\beta,y))), \end{aligned}$$

which explains E_5. For $|\Im s| \leqslant t_0$, we Taylor-expand $I(\xi - it\log y)$ as in the medium u range and obtain the contribution of E_1 (see (4.7)) We treat the contribution of

$|\Im s| \in [t_0, t_1]$. We replace $G(s, y) - G(\beta, y)$ with

$$|G(s,y) - G(\beta,y)| \leqslant |\Im s| \max_{0\leqslant|t|\leqslant|\Im s|} |G'(\beta+it,y)|.$$

The first two parts of lemma 2.6 show

$$\begin{aligned}
&\int_{\beta+it,\,|t|\in[t_0,t_1]} (G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s \\
&\quad\ll \max_{|t|\leqslant t_1} |G'(\beta+it,y)|x\rho(u)(\log y)\sqrt{u} \\
&\qquad\int_{|t|\in[t_0,t_1]} |t|\left(\exp\left(-\frac{t^2(\log x)(\log y)}{2\pi^2}\right)+\exp(-u/(\pi^2+\xi^2))\right)\mathrm{d}t \\
&\quad\ll \max_{|t|\leqslant t_1} |G'(\beta+it,y)|x\rho(u)\sqrt{u}\frac{\exp(-u^{1/3}/2\pi^2)}{\log x}.
\end{aligned}$$

This explains E_2. It remains to consider $t_2 \geqslant |\Im s| \geqslant t_1$. We use the third part of lemma 2.6 to replace $\hat{\rho}((s-1)\log y)$, appearing in $F(s,\,y)$, with its approximation:

$$\begin{aligned}
&\int_{\beta+it,\,|t|\in[t_1,t_2]} (G(s,y)-G(\beta,y))F(s,y)\frac{x^s}{s}\,\mathrm{d}s \\
&\quad= (\log y)x^\beta \int_{s=\beta+it,\,|t|\in[t_1,t_2]} K(s-1)x^{it}(G(s,y) \\
&\qquad -G(\beta,y))\left(\frac{i}{t\log y}+O\left(\frac{u\log(u+1)}{t^2\log^2 y}\right)\right)\mathrm{d}s. \qquad (4.11)
\end{aligned}$$

Recall $x^\beta \ll x\rho(u)\sqrt{u}\exp(-I(\xi(u)))$ by lemma 2.2, and that $I(\xi(u)) \sim u$ since a change of variables shows $I(r) = \mathrm{Li}(e^r) + O(\log r) \sim e^r/r$. The contribution of the error term in the right-hand side of (4.11) is

$$\begin{aligned}
&\ll x^\beta \log y \int_{s=\beta+it,\,|t|\in[t_1,t_2]} |K(s-1)x^{it}(G(s,y)-G(\beta,y))|\frac{u\log(u+1)}{t^2\log^2 y}|\,\mathrm{d}s| \\
&\ll \frac{x\rho(u)\exp(-2u/3)}{\log y}\int_{|t|\in[t_1,t_2]} |G(\beta+it)-G(\beta,y)|\frac{|\zeta(\beta+it)||\beta+it-1|}{t^2|\beta+it|}\,\mathrm{d}t.
\end{aligned}$$

If $|t| \leqslant 2$ we use $|\zeta(\beta+it)(\beta+it-1)| \ll 1$ while if $|t| \geqslant 2$ we use lemma 2.4, to obtain an error term of size E_3. The main term of (4.11) gives E_4. □

4.1. Proof of theorem 1.1: medium u

Here we prove theorem 1.1 in the range (1.19). We obtain from proposition 4.1 that *unconditionally*

$$\Psi(x,y) = \Lambda(x,y)G(\beta,y)(1+E) \qquad (4.12)$$

for

$$E \ll_\varepsilon \frac{\max_{|v|\leqslant 1}|G'(\beta+iv,y)|}{G(\beta,y)\log x} + \frac{\max_{|v|\leqslant 1}|G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)} + \frac{1}{y}. \qquad (4.13)$$

Because we assume $y \geqslant (\log x)^{2+\varepsilon}$, we have $\beta \geqslant 1/2 + c_\varepsilon$. Under RH, $\log G(\beta,\, y) = O_\varepsilon(1)$ by lemma 2.11. To bound the quantities appearing in E, we write $G(\beta + it,\, y)$ as $G_1(\beta + it,\, y)$ times $G_2(\beta + it,\, y)$. Lemma 2.10 and equation (2.11) tell us that

$$(\log G_2)^{(i)}(\beta + it, y) \ll_\varepsilon (\log y)^{i-1} y^{1/2-\beta} \tag{4.14}$$

for $i = 0,\, 1,\, 2$ and $t \in \mathbb{R}$. Corollary 2.9 says that under RH

$$\begin{aligned}(\log G_1)^{(i)}(\beta + it, y) &= (-1)^i (\log y)^{i-1} y^{-\beta-it}(\psi(y) - y + O_\varepsilon(y^{1/2})) \\ &\ll_\varepsilon (\log y)^{i+1} y^{1/2-\beta}\end{aligned} \tag{4.15}$$

for all $i = 0,\, 1,\, 2$ and $|t| \leqslant 1$. Putting these two together, one obtains (1.11).

4.2. Proof of theorem 1.1: small u

Here we prove theorem 1.1 for u in the range (4.8). In this range, $\beta = 1 + o(1)$ and $\Psi(x,\, y) = x^{1+o(1)}$. Moreover, $\log G(\beta,\, y) = O(1)$ unconditionally by corollary 2.9 and lemma 2.10. The hardest range of the proof will be $u \asymp 1$. Before proceeding with the actual proof, note that from proposition 4.2 and the triangle inequality, it follows that

$$\begin{aligned}\Psi(x, y) = \Lambda(x, y) G(\beta, y) \Bigg(1 + O\Bigg(t_2^{-4/5} &+ t_2 \max_{|t| \leqslant t_2} |G'(\beta + it, y)| \\ &+ \max_{|t| \leqslant 1} |G''(\beta + it, y)|\Bigg)\Bigg)\end{aligned} \tag{4.16}$$

holds unconditionally for $t_2 \in [(\log x)^5,\, y^{4/5}]$ and the range $x \geqslant y \geqslant C$, $u \leqslant (\log y)(\log\log y)^3$.

We obtain from proposition 4.2 with $t_2 = y^{4/5}$ that

$$\Psi(x, y) = \Lambda(x, y) G(\beta, y)(1 + E_1 + E_2 + E_3 + E_4 + y^{-3/5})$$

for E_i bounded in (4.10). We write $G(\beta + it,\, y)$ as $G_1(\beta + it,\, y)$ times $G_2(\beta + it,\, y)$. By lemma 2.10 and (2.11),

$$(\log G_2)^{(i)}(\beta + it, y) \ll (\log y)^{i-1} u \log(u + 1) y^{-1/2} \tag{4.17}$$

for $i = 0,\, 1,\, 2$ and $t \in \mathbb{R}$ where we simplified $y^{-\beta}$ using (2.1). From now on we assume RH. Corollary 2.9 implies

$$(\log G_1)^{(i)}(\beta + it, y) \ll \frac{(\log y)^{i-1} u \log(u + 1)}{y} (|\psi(y) - y| + y^{1/2}) \tag{4.18}$$

for $i = 0,\, 1,\, 2$ when $|t| \leqslant 1$. As in the medium u case, one can bound E_1 by an acceptable quantity using our estimates for $(\log G_1)^{(i)}$ and $(\log G_2)^{(i)}$. Recall

$$E_2 \ll \frac{\max_{|v| \leqslant t_1} |G'(\beta + iv, y)| \exp(-u^{1/3}/20)}{G(\beta, y) \log x}$$

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where $t_1 = u\log(u+1)/\log y$. If $t_1 \leqslant 1$ we bound E_2 in the same way we bounded E_1. Otherwise we use (2.8), which implies that

$$(\log G_1)^{(i)}(\beta+it,y) \ll (\log y)^{i+1}u\log(u+1)y^{-1/2} \tag{4.19}$$

holds for $i=0,\,1,\,2$ and $|t|\leqslant y^{9/10}$. This shows that, if $t_1>1$, i.e. $u\log(u+1)\geqslant \log y$,

$$E_2 \ll \frac{(\log y)^2 u\log(u+1)\exp(-u^{1/3}/20)}{y^{1/2}\log x} \ll \log(u+1)y^{-1/2}.$$

This is an acceptable contribution when $u\log(u+1)>\log y$. We now study E_3 and E_4. Due to $G(\beta+it,\,y)/G(\beta,\,y)$ being very close to 1 in our considered range by (4.17) and (4.19), we may replace

$$G(\beta+it,y)/G(\beta,y)-1$$

by

$$\log G(\beta+it,y)-\log G(\beta,y)$$

and incur a negligible error, in both E_3 and E_4. So to show E_3 is acceptable we need to prove

$$\int_{t_1\leqslant|t|\leqslant y^{4/5}} |\log G(\beta+it,y)-\log G(\beta,y)|\frac{\log(|t|+2)}{t^2}\,\mathrm{d}t \ll \frac{e^{u/3}}{y}(|\psi(y)-y|+y^{1/2}). \tag{4.20}$$

This is shown using the bound

$$\log G(\beta+it,y) \ll \frac{u\log(u+1)}{y\log y}(|\psi(y)-y|+y^{1/2}\log^2(|t|+2)), \quad |t|\leqslant y^{9/10}, \tag{4.21}$$

which is a consequence of (2.8) and (4.17). To handle E_4 it remains to prove

$$\int_{t_1\leqslant|t|\leqslant y^{4/5}} K(\beta+it-1)x^{it}(\log G(\beta+it,y)-\log G(\beta,y))\frac{\mathrm{d}t}{t} \ll_\varepsilon \frac{e^{u/2}}{y\log y}(|\psi(y)-y|+y^{1/2}). \tag{4.22}$$

Here we cannot use the triangle inequality and put absolute value inside the integral. Indeed, if we use the pointwise bound (4.21), along with our bounds for ζ (lemmas 2.4 and 2.5), we get a bound which falls short by a factor of $(\log y)^3$. We shall overcome this by several integrations by parts as we now describe.

To deal with the contribution of $\log G(\beta,\,y)$ to (4.22) we use (4.21) with $t=0$ along with the bound

$$\int_{t_1\leqslant|t|\leqslant y^{4/5}} K(\beta+it-1)x^{it}\frac{\mathrm{d}t}{t} \ll u^2$$

which follows by integration by parts, where we replace x^{it} by its antiderivative $x^{it}/\log x$.

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Note that due to integration by parts, derivatives of ζ arise. This means that in addition to lemmas 2.4 and 2.5 we need the bounds $\zeta^{(k)}(s) \ll_k (1+(|t|+4)^{1-\sigma})\log^{k+1}(|t|+4)$ and $\int_1^T |\zeta^{(k)}(\sigma+it)|^2 \,\mathrm{d}t \ll_k T$ for $\sigma \in [2/3, 1]$ and $T, |t| \geqslant 1$. These bounds follow from lemmas 2.4 and 2.5 through Cauchy's integral formula.

To deal with the contribution of $\log G(\beta+it, y)$ to (4.22) we write it $\log G_1(\beta+it, y) + \log G_2(\beta+it, y)$ and obtain two integrals which we bound separately.

4.2.1. Treatment of $\log G_1$ Recall we assume $y \leqslant x^{1-\varepsilon}$. We want to show

$$\int_{t_1 \leqslant |t| \leqslant y^{4/5}} K(\beta+it-1)x^{it} \log G_1(\beta+it, y)\frac{\mathrm{d}t}{t} \ll_\varepsilon \frac{e^{u/2}}{y\log y}(|\psi(y)-y|+y^{1/2}). \tag{4.23}$$

We integrate by parts, replacing x^{it} by its antiderivative, reducing matters to showing

$$\frac{1}{\log x}\int_{t_1 \leqslant |t| \leqslant y^{4/5}} K(\beta+it-1)x^{it} \frac{G_1'}{G_1}(\beta+it, y)\frac{\mathrm{d}t}{t} \ll_\varepsilon \frac{e^{u/2}}{y\log y}(|\psi(y)-y|+y^{1/2}). \tag{4.24}$$

We divide and multiply the integrand by y^{it}, so the left-hand side of (4.23) is now

$$\frac{1}{\log x}\int_{t_1 \leqslant |t| \leqslant y^{4/5}} K(\beta+it-1)(x/y)^{it} H(t)\frac{\mathrm{d}t}{t} \tag{4.25}$$

where $H(t) := y^{it}(G_1'/G_1)(\beta+it, y)$. From lemma 2.8,

$$y^\beta \cdot H(t) = \sum_{|\Im(\rho)-t|\leqslant 2y^{4/5}} \frac{y^\rho}{\rho-\beta-it} + O(y^{2/5}) \ll |\psi(y)-y| + y^{1/2}\log^2(|t|+2)$$

and, for $k = 1, 2, 3$,

$$y^\beta \cdot H^{(k)}(t) = (k+1)!i^k \sum_{|\Im(\rho)-t|\leqslant 2y^{4/5}} \frac{y^\rho}{(\rho-\beta-it)^{k+1}} + O(y^{2/5}) \ll y^{1/2}\log(|t|+2).$$

We integrate by parts 3 times, replacing $(x/y)^{it}$ by its antiderivative. We are guaranteed to get enough saving since $\log(x/y) \gg_\varepsilon \log x$.

4.2.2. Treatment of $\log G_2$ The function $\log G_2(\beta+it, y)$ is given as a sum over proper primes powers. As the cubes and higher powers contribute at most $\ll y^{-2/3+o(1)}$ to it by the prime number theorem (see [**9**]), we can replace $\log G_2(\beta+it, y)$ with the prime sum $\sum_{y^{1/2}<p\leqslant y} p^{-2(\beta+it)}/2$, so we are left to show

$$\sum_{y^{1/2}<p\leqslant y} p^{-2\beta} \int_{t_1 \leqslant |t| \leqslant y^{4/5}} K(\beta+it-1)(x/p^2)^{it}\frac{\mathrm{d}t}{t} \ll \frac{e^{u/2}}{y^{1/2}\log y}.$$

For a given p, the pointwise bound $(x/p^2)^{it} \ll 1$ leads to the above integral being bounded by $\ll \log y$. This is good enough for the primes $p \in [y^{1/2}\log y, y]$, since

$$\sum_{y^{1/2}\log y \leqslant p \leqslant y} p^{-2\beta}\log y \asymp \frac{u\log(u+1)}{y^{1/2}\log y}.$$

For the primes $p \in (y^{1/2}, y^{1/2}\log y)$ we integrate by parts, replacing $(x/p^2)^{it}$ by its antiderivatives.

4.3. Proof of theorem 1.2

Suppose $(\log x)^3 \geqslant y \geqslant (\log x)^{4/3+\varepsilon}$. It follows from proposition 4.1 that $\Psi(x, y) = \Lambda(x, y)G(\beta, y)(1+E)$ holds unconditionally for

$$E \ll_\varepsilon \frac{|G'(\beta,y)|}{G(\beta,y)\log x} + \frac{\max_{|v|\leqslant t_0}|G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)} + \frac{\max_{|v|\leqslant \frac{1}{\log y}}|G'(\beta+iv,y)|}{G(\beta,y)\exp(u^{1/3}/20)} + \frac{1}{y} \tag{4.26}$$

where t_0 is given in the proposition. It remains to bound the quantities appearing in E. From now on we assume RH. Let $A := (\log x)/y^{1/2}$. We will prove the stronger bound

$$\begin{aligned} E \ll_\varepsilon {} & \frac{|\psi(y)-y|+y^{1/2}}{y}\left(1+u\frac{|\psi(y)-y|+y^{\frac{1}{2}}}{y}\right) \\ & + \frac{\max\{A,A^2\}}{u\max\{1,|\log A|\}}\left(1+\frac{\max\{A,A^2\}}{\max\{1,|\log A|\}}\right), \end{aligned} \tag{4.27}$$

which implies the theorem using $\psi(y)-y \ll y^{1/2}\log^2 y$. Recall we can always simplify $y^{-\beta}$ using (2.1) as $\asymp_\varepsilon (\log x)/y$. In particular, $y^{1/2-\beta} \asymp_\varepsilon A$. Recall $G = G_1G_2$. Lemma 2.10 and equation (2.11) tell us that

$$(\log G_2)^{(i)}(\beta+it,y) \ll (\log y)^i \frac{\max\{A,A^2\}}{\max\{1,|\log A|\}} \tag{4.28}$$

for $i = 0, 1, 2$ and $t \in \mathbb{R}$. Corollary 2.9 says that under RH

$$(\log G_1)^{(i)}(\beta+it,y) \ll (\log y)^{i-1}\frac{\log x}{y}(|\psi(y)-y|+y^{1/2}) \tag{4.29}$$

for $i = 0, 1, 2$ and $|t| \leqslant 1$. Applying (4.28) and (4.29) with $i = 1$ shows

$$\frac{|G'(\beta,y)|}{G(\beta,y)}\frac{1}{\log x} \ll \frac{|\psi(y)-y|+y^{1/2}}{y} + \frac{\max\{A,A^2\}}{u\max\{1,|\log A|\}}$$

which treats the first quantity in (4.26). We now consider the third term in (4.26). Observe

$$\begin{aligned} \frac{\max_{|v|\leqslant 1/\log y}|G'(\beta+iv,y)|}{G(\beta,y)\exp(u^{1/3}/20)} \leqslant {} & \frac{\max_{|v|\leqslant 1/\log y}|G(\beta+iv,y)|}{G(\beta,y)\exp(u^{1/3}/20)} \\ & \cdot \max_{|v|\leqslant 1}|(\log G)'(\beta+iv,y)|. \end{aligned} \tag{4.30}$$

From (4.28) and (4.29) we have

$$\max_{|v|\leqslant 1}|(\log G)'(\beta+iv,y)| \ll (\log x)^4, \tag{4.31}$$

say, and, by (2.11) and (4.29),

$$\frac{\max_{|v|\leqslant 1/\log y} |G(\beta+iv,y)|}{G(\beta,y)} \leqslant \exp(C_\varepsilon (\log y)^2 (\log x)/y^{1/2}), \tag{4.32}$$

so that (4.30) leads to

$$\frac{\max_{|v|\leqslant 1/\log y} |G'(\beta+iv,y)|}{G(\beta,y)\exp(u^{1/3}/20)} \ll_\varepsilon \frac{\exp(C_\varepsilon (\log y)^2 (\log x)/y^{1/2})}{\exp(u^{1/3}/40)} \ll_\varepsilon \frac{1}{y}.$$

It remains to bound the second term in (4.26). Observe

$$\frac{\max_{|v|\leqslant t_0} |G''(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)} \leqslant \frac{\max_{|v|\leqslant t_0} |G(\beta+iv,y)|}{G(\beta,y)(\log x)(\log y)} \cdot \big(\max_{|v|\leqslant 1} |(\log G)''(\beta+iv,y)| + \max_{|v|\leqslant 1} |(\log G)'(\beta+iv,y)|^2\big). \tag{4.33}$$

By (2.11) we can bound the fraction in the right-hand side of (4.33) by $O_\varepsilon(1)$:

$$\frac{\max_{|v|\leqslant t_0} |G(\beta+iv,y)|}{G(\beta,y)} \leqslant \frac{\max_{|v|\leqslant t_0} |G_1(\beta+iv,y)|}{G_1(\beta,y)} \leqslant \exp\Big(\int_{-t_0}^{t_0} |G_1'/G_1|(\beta+iv,y)\,\mathrm{d}v\Big) \leqslant \exp(C_\varepsilon t_0 (\log y)^2 (\log x)/y^{1/2}) \ll_\varepsilon 1.$$

The derivatives of $\log G$ in the right-hand side of (4.33) are handled by (4.28) and (4.29), giving

$$\begin{aligned}\max_{|v|\leqslant 1} &|(\log G)''(\beta+iv,y)| + \max_{|v|\leqslant 1} |(\log G)'(\beta+iv,y)|^2 \\ &\ll \frac{(\log y)(\log x)}{y}(|\psi(y)-y| + y^{1/2}|) + \frac{(\log x)^2}{y^2}(|\psi(y)-y|+y^{1/2})^2 \\ &+ (\log y)^2 \left(\frac{\max\{A,A^2\}}{\max\{1,|\log A|\}} + \frac{\max\{A,A^2\}^2}{\max\{1,|\log A|\}^2}\right).\end{aligned}$$

Dividing this by $(\log x)(\log y)$ gives a bound for the second term in (4.26).

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Appendix A. Review of $\Lambda(x, y)$

Appendix A.1. λ_y and its Laplace transform

Saias [**17**, Lem. 4(iii)] proved that $\lambda_y(v) \ll \rho(v)v^3 + e^{2v}y^{-v}$ holds for $y \geqslant 2$, $v \geqslant 1$. The following is a weaker version of his result which suffices for us.

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LEMMA A.1 Saias. *If* $u \geqslant \max\{C,\, y+1\}$ *we have* $\lambda_y(u) \ll (C/y)^u$.

Proof. The condition $u \geqslant \max\{C,\, y+1\}$ ensures $e^{\xi(u-1)} \geqslant y$:

$$e^{\xi(u-1)} \geqslant (u-1)\xi(u-1) \geqslant y\xi(u-1) \geqslant y.$$

Integrating the definition of λ_y by parts gives

$$\lambda_y(u) = \rho(u) + \int_0^{u-1} (-\rho'(u-v))\{y^v\}y^{-v}\,\mathrm{d}v + O(y^{-u}). \tag{A.1}$$

By (A.1) and the definition of ρ we have

$$\begin{aligned}\frac{\lambda_y(u)}{\rho(u)} &= 1 - \int_0^{u-1} \frac{\rho'(u-v)}{\rho(u)}\frac{\{y^v\}}{y^v}\,\mathrm{d}v + O(y^{-u})\\ &= \int_0^{u-1} \frac{\rho(u-v-1)}{(u-v)\rho(u)}\frac{\{y^v\}}{y^v}\,\mathrm{d}v + O(1).\end{aligned} \tag{A.2}$$

One has $\rho(u-v) \ll \rho(u)e^{v\xi(u)}$ uniformly for $0 \leqslant v \leqslant u$ [**14**, Cor. 2.4]. Hence the integral on the right-hand side of (A.2) is

$$\ll \frac{\rho(u-1)}{\rho(u)} \int_0^{u-1} \left(\frac{e^{\xi(u-1)}}{y}\right)^v \mathrm{d}v \leqslant \frac{\rho(u-1)}{\rho(u)}(u-1) \ll ue^{\xi(u)}$$

which is $\ll u^2 \log(u+1)$ by lemma 2.1. Hence

$$\begin{aligned}\lambda_y(u) &\ll \rho(u)u^2\log(u+1) \ll u^{3/2}\log(u+1)\exp(I(\xi(u))e^{-u\xi(u)}\\ &\leqslant u^{3/2}\log(u+1)\exp(I(\xi(u))y^{-u}\end{aligned}$$

using lemma 2.2. We have $I(\xi(u)) \ll u$. As $u^{3/2}\log(u+1)$ may be absorbed in C^u, we are done. □

By lemma A.1, the contribution of $v \geqslant \max\{C,\, y+1\}$ to (1.17) is

$$\int_{\max\{C,y+1\}}^{\infty} |e^{-sv}\lambda_y(v)|\,\mathrm{d}v \ll \int_{\max\{C,y+1\}}^{\infty} (e^{-\Re s}C/y)^v\,\mathrm{d}v < \infty.$$

This establishes

COROLLARY A.2. *Fix* $\varepsilon > 0$. *If* $y \geqslant C_\varepsilon$ *then* $\hat{\lambda}_y$ *converges absolutely for* $\Re s > -(\log y)/(1+\varepsilon)$.

Appendix A.2. Asymptotics of Λ

We define $r\colon [1,\,\infty) \to \mathbb{R}$ by $r(t) := -\rho'(t)/\rho(t) = \rho(t-1)/(t\rho(t))$.

LEMMA A.3. *[**8**, Eq. (6.3)] For* $0 \leqslant v \leqslant u-1$ *and* $u \geqslant 1$ *we have*

$$\rho'(u-v) - \rho'(u)e^{vr(u)} \ll \frac{\rho(u)ve^{vr(u)}}{u}(1 + v\log(u+1)).$$

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LEMMA A.4. *[4, Lem. 3.7] For $u \geqslant 1$ we have $r(u) = \xi(u) + O(1/u)$.*

PROPOSITION A.5. *Fix $\varepsilon > 0$. Suppose $x \geqslant C_\varepsilon$. For $x \geqslant y \geqslant (\log x)^{1+\varepsilon}$,*

$$\Lambda(x,y) = x\rho(u)K\left(-\frac{r(u)}{\log y}\right)\left(1 + O_\varepsilon\left(\frac{1}{(\log x)(\log y)} + \frac{y}{x\log x}\right)\right).$$

Equation (1.5) follows from proposition A.5 using lemma A.4. Proposition A.5, in slightly weaker form, is implicit in [**5**, pp. 176–177], and the proof given below follows these pages.

Proof. For $u = 1$ the claim is trivial since $\Lambda(x, x) = \lfloor x \rfloor$ [**3**, Eq. (3.2)], so we assume $u > 1$. Recall the integral representation $\zeta(s) = s/(s-1) - s\int_1^\infty \{t\}\,\mathrm{d}t/t^{1+s}$ for $\Re s > 0$ [**15**, Eq. (1.24)]. We apply it with $s = 1 - r(u)/\log y$ and perform the change of variable $t = y^v$ to obtain

$$K(-r(u)/\log y) = 1 + r(u)\int_0^\infty e^{r(u)v}\{y^v\}y^{-v}\,\mathrm{d}v. \tag{A.3}$$

From (A.3) and (A.1) we deduce

$$x\rho(u)K(-r(u)/\log y) - \Lambda(x,y) = x\int_0^\infty (\rho'(u-v) - \rho'(u)e^{r(u)v})\{y^v\}y^{-v}\,\mathrm{d}v + O(1). \tag{A.4}$$

It remains to show that the right-hand side of (A.4) is

$$\ll_\varepsilon x\rho(u)\left(\frac{1}{(\log x)(\log y)} + \frac{y}{x\log x}\right).$$

It is convenient to set

$$a := \log\left(\frac{y}{e^{r(u)}}\right) = (\log y) - r(u) \geqslant \frac{\varepsilon}{2}\log y, \tag{A.5}$$

where the inequality is due to lemmas A.4 and 2.1 and our assumptions on x and y. By lemma A.3, the contribution of $0 \leqslant v \leqslant u-1$ to the right-hand side of (A.4) is

$$\begin{aligned} &\ll \frac{x\rho(u)}{u}\int_0^{u-1}\left(\frac{e^{r(u)}}{y}\right)^v v(1 + v\log(u+1))\,\mathrm{d}v \\ &= \frac{x\rho(u)}{u}\left(-e^{-av}\left(\frac{\log(u+1)}{a}v^2 + \frac{2\log(u+1)+a}{a^2}v + \frac{2\log(u+1)+a}{a^3}\right)\right)\Bigg|_{v=0}^{v=u-1}. \end{aligned}$$

Using $e^{(u-1)a} \gg \max\{(u-1)a, (u-1)^2a^2\}$ and (A.5) we find that the last quantity is $\ll_\varepsilon x\rho(u)/((\log x)(\log y))$ which is acceptable. For $v > u-1$, $\rho'(u-v) = 0$ and that part of the integral (times x) is estimated as

$$\ll x(-\rho'(u))\int_{u-1}^\infty e^{-av}\,\mathrm{d}v = x\rho(u)r(u)\frac{e^{-a(u-1)}}{a} \ll_\varepsilon x\rho(u)\log(u+1)\frac{e^{-a(u-1)}}{\log y}.$$

If $u \geqslant 2$ this is $\ll_\varepsilon x\rho(u)/((\log x)(\log y))$, otherwise this is $\ll x\rho(u)(y/x)/\log x$. Both cases give an acceptable contribution. □

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