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PROJECTIONS IN SPACES OF BIMEASURES

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ABSTRACT. Let X and Y be metrizable compact spaces and μ and ν be nonzero continuous measures on X and Y, respectively. Then there is no bounded operator from the space of bimeasures $BM(X, Y)$ onto the closed subspace of $BM(X, Y)$ generated by $L^1(\mu \times \nu)$; in particular, if X and Y are nondiscrete locally compact groups, then there is no bounded projection from $BM(X, Y)$ onto the closed subspace of $BM(X, Y)$ generated by $L^1(X \times Y)$.

0. **Introduction and Statement of Results.** Let X, Y and Z be locally compact Hausdorff spaces. The space of bounded, regular Borel measures on X is denoted by $M(X)$. The *tensor algebras* $V_0(X, Y)$ and $V_0(X, Y, Z)$ are the respective closures,

$$C_0(X) \hat{\otimes} C_0(Y) \quad \text{and} \quad C_0(X) \hat{\otimes} C_0(Y) \hat{\otimes} C_0(Z),$$

in the greatest cross-norm (projective norm), of the tensor products of the indicated C_0-spaces. The space $BM(X, Y)$ of *bimeasures* on $X \times Y$ constitutes the dual space of $V_0(X, Y)$; the dual space of $V_0(X\ Y, Z)$ will be denoted by $BM(X, Y, Z)$ and its elements will be called *trimeasures*. Given a measure ω, we consistently identify $L^1(\omega)$ with the space of measures that are absolutely continuous with respect to ω. We denote Haar measure on the locally compact group G by m_G.

Let $\mathscr{L}^\infty(X)$, $\mathscr{L}^\infty(Y)$, and $\mathscr{L}^\infty(Z)$ denote the Banach spaces of bounded, Borel-measurable functions on X, Y, and Z, respectively. Recall that there is a canonical extension of each bimeasure on $X \times Y$ to an element of $(\mathscr{L}^\infty(X) \hat{\otimes} \mathscr{L}^\infty(Y))^*$. The extension is implemented as follows. For $u \in BM(X, Y)$, let $S_u: C_0(X) \to C_0(Y)^*$ be the operator given by

$$\langle g, S_u(f) \rangle = u(f \otimes g), \qquad f \in C_0(X), g \in C_0(Y).$$

Thus $S_u^{**}: C_0(X)^{**} \to C_0(Y)^{***}$. For $\Phi \in C_0(X)^{**}$ and $\Psi \in C_0(Y)^{**}$, set

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$$u^{**}(\Phi \otimes \Psi) = \langle \Psi, T_u^{**}(\Phi) \rangle.$$

Then $\|u^{**}\| = \|u\|$. Since we may consider $\mathscr{L}^\infty(X) \subset C_0(X)^{**}$ and $\mathscr{L}^\infty(Y) \subset C_0(Y)^{**}$, restricting u^{**} to the respective $\mathscr{L}^\infty$-spaces and extending to the associated projective tensor products provides the desired extension, which we also denote by u. It is easy to check that if u is the bimeasure represented by integration with respect to a measure ω on $X \times Y$, then the extension of u to bounded, Borel-measurable functions is still represented by integration with respect to ω. Now if X and Y are locally compact abelian (LCA) groups with character groups $\hat{X}$ and $\hat{Y}$, respectively, then for $u \in BM(X, Y)$ the Fourier transform of u is defined via the canonical extension by

$$\hat{u}(\chi, \eta) = u(\chi \otimes \eta), \qquad \chi \in \hat{X}, \eta \in \hat{Y}.$$

For background on tensor algebras, see [3, Chap. 11]. For information about bimeasures and trimeasures on locally compact groups, see [2] and [4].

THEOREM 1. *Let X and Y be locally compact spaces, and let μ and ν be nonzero continuous measures on X and Y, respectively. Let L be the closure in $BM(X, Y)$ of $L^1(\mu \times \nu)$. Then there is no bounded operator from $BM(X, Y)$ onto L.*

COROLLARY 2. *Let G and H be nondiscrete locally compact groups. Then the closure of $L^1(m_G \times m_H)$ in $BM(G, H)$ is not a direct summand of $BM(G, H)$.*

DEFINITION 3. *We shall now define the canonical extension for elements of $BM(X, Y, Z)$. For $u \in BM(X, Y, Z)$, let $T_u : C_0(X) \to BM(Y, Z)$ be defined by*

$$\langle g \otimes h, T_u(f) \rangle = u(f \otimes g \otimes h)$$

for $f \in C_0(X)$, $g \in C_0(Y)$, and $h \in C_0(Z)$. Then

$$T_u^{**} : C_0(X)^{**} \to BM(Y, Z)^{**}.$$

For $\Phi \in C_0(X)^{**}$ and $\Psi \in V_0(Y, Z)^{**} = BM(Y, Z)^*$, set

$$u^{**}(\Phi, \Psi) = \langle \Psi, T_u^{**}(\Phi) \rangle,$$

so that $\|u^{**}\| = \|u\|$. Now, each element of $C_0(Y)^{**} \hat{\otimes} C_0(Z)^{**}$ induces an element of $BM(Y, Z)^*$, as described earlier. Thus we have defined u^{**} on $C_0(X)^{**} \hat{\otimes} C_0(Y)^{**} \hat{\otimes} C_0(Z)^{**}$. We now restrict to the appropriate $\mathscr{L}^\infty$-spaces and call our extension the canonical extension of u to $\mathscr{L}^\infty(X) \hat{\otimes} \mathscr{L}^\infty(Y) \hat{\otimes} \mathscr{L}^\infty(Z)$ and continue to refer to this extension as u. As above, if X, Y, and Z are LCA *groups and $u \in BM(X, Y, Z)$, we use the canonical extension to define the Fourier transform by*

$$\hat{u}(\chi, \eta, \zeta) = u(\chi \otimes \eta \otimes \zeta), \quad \chi \in \hat{X}, \eta \in \hat{Y}, \zeta \in \hat{Z}.$$

Again it is easy to see that the extension of the trimeasure represented by integration with respect to a measure on $X \times Y \times Z$ is still represented as such.

COROLLARY 4. *Let G and H be infinite, compact, abelian groups. Let K be a noncompact, abelian group. Then there is an element of $BM(G, H, K)$ whose Fourier transform is not uniformly continuous.*

THEOREM 5. *Let X and Y be locally compact spaces that support continuous measures, and let Z be a locally compact space that is not countably compact. Then the compactly supported elements of $BM(X, Y, Z)$ are not norm dense.*

THEOREM 6. *Let G, H, and K be nondiscrete locally compact abelian groups. There exist elements $u, v \in BM(G, H, K)$ such that $\hat{u}\hat{v}$ is not the Fourier transform of an element of $BM(G, H, K)$. In fact, convolution on $M(G \times H \times K)$ is not continuous in the trimeasure norm.*

Theorem 1 is proved in Section 1. Corollary 2 is immediate. The remaining results are proved in Section 2. Comments and credits end this section.

In [4] the authors showed that if G and H are infinite, locally compact, abelian groups, then the closure of $L^1(m_G \times m_H)$ in $BM(G, H)$ plays a role in $BM(G, H)$ analogous to that played by $L^1(m_G)$ in the measure algebra $M(G)$; for example the bimeasures for which translation is a norm-continuous function on $G \times H$ are precisely those in that closure. Analogous results for nonabelian groups were obtained in [2], which also includes a proof that the continuous bimeasures form an ideal under convolution.

A proof of Corollary 2 for the case $G = H$ and G abelian was given in [4]; that proof used the Fourier transform and does not appear to be directly adaptable to the nonabelian case. It also seemed that Haar measure on $G \times G$ played a special role. The harmonic analysis is absent from the present proof; only an l^2 argument remains.

That the closure of $L^1(\mu \times \nu)$ contains c_0 as a direct summand is due to Bessaga and Pełczynski [1]. Our proof of Theorem 1 contains a version of their argument. We are grateful to Professor Pełczynski for bringing [1] to our attention. Theorem 5 is essentially proved in the proof of [7, Theorem 2]; the assertion of Theorem 1 is that $BM(X, Y)$ does not satisfy the condition $\mathscr{P}$ of [7], the hypothesis of Saeki's result.

1. **Proof of Theorem 1.** We may assume that μ and ν are probability measures. A standard construction, using the continuity of the probability measure μ, shows that there is a sequence $\{f_n\}$ of Borel functions on X such that for all n, $f_n^2 = 1$ everywhere and such that $\{f_n\}$ is an orthonormal sequence in $L^2(\mu)$. (That is simply an abstract version of the construction of the Rademacher functions.) There is a similar sequence $\{g_n\}$ of functions on Y. For each $u \in BM(X, Y)$ and each pair m, n of integers, we define $u_{m,n}$ by $u_{m,n} = \langle f_m \otimes g_n, u \rangle$. We claim that the mapping

$$f \otimes g \mapsto \langle f \otimes g, Pu \rangle = \sum u_{m,m} \int f_m f d\mu \int g_m g d\nu$$

defines an element of $BM(X, Y)$. (The definition is justified via the canonical extension of each bimeasure to a bilinear functional on the bounded Borel functions, as indicated above.) Indeed, by the Cauchy-Schwarz inequality,

$$\left|\sum u_{m,m} \int f_m f d\mu \int g_m g d\nu\right| \leqq \sup|u_{m,m}| \, \|f\|_2\|g\|_2$$
$$\leqq \sup|u_{m,m}| \, \|f\|_\infty\|g\|_\infty$$
$$\leqq \|u\|_{BM}\|f\|_\infty\|g\|_\infty.$$

It is obvious that $P(Pu) = Pu$, so $u \mapsto Pu$ is a projection from bimeasures to bimeasures. The first two inequalities above show that the sequence $\{u_{m,m}\}$ may be any bounded sequence: that is, the image of P may be identified isometrically with l^∞. Now, if $u \in L^2(\mu) \otimes L^2(\nu)$, then clearly, $\{u_{m,m}\} \in c_0$. Since $L^2(\mu) \otimes L^2(\nu)$ is dense in $L^1(\mu \times \nu)$, every element of $L^1(\mu \times \nu)$ gives rise to a sequence in c_0. In the subspace $P(BM(X, Y))$ the norm corresponds to the supremum norm of the coefficients $u_{m,m}$, so the closure of $L^1(\mu \times \nu) \cap P(BM(X, Y))$ corresponds to all of c_0.

Let us suppose that there were a bounded operator Q from $BM(X, Y)$ onto L. Then PQP is easily seen to be a bounded operator from the image $P(BM(X, Y))$ onto $P(L)$. Since those last two spaces are isomorphic with l^∞ and c_0, respectively, we would have a bounded operator from l^∞ onto c_0. But l^∞ does not have c_0 as a quotient space, since every separable quotient space of l^∞ is reflexive [6, p. 42]. That ends the proof of Theorem 1.

2. **Proofs of results 4-6.**

PROOF OF COROLLARY 4. We use notation similar to that of the proof of Theorem 1, with G, H, K, m_G, and m_H in place of X, Y, Z, μ, and ν, respectively. Take $\{f_m\}$ to be a sequence of distinct characters on G and $\{g_m\}$ to be such a sequence on H. Since K is not compact, there exists an infinite sequence $\{z_j\} \subset K$ with no accumulation points. The mapping that assigns to each triple $f \in C(G)$, $g \in C(H)$, and $h \in C_0(K)$ the number

$$\langle f \otimes g \otimes h, \nu\rangle = \sum h(z_m) \int f_m f dm_G \int g_m g dm_H$$

defines an element of $BM(G, H, K)$, since

$$|\langle f \otimes g \otimes h, \nu\rangle| \leqq \sum \left|h(z_m) \int f_m f dm_G \int g_m g dm_H\right|$$
$$\leqq \sup_m |h(z_m)| \, \|f\|_2\|g\|_2$$
$$\leqq \|f\|_\infty\|g\|_\infty\|h\|_\infty.$$

The Fourier transform of ν equals $\langle z_m, h\rangle$ on the coset $(f_m, g_m) \times K$. Since $\{z_m\}$ is not relatively compact, the functions $h \mapsto \langle z_m, h\rangle$ are not uniformly continuous. That ends the proof of Corollary 4.

PROOF OF THEOREM 5. Let f_m and g_m be as in the proof of Theorem 1. Since Z is not countably compact, there exists an infinite sequence $\{z_j\} \subset Z$ with no accumulation points. That the mapping v assigning to each triple $f \in C_0(X)$, $g \in C_0(Y)$, and $h \in C_0(Z)$ the number determined by

$$\langle f \otimes g \otimes h, v \rangle = \sum h(z_m) \int f_m f d\mu \int g_m g d\nu$$

defines an element of $BM(X, Y, Z)$ follows exactly as in the proof of Corollary 4. Let w be an element of $BM(X, Y, Z)$ with compact support. There is an m and a neighborhood U of z_m such that $(x, y, z) \notin \operatorname{supp} w$ for all $z \in U$, and $z_n \notin U$ for all $n \neq m$. Let $h \in C_0(Z)$ be such that $h(z_m) = 1 = \|h\|_\infty$ and $h(z) = 0$ for all $z \notin U$. Choose $f \in C_0(X)$ such that $-1 \leqq f \leqq 1$ and $\int f_m f d\mu > 1/2$, and similarly choose $g \in C_0(Y)$. Then since $\langle f \otimes g \otimes h, w \rangle = 0$,

$$\begin{aligned} \|v - w\| &\geqq \langle f \otimes g \otimes h, v \rangle \\ &= h(z_m) \int f_m f d\mu \int g_m g d\nu \\ &\geqq 1/4. \end{aligned}$$

Theorem 5 now follows.

REMARK 7. The requirement that Z not be countably compact is needed in the assertion of Theorem 5 because of the existence of spaces that are countably compact but not compact. (See, for example, [5], pp. 162-3].) We do not know whether the conclusion of Theorem 5 holds when such spaces are involved.

PROOF OF THEOREM 6. We begin with a special case of the theorem. After establishing the special case, we will show how tensor algebra methods (based on independent sets) give the general result.

Let $\mathbf{T}$ denote the circle group. We shall show that there exist bounded sequences of finitely supported trimeasures

$$\{u_m\}, \{v_m\} \in BM(\mathbf{T}^2, \mathbf{T}^2, \mathbf{T}^2)$$

and a constant $c > 0$ such that $\|u_m * v_m\| > c \log m$. That will prove Theorem 6 in the case $G = H = K = \mathbf{T}^2$. Fix $m \geqq 1$. We shall denote the character $\exp(2\pi i k x)$ by $\chi_k(x)$. Let

$$u_m = \sum_{k=1}^{m} (\chi_k m_{\mathbf{T}} \times \delta_0) \times (\chi_k m_{\mathbf{T}} \times \delta_0) \times (\delta_{1/k} \times \delta_0)$$

and

$$v_m = \sum_{k=1}^{m} (\delta_0 \times \chi_k m_{\mathbf{T}}) \times (\delta_0 \times \delta_{1/k}) \times (\delta_0 \times \chi_k m_{\mathbf{T}}).$$

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Then u_m and v_m both have norm one by a simple variant of the l^2 estimate used in the proof of Corollary 3. For simplicity of notation, we drop the subscripts "m" on u_m and v_m.

The (j, k)-term of $u * v$ is concentrated on

$$\mathbf{T}^2 \times (\mathbf{T} \times \{1/k\}) \times (\{1/j\} \times \mathbf{T}).$$

By repeated application of [3, 11.1.4], there exists a function $f \in V(\mathbf{T}, \mathbf{T})$ such that

$$f(1/j, 1/j) = 1 \quad \text{for } 1 \leqq j \leqq m,$$
$$f(1/j, 1/k) = 0 \quad \text{for } 1 \leqq j \neq k \leqq m,$$

and $\|f\| \leqq 2$. We can extend f to a function g on $\mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{T}^2$ by the formula $g(x_1, x_2, y_1, y_2, z_1, z_2) = f(y_2, z_1)$.

It is obvious that $g \in V(\mathbf{T}^2, \mathbf{T}^2, \mathbf{T}^2)$ and $\|g\| \leqq 2$. Then $\|g(u * v\| \leqq 2\|u * v\|$, and

$$g(u * v) = \sum_{k=1}^{m} (\chi_k m_{\mathbf{T}} \times \chi_k m_{\mathbf{T}}) \times (\chi_k m_{\mathbf{T}} \times \delta_{1/k}) \times (\delta_{1/k} \times \chi_k m_{\mathbf{T}}).$$

The preceding sum consists of terms whose supports have pairwise disjoint projections on two different coordinates. For each k, let p_k and q_k be continuous functions on $\mathbf{T}^2$ having pairwise disjoint supports, each of norm one and such that

$$\int p_k d(\chi_k m_{\mathbf{T}} \times \delta_{1/k}) = 1 \quad \text{and} \quad \int q_k d(\delta_{1/k} \times \chi_k m_{\mathbf{T}}) = 1.$$

Because of the condition on the supports of p_k and q_k, [3, 11.1.4] applies, so the sum $r = \sum_1^m (p_k \otimes q_k)$ has norm one. Define a measure μ on $\mathbf{T}^2$ by

$$\int h d\mu = \langle h \otimes r, g(u * v) \rangle.$$

Then $\int h d\mu = \sum_1^m \hat{h}(k, k)$, so that $\|\mu\| \geqq c \log m$, for some $c \neq 0$. It follows that

$$\|u * v\| \geqq (1/2) \|g(u * v)\| \geqq (c/2) \log m.$$

Theorem 6 now follows for the special case under consideration.

The general case is obtained as follows. Let u_r and v_s be finitely supported approximants to u and v with $\|u_r\| = \|v_s\| = 1$. We may assume that u_r is supported on $U_1 \times U_2 \times U_3$ and v_s is supported on $V_1 \times V_2 \times V_3$, where $U_j \cup V_j$ is a disjoint union whose result is an independent set, for $j = 1, 2, 3$. Such a choice of u_r and v_s is possible because the finitely supported trimeasures of (trimeasure) norm one are weak-$*$ dense in the unit ball of $BM(G, H, K)$.

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Because of the independence of the sets $U_j \cup V_j$, the mass distribution of $u_r * v_s$ is independent of the underlying group structure. We claim further that u_r and v_s can be found so that the trimeasure norm of $u_r * v_s$ will be approximately $\|u * v\|$. Indeed, because convolution is weak-* continuous in each variable separately, v_s can be chosen so that $\|u * v_s\|$ is large. Now u_r is chosen so that $\|u_r * v_s\|$ is large. All that occurs, we stress, independently of the underlying groups' structure.

We now map U_j and V_j one-to-one onto sets in any other LCA groups, $U_j', V_j' \subset G_j'$, such that $U_j' \cup V_j'$ is a disjoint union whose result is independent, for $j = 1, 2, 3$. Then u_r, v_s, and $u_r * v_s$ are mapped onto elements u_r', v_s', and $u_r' * v_s'$ of $BM(G_1', G_2', G_3')$, with no change in norms. It follows that the norm of the convolution of two finitely supported trimeasures in $BM(G_1', G_2', G_3')$ is not bounded by a (fixed) constant times the product of the norms of the factors. Therefore, $BM(G_1', G_2', G_3')$ is not closed under convolution.

We leave the remaining details to the reader. That ends the proof of Theorem 6.

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