ON INVARIANT MEANS WHICH ARE NOT INVERSE INVARIANT

M. RAJAGOPALAN AND K. G. WITZ

In **(1)** R. G. Douglas says: "For a finite abelian group there exists a unique invariant mean which must be inversion invariant. For an infinite torsion abelian group it is not clear what the situation is." It is not hard to see that if every element of an abelian group G is of order 2, then every invariant mean on G is also inversion invariant **(**see **1**, remark 4**)**. In this note we prove the following theorem (Theorem 1 below): *An abelian torsion group G has an invariant mean which is not inverse invariant if, and only if, $2G$ is infinite.* This result, together with the theorems of Douglas, answers completely the question of the existence (on an arbitrary abelian group) of invariant means which are not inverse invariant.

Definitions and notations. Suppose G is an abelian group written additively. Then G is a *torsion* group if, and only if, every element of G is of finite order. If $E \subset G$ and $y \in G$ we write $-E = \{-x: x \in G\}$ and $y + E = \{y + x: x \in E\}$, and if n is a positive integer, then $nE = \{nx: x \in E\}$. Let $m(G)$ be the Banach space of all bounded real-valued functions in G with the sup norm. A *mean* on $m(G)$ is a bounded linear functional μ on $m(G)$ such that $\mu(1) = ||\mu|| = 1$, where 1 is the constant 1 function on G. A mean μ is *invariant* if $\mu(f) = \mu(f_x)$ $(f \in m(G))$ where f_x is the translate of f by

$$x: f_x(y) = f(x + y) \qquad (y \in G).$$

Finally a mean μ is *inverse invariant* if, and only if, $\mu(f) = \mu(f^*)$ $(f \in m(G))$ where $f^*(x) = f(-x)$ $(x \in G)$.

LEMMA. *Let G be a countable abelian torsion group and suppose $2G$ is an infinite set. Then there is a set $E \subset G$ such that*

(i) $E \cap -E = \emptyset$ *and*

(ii) *for every finite set $S \subset G$ there is an element $s \in G$ such that $s + S \subset E$.*

Proof. Let $y_0, y_1, \ldots$ be an enumeration of all elements of G with $y_0 = 0$. Put $H_0 = \{y_0\}$. Let n_1 be the first positive integer k such that $2y_k \neq 0$ and let H_1 be the group generated by $y_0, y_1, \ldots, y_{n_1}$. Since G is a torsion group, H_1 is finite. Let n_2 be the first positive integer k such that $k > n_1$ and $2y_k \notin H_1$; such a k exists because H_1 is finite and $2G$ is infinite. Let H_2 be the group generated by $y_0, y_1, \ldots, y_{n_2}$. In this way we get an infinite sequence

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$H_0 \subset H_1 \subset H_2 \ldots$ of finite subgroups of G whose union is G, and an infinite sequence $y_{n1}, y_{n2}, \ldots$ of elements in G such that

$$(1) \qquad 2y_{nk} \notin H_{k-1} \quad \text{and} \quad y_{nk} \in H_k \qquad (k = 1, 2, \ldots).$$

Let

$$(2) \qquad E_k = y_{nk} + H_{k-1} \qquad (k = 1, 2, \ldots)$$

and

$$(3) \qquad E = \bigcup_{k=1}^{\infty} E_k.$$

Then E is a set which has the desired properties (i) and (ii) in the statement of the lemma.

To see (i) observe first that since H_{k-1} is a group and $2y_{nk} \in H_{k-1}$ we have both $y_{nk} \notin H_{k-1}$ and $-y_{nk} \notin H_{k-1}$ $(k = 1, 2, \ldots)$. Now if $k \neq l$, then $E_k \cap -E_l = \emptyset$ $(k, l = 1, 2, \ldots)$, for if $x \in E_k \cap -E_l$ and, say, $l > k$, then there would exist $x_k \in H_{k-1}$ and $x_l \in H_{l-1}$ such that $x = y_{nk} + x_k = -(y_{nl} + x_l)$ and hence $y_{nl} = -(x_l + x_k) - y_{nk} \in H_{l-1}$, a contradiction. Similarly, $E_k \cap -E_k = \emptyset$ $(k = 1, 2, \ldots)$ because $2y_k \notin H_{k-1}$. Therefore

$$(4) \qquad E \cap -E = \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=1}^{\infty} - E_k\right) = \emptyset.$$

(ii) is trivial: if S is a finite subset of G, then there is a positive integer k, such that $S \subset H_{k-1}$. Then $y_{nk} + S \subset y_{nk} + H_{k-1} = E_k \subset E$, i.e., (ii) holds with $s = y_{nk}$.

THEOREM 1. *An abelian torsion group G has an invariant mean which is not inverse invariant if and only if the set $2G$ is infinite.*

Proof. Suppose $2G$ is infinite and suppose also that G is countable. A basic theorem of Mitchell (**2**, Theorem 8) states that if E is a subset of a left amenable semigroup Σ, then there is a left invariant mean μ on Σ such that $\mu(\chi_E) = 1$ if and only if for each finite subset F of Σ there exists $\sigma \in \Sigma$ such that $F\sigma \subset E$. Here χ_E is the characteristic function of E. Hence letting E be a subset of G that has properties (i) and (ii) of the lemma, there is an invariant mean μ on G such that $\mu(\chi_E) = 1$. Since $\mu(\chi_E^*) = \mu(\chi_{-E}) = 0$, μ is not inverse invariant.

If G is not countable, then there is a homomorphism ϕ of G into some countably infinite group D such that both $\phi(G)$ and $\phi(2G)$ are infinite. The argument for this goes as in (**3**, p. 45): let Λ be a countably infinite subgroup of G, embed Λ in a countable divisible group D, and extend the embedding to a homomorphism ϕ from G into D. Since $\phi(G)$ is a countable torsion group and $2\phi(G) = \phi(2G)$ is infinite, there is an invariant mean μ on $\phi(G)$ which is not inverse invariant. Define a functional γ by $\gamma(f[\phi]) = \mu(f)$ $(f \in m(\phi(G)))$. Then γ can be extended to an invariant mean $\bar{\gamma}$ defined on all of $m(G)$ and $\bar{\gamma}$ is not inverse invariant.

On the other hand, if $2G = \{g_1, \ldots, g_n\}$ and if $f \in m(G)$, then, since $x + 2G = -x - 2G$,

$$\sum_{i=1}^{n} f_{g_i}(x) = \sum_{i=1}^{n} f(g_i + x) = \sum_{i=1}^{n} f(-g_i - x) = \sum f_{g_i}^{*}(x) \qquad (x \in G)$$

and hence for any invariant mean μ

$$\mu(f) = \frac{1}{n}\mu\left(\sum f_{g_i}\right) = \frac{1}{n}\mu\left(\sum f_{g_i}^{*}\right) = \mu(f^{*}).$$

Remark. Douglas showed that in a non-torsion abelian group (i.e., one in which at least one element is of infinite order) there always exists an invariant mean that is not inverse invariant. Since in such a group G the set $2G$ is always infinite, we can summarize the whole situation with

THEOREM 2. *An abelian group G has an invariant mean which is not inverse invariant if and only if the set $2G$ is infinite.*

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University of Illinois,
Urbana, Illinois

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