J. Austral. Math. Soc. (Series A) 47 (1989), 307–312

ON PROPERLY SEPARABLE QUOTIENTS OF STRICT (LF) SPACES

W. J. ROBERTSON

(Received 28 March 1988)

Communicated by S. Yamamuro

Abstract

All known Banach spaces have an infinite-dimensional separable quotient and so do all non-normable Fréchet spaces, although the general question for Banach spaces is still open. A properly separable topological vector space is defined, in such a way that separable and properly separable are equivalent for an infinite-dimensional complete metrisable space. The main result of this paper is that the strict inductive limit of a sequence of non-normable Fréchet spaces has a properly separable quotient.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 46 A 05, 46 A 07, 46 A 12.

1. Introduction

The problem of the existence of an infinite-dimensional separable quotient of a Banach space has been much studied: see, for example, H. E. Lacey [2] and S. A. Saxon and A. Wilansky [5]. A result of G. Köthe [1, page 431] includes a proof that any non-normable Fréchet space has a quotient isomorphic to ω, the product of a countable number of copies of the scalar field, and so separable.

Clearly an infinite-dimensional topological vector space is separable if and only if it has a dense vector subspace of countable dimension. Here, dimension means the cardinal of a Hamel basis; for convenience, countable is taken to mean infinite and countable.

We call a topological vector space *properly separable* if and only if it has a proper dense vector subspace of countable dimension. Since a complete metrisable topological vector space cannot have countable dimension, it is properly separable if and only if it is separable. In fact this is also true, for the same reason, of a metrisable barrelled locally convex space. For the only barrelled space of countable dimension is the space φ, the topological direct sum of a countable number of copies of the scalar field, which is not metrisable.

Properly separable quotients of barrelled spaces are discussed in [3] by W. J. Robertson and P. P. Narayanaswami, in connection with the main result of [5].

The notation and terminology are, for the most part, those of [4]. We take *quotient* to imply quotient by a closed vector subspace.

2. The main theorem

LEMMA 1. *Suppose that E is a metrisable locally convex space, that F is a closed vector subspace of E, and that $\{V_n: n = 0, 1, 2, \dots\}$ is a decreasing base of absolutely convex neighbourhoods of o in F. Then there is a decreasing base $\{U_n: n = 0, 1, 2, \dots\}$ of absolutely convex neighbourhoods of o in E with $U_n \cap F = V_n$ for all n.*

PROOF. If $\{W_n: n = 0, 1, 2, \dots\}$ is any base of absolutely convex neighbourhoods of o in E and if, for each n, X_n is an absolutely convex neighbourhood of o in E with $X_n \cap F = V_n$, then taking U_n to be the absolutely convex envelope of $V_n \cup (X_n \cap W_n)$ will suffice.

COROLLARY. *With the notation of the lemma, suppose that, for each $n \geq 1$, f_n is a continuous linear form on F with $f_n \in V_n^0$ but $f_n \notin \operatorname{span} V_{n-1}^0$. Then there is an extension $\bar{f}_n$ of f_n to a continuous linear form on E, with $\bar{f}_n \in U_n^0$ but $\bar{f}_n \notin \operatorname{span} U_{n-1}^0$.*

PROOF. If p_n is the gauge of U_n, then $|f_n(x)| \leq p_n(x)$ on F. The Hahn-Banach theorem then ensures the existence of an extension $\bar{f}_n$ with the required properties.

The next lemma is more or less contained in the result of Köthe mentioned before. We give a brief proof of it in the following formulation, convenient for our purpose.

Lemma 2. *Suppose that E is a non-normable Fréchet space. Then*

(a) *there is a base $\{U_n: n = 0, 1, 2, \dots\}$ of absolutely convex neighbourhoods of o with $U_n^0 \not\subseteq \operatorname{span} U_{n-1}^0$ for $n = 1, 2, \dots$;*

(b) *if $(f_n)_1^\infty$ is a sequence of continuous linear forms with $f_n \in U_n^0 \backslash (\operatorname{span} U_{n-1}^0)$, then $\{f_n: n = 1, 2, \dots\}$ is linearly independent;*

(c) *if $G = \operatorname{span}\{f_n: n = 1, 2, \dots\}$, then G is $\sigma(E', E)$-closed.*

Proof. (a) The fact that E is metrisable but non-normable gives a decreasing base $\{U_n: n = 0, 1, 2, \dots\}$ such that, for each $n \geq 1$, $U_n^0 \not\subseteq \lambda_n U_{n-1}^0$ for any λ_n. If for some n, $\operatorname{span} U_n^0 = \operatorname{span} U_{n-1}^0 = L$, then with unit ball U_n^0, L is a Banach space. Similarly L is a Banach space with unit ball U_{n-1}^0. But these norm topologies cannot coincide since $U_n^0 \not\subseteq \lambda_n U_{n-1}^0$ for any λ_n; hence $U_n^0 \not\subseteq \operatorname{span} U_{n-1}^0$.

(b) This is immediate from (a).

(c) It is easy to see that $G \cap U_n^0 \subseteq \operatorname{span}\{f_1, \dots, f_n\}$ for each $n \geq 1$. Hence $G \cap U_n^0 = \operatorname{span}\{f_1, \dots, f_n\} \cap U_n^0$, which is $\sigma(E', E)$-closed. Since E is fully complete, being a Fréchet space, G is $\sigma(E', E)$-closed.

Theorem. *Suppose that E is the strict inductive limit of a strictly increasing sequence (E_m) of Fréchet spaces, and that some E_m is non-normable. Then E has a properly separable quotient.*

Proof. We may suppose that E_1 is non-normable; then so are all E_m. First, we construct a sequence (f_n) of continuous linear forms on E.

By Lemma 2(a), applied to E_1, there is a base $\{U_{1n}: n = 0, 1, \dots\}$ of absolutely convex neighbourhoods of o in E, such that $U_{1n}^0 \not\subseteq \operatorname{span} U_{1,n-1}^0$, for $n = 1, 2, \dots$. Hence for each $n \geq 1$ there is some f_{1n} continuous on E_1, with $f_{1n} \in U_{1n}^0 \backslash (\operatorname{span} U_{1,n-1}^0)$. Now apply Lemma 1 with $F = E_1$, $E = E_2$ and $V_n = U_{1n}$ for each n. Then there is a base $\{U_{2n}: n = 0, 1, \dots\}$ of absolutely convex neighbourhoods of o in E_2 with $U_{2n} \cap E_1 = U_{1n}$ for all n. By the Corollary to Lemma 1, there is an extension f_{2n} of f_{1n} to E_2 with $f_{2n} \in U_{2n}^0 \backslash (\operatorname{span} U_{2,n-1}^0)$ for all $n \geq 1$.

Continue up the (E_m), by induction, to get, for each m, a base $\{U_{mn}: n = 0, 1, \dots\}$ of absolutely convex neighbourhoods of o in E_m, and a sequence $\{f_{mn}: n = 1, 2, \dots\}$ of continuous linear forms on E_m with $f_{mn} \in U_{mn}^0 \backslash (\operatorname{span} U_{m,n-1}^0)$. By Lemma 2 (b), $\{f_{mn}: n = 1, 2, \dots\}$ is linearly independent.

On E, for each $n \geq 1$, define f_n by $f_n(x) = f_{mn}(x)$ for all x in E_m. Since $E = \bigcup_{m=1}^\infty E_m$ and $f_{mn} = f_{rn}$ on E_m if $r \geq m$, f_n is properly defined, and the

https://doi.org/10.1017/S1446788700031736 Published online by Cambridge University Press

restriction of f_n to E_m is f_{mn}. Thus f_n is continuous on each E_m and so on E.

Also $\{f_n : n = 1, 2, \ldots\}$ is linearly independent. For if $\sum_1^k \lambda_n f_n = o$ and no f_n for $1 \le n \le k$ is o, then for each n, $1 \le n \le k$, there is some x_n with $f_n(x_n) \neq 0$ and $x_n \in E_{m_n}$, say. Put $m = \max\{m_n : 1 \le n \le k\}$. Then no f_n is identically zero on E_m. But, restricting each f_n to E_m, $\sum_1^k \lambda_n f_{mn} = o$. Since $\{f_{mn} : n = 1, 2, \ldots\}$ is linearly independent, $\lambda_n = 0$ for $1 \le n \le k$.

Now let $G = \operatorname{span}\{f_n : n = 1, 2, \ldots\}$. Take $M = G^0$, the polar of G in E, so that M is a closed vector subspace of E. In the quotient topology on E/M, the dual is the polar M^0 of M in E'. Since E is barrelled, the quotient topology is $\tau(E/M, M^0)$, or $\tau(E/G^0, G^{\infty})$. We show next that $G^{00} = G$, that is, that G is $\sigma(E', E)$-closed.

For each m, write j_m for the injection of E_m into E; then j_m is continuous, j'_m is $\sigma(E', E)$-$\sigma(E'_m, E_m)$ continuous and $j'_m(f)$ is the restriction of f to E_m. Thus $j'_m(f_n) = f_{mn}$ for all $n \ge 1$, so $j'_m(G) = G_m = \operatorname{span}\{f_{mn} : n = 1, 2, \ldots\}$. Hence $G \subseteq j'^{-1}_m(G_m)$, which is closed since j'_m is continuous and G_m is $\sigma(E'_m, E_m)$-closed, by Lemma 2(c). Therefore $G^{00} = \overline{G} \subseteq j'^{-1}_m(G_m)$ and so $j'_m(G^{00}) \subseteq G_m$.

Now let $g \in G^{00}$ ($g \neq o$). There is some m for which g is not identically zero on $j_m(E_m)$; let r be the least such m. Then $j'_r(g) \in G_r$ and so

$$j'_r(g) = \sum_{n=1}^{k} \lambda_n f_{rn} = j'_r\left(\sum_{n=1}^{k} \lambda_n f_n\right) \qquad (\lambda_k \neq 0).$$

Now if $m > r$, $j'_m(g) \in G_m$ and so

$$j'_m(g) = \sum_{n=1}^{p(m)} \lambda_{mn} f_{mn} \qquad (\lambda_{mp(m)} \neq 0).$$

Hence, writing j for the injection of E_r into E_m, so that j' is the restriction of the elements of E'_m to E_r, we obtain

$$j_r = j_m \circ j \quad \text{and} \quad j'_r = j' \circ j'_m,$$

$$j'_r(g) = j'\left(\sum_{n=1}^{p(m)} \lambda_{mn} f_{mn}\right) = \sum_{n=1}^{p(m)} \lambda_{mn} f_{rn}$$

since f_{mn} is an extension of f_{rn} from E_r to E_m. Thus

$$\sum_{n=1}^{k} \lambda_n f_{rn} = \sum_{n=1}^{p(m)} \lambda_{mn} f_{rn}.$$

Now by Lemma 2(c), $\{f_{rn}\colon n = 1,2,\dots\}$ is linearly independent and so $p(m) = k$ and $\lambda_{mn} = \lambda_n$ for all n. Hence $j'_m(g) = \sum_1^k \lambda_n f_{mn} = j'_m(\sum_1^k \lambda_n f_n)$ and $g - \sum_1^k \lambda_n f_n \in j'^{-1}_m(o)$ for all $m \geq r$. Thus $g - \sum_1^k \lambda_n f_n$ is zero on each $j_m(E_m)$ for $m \geq r$, and so is zero on E (and in fact $r = 1$). Hence $g \in G$, and G is closed.

Finally, we show that E/G^0 is properly separable. Since $G^{00} = G$, the quotient topology is $\tau(E/G^0, G)$. The dimension of G is countable and so $\tau(E/G^0, G)$ is metrisable, and equal to $\sigma(E/G^0, G)$. Thus E/G^0 is a dense vector subspace of G^* under $\sigma(G^*, G)$. But since G has countable dimension, $G^* \cong \omega$ and so is separable and metrisable. Hence E/G^0 is metrisable, separable and also barrelled; thus E/G^0 is properly separable (as shown in the remarks following the definition of properly separable).

Corollary. *With the hypotheses of the theorem, E has a metrisable properly separable quotient (from the proof).*

Example. It is shown in [3] that if there exists a Banach space F with no separable quotient, then there also exists a strict inductive limit of Banach spaces with no properly separable quotient (namely $F \times \varphi$, isomorphic to the strict inductive limit of the spaces $F \times \varphi_n$, where φ_m is m-dimensional).

Added in Proof

The following questions are still open:

(1) If E is the strict inductive limit of a sequence of Banach spaces, each with a separable quotient, has E a properly separable quotient?

(2) If E is the strict inductive limit of a sequence of metrisable barrelled spaces, each with a separable quotient, has E a properly separable quotient?

References

[1] G. Köthe, *Topological vector spaces* I (Springer-Verlag, 1969).

[2] H. E. Lacey, 'Separable quotients of Banach spaces', *An. Acad. Brasil. Cienc.* **44** (1972), 185–189.

[3] W. J. Robertson and P. P. Narayanaswami, *On properly separable quotients and barrelled spaces*, (Department of Mathematics, The University of Western Australia, Research Report, March 1988/1).

[4] A. P. Robertson and W. J. Robertson, *Topological vector spaces*, 2nd ed. (Cambridge University Press, 1973).
[5] S. A. Saxon and A. Wilansky, 'The equivalence of some Banach space problems', *Colloq. Math.* **37** (1977), 217–226.

Department of Mathematics
University of Western Australia
Nedlands, W. A. 6009
Australia