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TOPOLOGICAL SEQUENCE ENTROPY AND TOPOLOGICALLY WEAK MIXING

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A characterisation of topologically weak mixing is given by using the topological sequence entropy.

1. INTRODUCTION

The notion of sequence entropy was first introduced by Kushnirenko [5]. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an invertible measure preserving transformation of a probability space and $A = \{t_k\}_{k=1}^{\infty}$ be a sequence of integer. Let $\mathcal{P}$ be the set of measurable partitions of X with finite entropy. For $\xi \in \mathcal{P}$, define

$$h_A(T,\xi) = \limsup \frac{1}{n} H\Big(\bigvee_{i=1}^{n} T^{t_i}\xi\Big)$$
$$h_A(T) = \sup_{\xi\in\mathcal{P}} h_A(T,\xi);$$

$h_A(T)$ is called the *sequence entropy* of T with respect to the sequence A. When $A = \{k-1\}_{k=1}^{\infty}$, $h_A(T)$ is the usual entropy of T.

It is known that sequence entropy is an useful invariant of measurable dynamical systems and has close relationships with the spectrum of the systems. In [**4, 6**] the authors proved the following theorem.

THEOREM 1. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an automorphism of a Lebesgue space. Then T is weak mixing if and only if there exists an increasing sequence of natural numbers A such that $h_A(T,\xi) = H(\xi)$ for all $\xi \in \mathcal{P}$.*

On the other hand, topological entropy was also extended to topological sequence entropy by Goodman [3]. Let X be a compact metric space and $f : X \to X$ be a continuous map. For an open cover α of X, denote by $N(\alpha)$ the minimal cardinality of any subcover of α. The entropy $H(\alpha)$ of α is defined to be $\log N(\alpha)$. Let $A = \{t_k\}_{k=1}^{\infty}$ be a sequence of non-negative integers. We write

$$h_A(f,\alpha) = \limsup_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=1}^{n} f^{-t_i}\alpha\Big).$$

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The *topological sequence entropy* of f with respect to A is defined by

$$h_A(f) = \sup_\alpha h_A(f, \alpha)$$

where α ranges over all open covers of X. When $A = \{k-1\}_{k=1}^{\infty}$, $h_A(f)$ is the topological entropy of f.

Let $f : X \to X$ be a continuous map of a compact metric space. Then f is called *topologically transitive* if for any nonempty open sets U, V of X, there exists an integer $n > 0$ such that $U \cap f^{-n}V \neq \emptyset$. Also f is called *topologically weak mixing* if $f \times f$ is topologically transitive. For a finite open cover $\alpha = \{U_1, U_2, \dots U_n\}$ of X, define $I(U_i) = \left(\bigcup_{j \neq i} U_j\right)^c$. The cover α is called *regular* if $I(U_i)$ has nonempty interior for each i. It is easy to see that if an n-element open cover is regular then $N(\alpha) = n$, but the converse is not true. We shall prove the following theorem which is a topological version of Theorem 1.

THEOREM 2. *Let $f : X \to X$ be a continuous map of a compact metric space. Then f is topologically weak mixing if and only if for any regular open cover α of X, there exists an increasing sequence of non-negative integers $A = \{t_k\}_{k=1}^{\infty}$ such that $h_A(f, \alpha) = H(\alpha)$.*

2. Some lemmas

LEMMA 1. *Suppose $\alpha = \{U_1, U_2, \dots, U_n\}$ and $\beta = \{V_1, V_2, \dots, V_m\}$ are regular open covers of the compact metric space. If $I(U_i) \cap I(U_j) \neq \emptyset$ for any i, j, then*

$$N(\alpha \vee \beta) = N(\alpha)N(\beta).$$

PROOF: Obviously $N(\alpha \vee \beta) \leqslant N(\alpha)N(\beta)$. If the equality is not true, then there exists i_0 and j_0 such that $U_{i_0} \cap V_{j_0} \subset \bigcup_{(i,j) \neq (i_0, j_0)} U_i \cap V_j$. Since $I(U_{i_0}) \cap U_i = \emptyset\,(i \neq i_0), I(V_{j_0}) \cap V_j = \emptyset\,(j \neq j_0)$, we have

$$\big(I(U_{i_0}) \cap I(V_{j_0})\big) \cap (U_i \cap V_j) = \emptyset \quad \text{if} \quad (i,j) \neq (i_0, j_0).$$

But this contradicts

$$\emptyset \neq I(U_{i_0}) \cap I(V_{j_0}) \subset U_{i_0} \cap V_{j_0} \subset \bigcup_{(i,j) \neq (i_0, j_0)} U_i \cap V_j.$$

∎

For a subset Y of X, denote by $\operatorname{int}(Y)$ the interior of Y.

LEMMA 2. *Let $\alpha = \{U_1, \dots, U_n\}$ and $\beta = \{V_1, \dots, V_n\}$ be regular open covers of compact metric X. If $\operatorname{int}\big(I(U_i) \cap I(V_j)\big) \neq \emptyset$, then $\alpha \vee \beta$ is also a regular open cover.*

PROOF: It follows immediately from that $I(U_i \cap V_j) = I(U_i) \cap I(V_j)$. ∎

Lemma 3. ([2]) *Let $f : X \to X$ be a continuous map of a compact metric space. If f is topologically weak mixing, then $f \times f \times \ldots \times f$ (n times) is topologically transitive for any $n > 0$.*

Lemma 4. ([1]) *Let $f : X \to X$ be a continuous map of a compact metric space. If for any nonempty open sets U, V of X, there exists an integer $n > 0$ such that $U \cap f^{-n}(U) \neq \emptyset$ and $U \cap f^{-n}(V) \neq \emptyset$, then f is topologically weak mixing.*

3. Proof of Theorem 2

Proof: First suppose that $f : X \to X$ is topologically weak mixing and $\alpha = \{U_1, U_2, \ldots, U_n\}$ is a regular open cover of X. Let $V_i = \operatorname{int}(I(U_i))$. Then V_i is a nonempty open set of X as α is regular. Set $t_1 = 0$. Suppose

$$0 = t_1 < t_2 < \cdots < t_{k-1}$$

have been defined and that $\bigvee_{i=1}^{k-1} f^{-t_i}\alpha$ is a regular open cover and $N\left(\bigvee_{i=1}^{k-1} f^{-t_i}\alpha\right) = N(\alpha)^{k-1}$. For every $W'_j \in \bigvee_{i=1}^{k-1} f^{-t_i}\alpha$, let $W_j = \operatorname{int}(I(W'_j))$. Then W_j is nonempty as $\bigvee_{i=1}^{k-1} f^{-t_i}\alpha$ is regular. Since f is topologically weak mixing, by Lemma 3, there exists an integer $t_k > t_{k-1}$ such that $W_j \cap f^{-t_k}V_i \neq \emptyset$ $(j = 1, 2, \ldots, n^{k-1}, i = 1, 2, \ldots, n)$. By Lemma 1 and Lemma 2, $\bigvee_{i=1}^{k} f^{-t_i}\alpha$ is regular and $N\left(\bigvee_{i=1}^{k} f^{-t_i}\alpha\right) = N(\alpha)^k$. By induction we can choose an increasing sequence $A = \{t_i \mid i = 1, 2, \ldots\}$ such that, for any, $k > 0$ we have $N\left(\bigvee_{i=1}^{k} f^{-t_i}\alpha\right) = N(\alpha)^k$ and therefore $h_A(f, \alpha) = \log N(\alpha) = H(\alpha)$.

Conversely if f is not topologically weak mixing, then, by Lemma 4, there exist nonempty open sets U, V of X such that for any $k > 0$,

$$(*) \qquad U \cap f^{-k}U = \emptyset \quad \text{or} \quad U \cap f^{-k}V = \emptyset.$$

Without loss of generality we may assume that $U \cap V = \emptyset$ (if $U \cap V \neq \emptyset$, we may replace U, V by $U \cap V$ and $f^{-1}(U \cap V)$ respectively). Choose nonempty open sets U_1, V_1 of X such that $\overline{U_1} \subset U, \overline{V_1} \subset V$. Let $U' = \overline{U_1}^c, V' = \overline{V_1}^c$. Then $\alpha = \{U', V'\}$ is a regular open cover and $U' \supset U^c, V' \supset V^c$. Now let $A = \{t_i\}_{i=1}^{\infty}$ be an arbitrary non-negative increasing sequence of integers. By $(*)$ we have that for any $m > 0$,

$$U \subset W_0 \cap f^{-1}W_1 \cap f^{-2}W_2 \cap \ldots \cap f^{-m}W_m,$$

where

$$W_i = \begin{cases} U' & \text{if} \quad U \cap f^{-i}U = \emptyset \\ V' & \text{if} \quad U \cap f^{-i}V = \emptyset. \end{cases}$$

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Now given any $k > 0$,

$$U \subset f^{-t_1}W_{t_1} \cap f^{-t_2}W_{t_2} \cap \ldots \cap f^{-t_k}W_{t_k}.$$

For $x \in U^c$, if there exists an $1 \leqslant j \leqslant k$ such that $f^{t_j}(x) \in U$, we let

$$i = i(x) = \min\{j \mid 1 \leqslant j \leqslant k, f^{t_j}(x) \in U\}.$$

Then

$$\begin{aligned} x &\in f^{-t_1}U' \cap f^{-t_2}U' \cap \ldots \cap f^{-t_{i-1}}U' \cap f^{-t_i}U \\ &\subset f^{-t_1}U' \cap f^{-t_2}U' \cap \ldots \cap f^{-t_{i-1}}U' \cap f^{-t_i}(W_0 \cap f^{-(t_{i+1}-t_i)}W_{t_{i+1}-t_i} \\ &\qquad \cap f^{-(t_{i+2}-t_i)}W_{t_{i+2}-t_i} \cap \ldots \cap f^{-(t_k-t_i)}W_{t_k-t_i}) \\ &= f^{-t_1}U' \cap f^{-t_2}U' \cap \ldots \cap f^{-t_{i-1}}U' \cap f^{-t_i}W_0 \cap f^{-t_{i+1}}W_{t_{i+1}-t_i} \cap \ldots \cap f^{-t_k}W_{t_k-t_i} \\ &\in \bigvee_{j=1}^{k} f^{-t_j}\alpha. \end{aligned}$$

If for any $1 \leqslant j \leqslant k, f^{t_j}(x) \notin U$, then

$$x \in f^{-t_1}U' \cap f^{-t_2}U' \cap \ldots \cap f^{-t_k}U' \in \bigvee_{j=1}^{k} f^{-t_j}\alpha.$$

Therefore

$$N(f^{-t_1}\alpha \vee \ldots \vee f^{-t_k}\alpha) \leqslant k+2 \quad \text{for any} \quad k > 0.$$

So $h_A(f,\alpha) = 0$. It contradicts the assumption of the theorem. ∎

COROLLARY 1. *Suppose X is a compact metric space which is not one point and $f : X \to X$ is topologically weak mixing. Then $\sup_A h_A(f) = \infty$ where the supremum is taken over all sequences of non-negative integers.*

PROOF: Since f is topologically weak mixing and X is not one point, X is infinite. We can choose regular open covers of X with n-elements for any $n > 0$. By theorem 2, $\sup_A h_A(f) = \infty$. ∎

EXAMPLE 1. The condition of the regularity of the cover of theorem 2 cannot be omitted. For example, let X be a compact metric space which is not one point and $f : X \to X$ be a topologically weak mixing homeomorphism. Let $x_0 \in X$ and $U_1 = X - \{x_0\}$. Let U_2 be another open set of X such that $\alpha = (U_1, U_2)$ is a regular open cover of X. It is easy to see that for any $k > 0$ and any sequence $A = \{t_i\}_{i=1}^{\infty}$,

$$N(f^{-t_1}\alpha \vee \ldots \vee f^{-t_k}\alpha) \leqslant k+1.$$

Therefore $h_A(f,\alpha) \neq H(\alpha)$.

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