BULL. AUSTRAL. MATH. SOC. 18E15, 18F20
VOL. 37 (1988) [43–59]

INJECTIVITY AND INJECTIVE HULLS OF ABELIAN GROUPS IN A LOCALIC TOPOS

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We prove the analogue of the Baer Criterion for injectivity in the category $AbSh\mathcal{L}$ of abelian groups in a topos of sheaves on a locale, that is, we show A is injective in $AbSh\mathcal{L}$ if and only if it is injective relative to all $S \rightarrowtail Z_{\mathcal{L}}$ where $Z_{\mathcal{L}}$ is the group of integers in $Sh\mathcal{L}$. for a well-ordered locale we describe the injective hulls in $AbSh\mathcal{L}$ in terms of injective hulls in Ab. Further we show that the global functor $A \to AE$ preserves injective hulls if and only if $\mathcal{L}$ is a finite boolean locale. Finally we characterise injectives in $AbSh\mathcal{L}$ for some special locales.

INTRODUCTION

This paper is devoted to the study of injectivity and injective hulls in the category $AbSh\mathcal{L}$ of abelian groups in a topos of sheaves on a local $\mathcal{L}$. We first prove the analogue of the Baer criterion for injectivity in $AbSh\mathcal{L}$ (Proposition 1.1) and show that injectivity is a local property (1.4). This is followed by a discussion on injective hulls, which we show is a local property (Proposition 2.1) but not a global one. For a well-ordered locale we describe the injective hulls in $AbSh\mathcal{L}$ in terms of injective hulls in Ab (Proposition 2.3). Further, in Proposition 2.7 we show that the global functor preserves injective hulls if and only if $\mathcal{L}$ is a finite boolean locale, that is, the topologies of finite discrete spaces.

Finally, we characterise in Propositions 3.1 and 3.3, the injectives in $AbSh\mathcal{L}$ for the following locales:

1) $\mathcal{L}$ with descending chain condition
2) $\mathcal{L}$ inversely well ordered

As a consequence we show that the direct sum of injectives in $AbSh\mathcal{L}$ is always injective for $\mathcal{L}$ inversely well ordered (Corollary 3.4). This does not hold for an arbitary $\mathcal{L}$ and a counter example is provided (3.5).

In Section 0 we describe briefly the background material required here, where as in Section 1 we derive general results on injectivity. In Section 2 we discuss injective hulls and finally we characterise injectives for some special locales in Section 3. The n^{th} result in the m^{th} Section will be numbered as $m.n$.

Received 4 March 1987
Abstracted partially from the dissertation submitted March 1983 to McMaster University for partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author is grateful to her supervisor Professor Bernhard Banaschewski, for his valuable assistance, encouragement and criticism throughout this work. Facilities for the preparation of this manuscript were provided by the Division of Computer Research and Technology at the National Institutes of Health, Bethesa, MD.

0. Background

Definition 0.1: Recall that a *locale* denote by $\mathcal{L}$ is a complete lattice satisfying the following;

$$U \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} (U \wedge U_i)$$

for all U and any family $\{U_i\}_{i \in I}$ in $\mathcal{L}$. We shall denote the bottom (=zero) of $\mathcal{L}$ by 0 and the top (=unit) of $\mathcal{L}$ by E. A morphism of locales $h\colon \mathcal{L} \to \mathcal{M}$ is a map which preserves arbitary joins and finite meets (hence preserves the zero and the unit). The obvious example of a locale is the topology $\mathcal{O}X$ (that is, the lattice of open sets) of any topological space X with joins as unions and meets as intersections. Other examples of locales are a complete chain, complete boolean algebra or a finite distributive lattice.

By the definition of continuity of maps between topological spaces, we get a contravariant functor $\mathcal{O}\colon TOP \to LOC$ where TOP is the category of topological spaces and continuous maps, and LOC is the category of locales and their morphisms. The functor $\mathcal{O}$ has an adjoint on the right, the contravariant functor $\Sigma\colon LOC \to TOP$ where $\Sigma\mathcal{L}$ is the space of completely prime filters F on $\mathcal{L}$, that is, F is a filter on $\mathcal{L}$ such that $\bigvee_{i \in I} U_i \in F$ for any family $\{U_i\}_{i \in I}$ in $\mathcal{L}$ implies $U_k \in F$ for some $k \in I$, and the sets $\Sigma_U = \{F | U \in F, F \in \Sigma\mathcal{L}\}$, $U \in \mathcal{L}$, form the open sets in this space. For any locale lattice homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ the corresponding continuous map $\Sigma h\colon \Sigma\mathcal{M} \to \Sigma\mathcal{L}$ sends $F \to h^{-1}(F)$. The space $\Sigma\mathcal{L}$ is called the spectrum of $\mathcal{L}$.

Definition 0.2: A locale $\mathcal{L}$ is called *spatial* if and only if the function $\mathcal{L} \to \mathcal{O}(\Sigma\mathcal{L})$ is an isomorphism. Since $\mathcal{O}_{\mathcal{L}}$ is always onto, a locale is spatial if and only if the completely prime filters separate points in $\mathcal{L}$. For more details refer to [**4**]. Note that any finite locale is spatial since in any distributive lattice the prime filters separate the elements (Balbes and Dwinger [**1**]), and for finite $\mathcal{L}$ the prime filters are completely prime. Also any totally ordered locale is spatial since the $U \geqslant V$, $V \in \mathcal{L}$ form a completely prime filter on $\mathcal{L}$ for any $U \in \mathcal{L}$.

Finally any $\mathcal{L}$ with descending chain condition is spatial: If $U \leqslant V$ in $\mathcal{L}$ and $W \in \mathcal{L}$ is minimal such that $U < W \leqslant V$ then $F = \{S | S \in \mathcal{L},\ U \vee S \geqslant W\}$ is a completely prime filter on $\mathcal{L}$ for which $U \notin F$ but $V \in F$.

Definition 0.3: A locale $\mathcal{L}$ is *boolean* if and only if every element in $\mathcal{L}$ has a complement. This is equivalent to saying that $\mathcal{L}$ has no dense elements other than E. That is, there is no $W \neq E$ such that $U \wedge W = 0$ implies $U = 0$. Note that a boolean locale is spatial if and only if $\mathcal{L}$ is atomic, [**1**], the non-trivial implication follows since any completely prime filter in a boolean locale is a principal filter given by an atom.

Definition 0.4: By $\mathit{AbSh}\mathcal{L}$ and $\mathit{AbPSh}\mathcal{L}$ we mean the categories of Sheaves and Presheaves on $\mathcal{L}$ with values in the category Ab of Abelian groups. These are

https://doi.org/10.1017/S0004972700004135 Published online by Cambridge University Press

Grothendieck categories with generator and hence have enough injective hulls [**12**]. $AbSh\mathcal{L}$ forms a full subcategory of $AbPSh\mathcal{L}$ and the inclusion $AbSh\mathcal{L} \to AbPSh\mathcal{L}$ has an exact left adjoint the sheaf reflection functor $AbPSh\mathcal{L} \to AbSh\mathcal{L}$. If A is the sheaf reflection of a given presheaf B, then we shall write $AU \doteq BU$, $U \in \mathcal{L}$. If $\mathcal{L} = \mathcal{O}(X)$ for some topological space X, then we shall write $AbShX$ for $AbSh\mathcal{O}(X)$. Also for any map $h\colon A \to B$ in $AbSh\mathcal{L}$ (and hence also in $AbPSh\mathcal{L}$) $h_U\colon AU \to BU$ will be the component of h at $U \in \mathcal{L}$. For any $a \in AU$ and $W \leqslant U$, the map $AU \to AW$ will be denoted $a \longrightarrow a|W$.

REMARK 0.5: $AbSh2 \cong Ab$ for the two element locale 2, and if X is a discrete topological space then $AbShX \cong Ab^{|X|}$.

DEFINITION 0.6: Any local lattice homomorphism $\phi\colon \mathcal{L} \to \mathcal{M}$ produces a pair of adjoint functors $AbSh\mathcal{M} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} AbSh\mathcal{L}$ where $(\phi_* A)U = A(\phi(U))$, and for any $V \in \mathcal{M}(\phi^* C)V \doteq \xrightarrow{\ell t}_{\phi(W) \geqslant V} CW (W \in \mathcal{L})$. Then ϕ^* is left exact left adjoint to ϕ_* and from well-known results it follows that ϕ_* preserves injectives.

DEFINITION 0.7: Special cases of local lattice homomorphisms.

1) If $\phi\colon 2 \to \mathcal{L}$ is the unique local lattice homomorphism, then it gives $AbSh\mathcal{L} \to AbSh2 \cong Ab$, where $(\phi_* A) = AE$ and $(\phi^* B)U \doteq B$. Notation $\phi^* B = B_{\mathcal{L}}$, $\phi_* = \Gamma$.

2) Any local lattice homomorphism $\phi\colon \mathcal{L} \to 2$ produces $Ab \to AbSh\mathcal{L}$ where

$$(\phi_* A)U = \begin{cases} A & \text{if } \phi(U) = 1 \\ 0 & \text{if } \phi(U) = 0 \end{cases}$$

and $\phi^* A = \ell t\, AU$ (All U such that $\phi(U) = 1$).

3) If $\mathcal{L} = \mathcal{O}(X)$ for some topological space X, and $x \in X$ is any point, then for the local lattice homomorphism $\hat{x}\colon \mathcal{L} \to 2$ given by $\hat{x}(U) = \operatorname{card}(U \cap \{x\})$, we get $(\hat{x})^* A = \xrightarrow{\ell t} AU (x \in U) = A_x$, the stalk of A at x.

4) For any $U \in \mathcal{L}$, $\phi\colon \mathcal{L} \to \downarrow U$ given by $\phi(W) = W \wedge U$ is a local lattice homomorphism, so we get $AbSh \downarrow U \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} AbSh\mathcal{L}$, where $(\phi_* A)V = A(V \wedge U)$ and $(\phi^* B)W \doteq \xrightarrow{\ell t}_{\phi(V) \geqslant W} BV = BW$, and so $\phi^* B$ is just the restriction of B to $\downarrow U$.

Notation : $\phi^* B = B|U = R_U B$.

Also ϕ^* has a left adjoint denoted by E_U, where

$$(E_U A)V \doteq \begin{cases} AV & \text{if } V \leqslant U \\ 0 & \text{if } V \nleqslant U. \end{cases}$$

Then E_U is left exact left adjoint to R_U. Since R_U is both a right adjoint as well as a left adjoint, it preserves all limits and colimits.

5) If $f\colon X \to Y$ is a continuous map of topological spaces, then it produces a local lattice homomorphism (also denoted by f) $f\colon \mathcal{O}Y \to \mathcal{O}X$, $V \to f^{-1}(V)$, and so correspondingly it gives $AbShX \to AbShY$. In particular for any topological space X, let $|X|$ be X with discrete topology. Then the identity map $i\colon |X| \to X$ is continuous, hence it produces $Ab^{|X|} \cong AbSh|X| \to AbShX$.

DEFINITION 0.8: $A \in AbSh\mathcal{L}$ is said to be a *divisible* group if for any $a \in AU$, and any $O \neq n \in N$ there exists a cover $U = \bigvee_{i\in I} U_i$ in $\mathcal{L}$, such that for all $i \in I$, $a \mid U_i = nb_i$ with $b_i \in AU_i$.

DEFINITION 0.9: For any $A \in AbSh\mathcal{L}$ the subgroup C of A generated by an element $a \in AU$, $U \in \mathcal{L}$, that is, the smallest subgroup $C \subseteq A$ such that $a \in CU$, is given by

$$CW \doteq \begin{cases} Z(a \mid W) & \text{if } W \subseteq U \\ 0 & \text{if } W \not\subseteq U. \end{cases}$$

DEFINITION 0.10: $B \supseteq A$ is an essential extension in $AbSh\mathcal{L}$ if and only if for any $0 \neq b \in BU$, there exists $V \leqslant U$ in $\mathcal{L}$ and $m \in Z$ such that $0 \neq mb \mid V \in AV$. To see this, one first notices that $B \supseteq A$ is essential if and only if $C \cap A \neq 0$ for any non zero subgroup $C \subseteq B$ (since a homomorphism in $AbSh\mathcal{L}$ is monic if and only if its kernel is 0), and then observe that it is sufficient to consider subgroups generated by a single non-zero $b \in BU$ for any $U \in \mathcal{L}$.

PROPOSITION 0.11. *For any $U \in \mathcal{L}$, the functors $R_U\colon AbSh\mathcal{L} \to AbSh \downarrow U$, and $E_U\colon AbSh \downarrow U \to AbSh\mathcal{L}$ preserve essential extensions.*

PROOF: Consider any essential extension $B \supseteq A$ in $AbSh\mathcal{L}$. Since R_U preserves all limits (0.7(4)), it follows that $B|U \supseteq A|U$. We claim this is an essential extension in $AbSh \downarrow U$. Let $0 \neq b \in BW = (B \mid U)(W)$ for some $W \in\downarrow U$. Since B is an essential extension of A in $AbSh\mathcal{L}$, there exists a $V \leqslant W$, $m \in Z$ such that $0 \neq mb \mid V \in AV = (A \mid U)(V)$. Hence $B \mid U \supseteq A \mid U$ is an essential extension in $AbSh \downarrow U$. To prove that E_U preserves essential extensions, consider an essential extension $P \supseteq Q$ in $AbSh \downarrow U$. Since E_U preserves monomorphisms (0.7(4)) it follows that $E_UP \supseteq E_UQ$. Let $0 \neq a \in (E_UP)W$, then by the definition of E_U, there exists a cover $W = \bigvee_{i\in I} W_i$, such that $0 \neq a \mid W_i \in PW_i$ for some $W_i \subseteq U$. But $P \supseteq Q$ is an essential extension in $AbSh \downarrow U$ so there exists $V \leqslant W_i$ and an $m \in Z$ such that $0 \neq m(a \mid W_i) \mid V = ma \mid V \in QV$. Hence $E_UP \supseteq E_UQ$ is an essential extension in $AbSh\mathcal{L}$. ∎

COROLLARY 0.12. *For any $U \in \mathcal{L}$, R_U preserves injectives and injective hulls.*

PROOF: Since R_U has a left adjoint E_U which preserves monomorphisms (0.8(4)), it follows that R_U preserves injectives. By the above proposition if follows R_U preserves injective hulls. ∎

COROLLARY 0.13. *For an injective group* $A \in AbSh\mathcal{L}$, AU *is an injective group in* Ab *for all* $U \in \mathcal{L}$.

PROOF: Clear from 0.12 and 0.7(1). ∎

REMARK 0.14: The composition $E_U R_U$ is denoted by $T_U \colon AbSh\mathcal{L} \to AbSh\mathcal{L}$, where

$$(T_U A)W \doteq \begin{cases} AW & \text{if } W \subseteq U \\ 0 & \text{if } W \not\subseteq U. \end{cases}$$

Since both E_U and R_U preserve essential extensions, it follows that T_U preserves essential extensions. Note, though, that T_U does not preserve injectives, as one can see by considering $\mathcal{L} = 3$.

REMARK 0.15: It is easily checked that $AbSh\mathcal{L}$ has the $T_U Z_{\mathcal{L}}$, $U \in \mathcal{L}$, as generating set where $Z_{\mathcal{L}}$ is the group of integers in $Sh\mathcal{L}$, that is, the sheaf reflection of the constant presheaf Z.

1. GENERAL RESULTS

THE BAER CRITERION FOR INJECTIVITY

PROPOSITION 1.1. $A \in AbSh\mathcal{L}$ *is injective if and only if it is injective relative to all* $S \rightarrowtail Z_{\mathcal{L}}$.

PROOF: $(\Rightarrow)$ is trivial.
$(\Leftarrow)$ Let A be injective relative to all $S \rightarrowtail Z_{\mathcal{L}}$. Consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \\ A & & \end{array}$$

where we may assume that $B \subseteq C$, and $f \colon B \rightarrowtail C$ is the natural embedding. Consider the family $\mathcal{A} = \{(B', g')\}$ where $B \subseteq B' \subseteq C$ and $g' \colon B' \to A$ such that $g' \mid B = g$. Then this family is non-empty since $(B, g) \in \mathcal{A}$. As usual, we introduce a partial ordering on this family by $(B', g') \leqslant (B'', g'')$ if and only if $B' \subseteq B''$ and $g'' \mid B' = g'$. If $\{(B_i, g_i)\}_{i \in I}$ is a linearly ordered family in $\mathcal{A}$, then it has an upper bound in $\mathcal{A}$ given by (D, h), where D is the join of B_i in the subgroup lattice of C. That is, D is the sheaf reflection of the presheaf $U \to \bigcup_{i \in I} B_i U$ and h is the corresponding sheaf reflection of the morphism to which the g_{i_U} extend. By Zorn's lemma, the family $\mathcal{A}$

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has a maximal element (P,p). We claim $P = C$. If not, then there is a $U \in \mathcal{L}$ and $c \in CU$ such that $c \notin PU$. Let H be the subgroup of C generated by c. Then H is the sheaf reflection of the presheaf

$$W \longrightarrow \begin{cases} Z(c \mid W) & \text{if } W \subseteq U \\ 0 & \text{if } W \not\subseteq U. \end{cases}$$

Since the presheaf defining $T_U Z_{\mathcal{L}}$ is given by $W \longrightarrow \begin{cases} Z & \text{if } W \subseteq U \\ 0 & \text{if } W \not\subseteq U \end{cases}$ therefore, there is an epimorphism of presheaves from that defining $T_U Z_{\mathcal{L}}$ to that defining H. The sheaf reflection preserves epimorphisms, and so $j\colon T_U Z_{\mathcal{L}} \to H$ is an epimorphism in $AbSh\mathcal{L}$. The diagram

$$\begin{array}{ccc} & & T_U Z_{\mathcal{L}} \\ & & \downarrow \\ P \cap H & \longrightarrow & H \end{array}$$

can be completed to a pull-back square,

$$\begin{array}{ccc} \bar{H} & \xrightarrow{\;\bar{i}\;} & T_U Z\mathcal{L} \\ {\scriptstyle \bar{j}}\downarrow & & \downarrow{\scriptstyle j} \\ P \cap H & \xrightarrow{\;i\;} & H \end{array}$$

where $\bar{\imath}$ is a mono since i is. Moreover since j is an epimorphism, $\bar{\jmath}$ is an epimorphism, and the above diagram is actually a push out diagram [**16**, p.33]. But $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ [**5**], and A is injective relative to all $\bar{H} \to Z_{\mathcal{L}}$, so there exists $\alpha\colon Z_{\mathcal{L}} \to A$ such that the outer triangle of the diagram

$$\begin{array}{ccccc} \bar{H} & \xrightarrow{\;\bar{\imath}\;} & T_U Z_{\mathcal{L}} & \longrightarrow & Z_{\mathcal{L}} \\ {\scriptstyle \bar{\jmath}}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \alpha} \\ P \cap H & \xrightarrow[\;i\;]{} & H & & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \beta} & & \| \\ P & & A & & A \\ {\scriptstyle p}\downarrow & & \| & & \| \\ A & = & A & = & A \end{array}$$

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commutes; that is, $(\alpha \mid T_U Z_{\mathcal{L}})\bar{\imath} = pk\bar{\jmath}$. The inner square is also a push out square, and so there exists a unique $\beta\colon H \to A$ such that $\beta j = \alpha \mid T_U Z_{\mathcal{L}}$ and $\beta i = pk$. Define another presheaf M by $MU = PU + HU \subseteq CU$, with the obvious restriction maps, then

$$\begin{array}{ccc} P \cap H & \xrightarrow{i} & H \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ P & \xrightarrow[n]{} & M \end{array}$$

is a push-out in $AbSh\mathcal{L}$, because at each $U \in \mathcal{L}$ it is a push-out in Ab. Since $P + H \doteq M$, and the sheaf reflection being a left adjoint preserves push out, it follows that

$$\begin{array}{ccc} P \cap H & \xrightarrow{i} & H \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ P & \xrightarrow[n]{} & P + H \end{array}$$

is a push-out in $AbSh\mathcal{L}$. Hence if we consider the diagram

$$\begin{array}{ccc} P \cap H & \xrightarrow{i} & H \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ P & \xrightarrow{n} & P + H \\ {\scriptstyle p}\big\downarrow & & \\ A & & \end{array}$$

then there is a unique $q\colon P+H \to A$ such that $qn = p$, and $q\tau = \beta$. Thus $(P + H, q) \in \mathcal{A}$, a contradiction, since $(P, p) \in \mathcal{A}$ is maximal and $P + H \supset P$. Thus $P = C$, and hence A is injective. ∎

REMARK: Although the analogue of the Baer Criterion for injectivity holds in $AbSh\mathcal{L}$, still the concepts of injectivity and divisibility do not coincide for an arbitary $\mathcal{L}$. In fact the two concepts coincide if and only if $\mathcal{L}$ is Boolean [**2**].

LEMMA 1.2. *For any cover $E = \bigvee_{i\in I} U_i$ of the unit in $\mathcal{L}$, the functor $R\colon AbSh\mathcal{L} \to \prod_{i\in I} AbSh \downarrow U_i$ given by $RB = (B \mid U_i)_{i\in I}$, $Rh = (h \mid U_i)_{i\in I}$ for $h\colon A \to B$ in $AbSh\mathcal{L}$, has the following two properties:*

(a) R preserves and reflects monomorphisms.

(b) R is faithful.

PROOF: (a) If $h\colon A \to B$ is a monomorphism in $AbSh\mathcal{L}$, then each $h \mid U_i\colon A \mid U_i \to B \mid U_i$ is a monomorphism in $AbSh \downarrow U_i$ (0.7(4)), hence $(h \mid U_i)\colon RA \to RB$ is a

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monomorphism in $\prod_{i\in I} AbSh \downarrow U_i$. Now suppose Rh is a monomorphism; we want to show that h is a monomorphism. Let $W \in \mathcal{L}$ be arbitary, and suppose $h_W(a) = h_W(b)$ for some $a, b \in AW$. Since h is a morphism of sheaves, therefore

$$\begin{array}{ccc} AW & \longrightarrow & A(W \wedge U_i) \\ \downarrow & & \downarrow \\ BW & \longrightarrow & B(W \wedge U_i) \end{array}$$

commutes for all $i \in I$, and hence $h_{W\wedge U_i}(a \mid W \wedge U_i) = h_{W\wedge U_i}(b \mid W \wedge W_i)$. But $h_{W\wedge U_i}$ is a monomorphism in $AbSh \downarrow U_i$ for all i, and therefore $a \mid W\wedge U_i = b \mid W\wedge U_i$ all i, which by the sheaf properties of A implies $a = b$. Hence h is a monomorphism. Thus R preserves and reflects monomorphisms.

(b) Suppose $Rf = Rg$ for some $f, g \colon A \to B$ in $AbSh\mathcal{L}$. Then $f \mid U_i = g \mid U_i$ for all $i \in I$. We claim $f = g$, that is $f_W = g_W$ for all $W \in \mathcal{L}$, For any $a \in AW$, we have $g_W(a) \mid W \wedge U_i = g_{W\wedge U_i}(a \mid W \wedge U_i) = f_{W\wedge U_i}(a \mid W \wedge U_i) = f_W(a) \mid W \wedge U_i$, all $i \in I$. Thus for the cover $W = \bigvee_{i\in I} W \wedge U_i$, we have $f_W(a) \mid W\wedge U_i = g_W(a) \mid W\wedge U_i$, all $i \in I$, hence $f_W(a) = g_W(a)$. Thus $f_W = g_W$ for all $W \in \mathcal{L}$ implies $f = g$. ∎

Proposition 1.3. *The functor R preserves and reflects injectives.*

Proof: If B is injective in $AbSh\mathcal{L}$, then each $B \mid U_i$ is injective in $AbSh \downarrow U_i$ (0.12), hence $RB = (B \mid U_i)_{i\in I}$ is injective in $\prod_{i\in I} AbSh \downarrow U_i$.

Assume RB is injective; we want to show that B is injective. Consider an essential extension D of B. Since R_{U_i} preserves essential extensions (0.11) it follows that each $D \mid U_i \supseteq B \mid U_i$ is an essential extension in $AbSh \downarrow U_i$. So if $0 \neq S \subseteq RD$ then for some $i \in I$, $0 \neq S_i \subseteq D \mid U_i$, hence $S_i \cap B \mid U_i \neq 0$. This means $S \cap RB \neq 0$ which shows RD is an essential extension of RB. But RB is given to be injective, and so $RB = RD$. Since R is faithful it reflects epimorphisms, hence the natural embedding $B \to D$ is an epimorphism and therefore $B = D$. Thus B has no proper essential extensions in $AbSh\mathcal{L}$, which means B is injective (0.4), hence the result. ∎

Proposition 1.4. *B is injective in $AbSh\mathcal{L}$ if and only if there is a cover $E = \bigvee_{i\in I} U_i$ such that $B \mid U_i$ is injective in $AbSh \downarrow U_i$, for all $i \in I$.*

Proof: $(\Rightarrow)$ Clear by taking the trivial cover of E.
$(\Leftarrow)$ For the converse assume $E = \bigvee U_i$ and each $B \mid U_i$ is injective. Then let $A \supseteq B$ be any essential extension. By 0.11 one then has an essential extension $A \mid U_i \supseteq B \mid U_i$, hence by hypothesis $A \mid U_i = B \mid U_i$ and then, finally $A = B$, showing that B is injective. ∎

Remark: The above proposition shows injectivity is a local property. This was also shown by Harting [**11**], but by an entirely different method. She considers the

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preservation of maximal partial morphisms by the restriction functors R_U for $U \in \mathcal{L}$, whereas our approach uses the preservation of essentialness by the functors R_U.

LEMMA 1.5. *If A is injective in $AbSh\mathcal{L}$, then for any $V \leqslant U$ in $\mathcal{L}$ the restriction $AU \to AV$ is a split epimorphism in Ab.*

PROOF: Consider the local lattice homomorphism $\phi: 3 \to \downarrow U$ with image $\begin{matrix} U \\ \downarrow \\ V \\ \downarrow \\ 0 \end{matrix}$.

Then since A is injective in $AbSh\mathcal{L}$, it follows that $\phi_* A = \begin{matrix} AU \\ \downarrow \\ AV \end{matrix}$ is an injective group in $AbSh3$ (0.6). But the injectives in $AbSh3$ are exactly the projections $\begin{matrix} P\times T \\ \downarrow \\ P \end{matrix}$ with divisible P and T [**2**], hence $\begin{matrix} AU \\ \downarrow \\ AV \end{matrix}$ is a split epimorphism in Ab.

2. INJECTIVE HULLS

Given A, B in $AbSh\mathcal{L}$, recall that B is the injective hull of A if and only if it is an essential injective extension of A.

PROPOSITION 2.1. *B is the injective hull of A in $AbSh\mathcal{L}$, if and only if there exists a cover $E = \bigvee_{i\in I} U_i$, such that $B \mid U_i$ is the injective hull of $A \mid U_i$ in $AbSh \downarrow U_i$.*

PROOF: ($\Rightarrow$) Clear, by taking the trivial cover.
($\Leftarrow$) Given that $B \mid U_i$ is the injective hull of $A \mid U_i$, all $i \in I$, it follows by 1.4 that B is injective in $AbSh\mathcal{L}$. So, it only remains to show that B is an essential extension of A. Let $D \subseteq B$ be a non zero subgroup of B, then $DU \neq 0$ for some $U \in \mathcal{L}$. Since $U = \bigvee_I (U \wedge U_i)$, it follows that $0 \neq DU \mapsto \prod D(U \wedge U_i)$ and so for some $i \in I$, $0 \neq D(U \wedge U_i) = (D \mid U_i)(U \wedge U_i)$. But $B \mid U_i$ is an essential extension of $A \mid U_i$ in $AbSh \downarrow U_i$, so $0 \neq D \mid U_i \subseteq B \mid U_i$ implies $D \mid U_i \cap A \mid U_i \neq 0$, and therefore $D \cap A \neq 0$. Hence B is an essential extension of A, and also being injective it is the injective hull of A. ∎

REMARK 2.2: In our next result, we describe the injective hull of any A in $AbSh\mathcal{L}$ where $\mathcal{L}$ is well-ordered, and so it might be appropriate to describe the topology of the spectrum of a well-ordered locale. If $\mathcal{L}$ is well-ordered, then without loss of generality we may assume $\mathcal{L} = \lambda + 1$, for some ordinal λ. We now show that the sets $W_\alpha = \{\gamma : \gamma$ not a limit ordinal, $0 < \gamma \leqslant \alpha\}$ for each $\alpha \in \lambda + 1$, form a topology $\mathcal{O}$ on the set X consisting of all the non-zero non-limit ordinals $\gamma \leqslant \lambda$. Now $W_0 = \emptyset$, $W_\lambda = |X|$, $W_\alpha \cap W_\beta = W_{\alpha\wedge\beta}$ since for $\alpha \leqslant \beta$, $W_\alpha \subseteq W_\beta$. To check $W_{\bigvee_I \alpha_i} = \bigcup_I W_{\alpha_i}$ for any family $\{\alpha_i\}_I$ in $\lambda + 1$, we consider $\gamma \in W_{\bigvee_I \alpha_i}$. Then $\gamma \leqslant \bigvee_I \alpha_i$, so if $\gamma \notin \bigcup_I W_{\alpha_i}$ then we must have $\alpha_i < \gamma$, for all $i \in I$. But $\gamma \in X$ and so $\gamma = \beta + 1$ for some

$\beta < \lambda$. Therefore $\alpha_i < \gamma$ implies $a_i \leqslant \beta$ for all $i \in I$, hence $\bigvee_I \alpha_i \leqslant \beta < \gamma$ a contradiction, since $\gamma \leqslant \bigvee_I \alpha_i$. Thus there is some $i \in I$ such that $\gamma \leqslant \alpha_i$ and so $\gamma \in \bigcup_I W_{\alpha_i}$. Therefore $W_{\bigvee_i \alpha_i} \subseteq \bigcup_I W_{\alpha_i}$. Moreover for all $i \in I$, $\alpha_i \leqslant \bigvee_I \alpha_i$ implies $\bigcup W_{\alpha_i} \subseteq W_{\bigvee_I \alpha_i}$ and hence $W_{\bigvee_i \alpha_i} = \bigcup_I W_{\alpha_i}$. Therefore $\mathcal{O}$ is indeed a topology on X. Now let $W_\alpha = W_\beta$ for some $\alpha, \beta \in \lambda + 1$, and suppose $\alpha < \beta$. Then $\alpha + 1 \leqslant \beta$ and so $\alpha + 1 \in W_\beta = W_\alpha$ which means $\alpha + 1 \leqslant \alpha$, a contradiction. Hence $W_\alpha = W_\beta$ implies $\alpha = \beta$. Therefore $\mathcal{L} = \lambda + 1$ is isomorphic to $\mathcal{O}$ by $\alpha \to W_\alpha$. Note that the completely prime filters on $\mathcal{L} = \lambda + 1$ are exactly $\uparrow \gamma$ for $\gamma \in X$, hence $\sum \mathcal{L}$ may be represented by the set X of these γ with the topology $W_\alpha = \{\gamma \in X \mid \gamma \leqslant \alpha\}$.

PROPOSITION 2.3. *For a well-ordered locale $\mathcal{L}$, the injective hull of any* $A = A_\lambda \xrightarrow{h_\gamma} \cdots \to A_2 \xrightarrow{h_2} A_1 \xrightarrow{h_1} A_0 (= 0)$ *in $Ab Sh \mathcal{L}$ is given by the group* $C = C_\lambda \to \cdots \to C_2 \to C_1 \to C_0 (= 0)$ *where* $C_\beta = CW_\beta = \prod_{\alpha \in W_\beta} E(\operatorname{Ker} h_\alpha)$ *for all* $\beta \in \lambda + 1$.

PROOF: Define a family $(B_\alpha)_{\alpha \in |X|}$ in $Ab^{|X|}$ by $B_\alpha = E(\operatorname{Ker} h_\alpha)$ for all $\alpha \in X$. Since the functor $F \colon Ab^{|X|} \to AbShX \cong AbSh\mathcal{L}$ preserves injectives (0.6, 0.7) it produces an injective C in $AbSh\mathcal{L}$, where $C = F\Big((B_\alpha)_{\alpha \in |X|}\Big)$, and so $C_\beta = CW_\beta = \prod_{\alpha \in W_\beta} E(\operatorname{Ker} h_\alpha)$, $\beta \in \lambda + 1$ with restrictions $CU_\beta \to CU_\gamma$ as projections for all $\gamma \leqslant \beta$. The morphism from A to C is obtained by induction as follows: For $n = 1$, $A_1 \to C_1 = E(\operatorname{Ker} h_1) = E(A_1)$ is the natural embedding. Assume $A_\alpha \to C_\alpha$ already defined for all $\alpha < \beta$. Then there are two possibilities:

Case (i) $\beta = \gamma + 1$ for some $\gamma \in \lambda + 1$;

Case (ii) β is a limit ordinal.

For case (i) we are given $A_\gamma \to C_\gamma$ and since $C_{\gamma+1} = C_\gamma \times E(\operatorname{Ker} h_{\gamma+1})$ with $C_{\gamma+1} \to C_\gamma$ the projection, we can define $\tau_{\gamma+1} \colon A_{\gamma+1} \to C_\gamma \prod E(\operatorname{Ker} h_{\gamma+1})$ as $\tau_\gamma h_{\gamma+1} \prod h_{\overline{\gamma+1}}$ where $h_{\overline{\gamma+1}} \colon A_{\gamma+1} \to E(\operatorname{Ker} h_{\gamma+1})$ is an extension of the natural embedding $\operatorname{Ker}(h_{\gamma+1}) \to E(\operatorname{Ker} h_{\gamma+1})$ to $A_{\gamma+1}$: then as required $P_\gamma \tau_{\gamma+1} = p_\gamma\Big(\tau_\gamma h_{\gamma+1} \Pi h_{\overline{\gamma+1}}\Big) = \tau_\gamma h_{\gamma+1}$; that is,

$$\begin{array}{ccccc} A_{\gamma+1} & \xrightarrow{\tau_{\gamma+1}} & C_{\gamma+1} & = & C_\gamma \times E(\operatorname{Ker} h_{\gamma+1}) \\ {\scriptstyle h_{\gamma+1}}\big\downarrow & & \big\downarrow{\scriptstyle p^\gamma} & & \\ A_\gamma & \xrightarrow{\tau_\gamma} & C_\gamma & & \end{array}$$

commutes.

Case (ii) $\beta = \bigvee_{\alpha < \beta} \alpha$, and so $C_\beta = \underset{\alpha<\beta}{\overleftarrow{\ell t}}\, C_\alpha$. Since $A_\beta = \underset{\alpha<\beta}{\overleftarrow{\ell t}}\, A_\alpha$, and by

https://doi.org/10.1017/S0004972700004135 Published online by Cambridge University Press

assumption all $A_\alpha \xrightarrow{\tau_\alpha} C_\alpha (\alpha < \beta)$ are defined, therefore we get a family of maps $A_\beta \to A_\alpha \xrightarrow{\tau_\alpha} C_\alpha (\alpha < \beta)$ and so by the definition of limit there is a unique $\tau_\beta \colon A_\beta \to C_\beta$ such that

$$\begin{array}{ccc} A_\beta & \xrightarrow{\tau_\beta} & C_\beta \\ \downarrow & & \downarrow \\ A_\alpha & \xrightarrow{\tau_\alpha} & C_\alpha \end{array}$$

commutes for all $\alpha < \beta$. Hence we can define a morphism $\tau \colon A \to C$ with components $\tau_\alpha \colon A_\alpha \to C_\alpha$ as defined above.

Now to check that τ is a monomorphism. Clearly τ_1, is a monomorphism, so assume τ_α is a mono for all $\alpha < \beta$.

Case (i) $\beta = \gamma + 1$, so if $\tau_{\gamma+1}(a) = 0$, then $\tau_\gamma h_{\gamma+1}(a) = 0 = h_{\overline{\gamma+1}}(a)$. This means $h_{\gamma+1}(a) = 0$, that is $a \in \operatorname{Ker} h_{\gamma+1}$. Hence $a = 0$, since $h_{\overline{\gamma+1}}(a) = a$ for $a \in \operatorname{Ker} h_{\gamma+1}$. Thus τ_β is a monomorphism.

Case (ii) β is a limit ordinal. If $\tau_\beta(a) = 0$, then $\tau_\alpha(a \mid \alpha) = 0$ for all $\alpha < \beta$. But τ_α is a monomorphism for each $\alpha < \beta$, so $a \mid \alpha = 0$ which by the sheaf properties implies that $a = 0$. Thus the morphism $\tau \colon A \to C$ is indeed a monomorphism.

Finally we want to show that C is an essential extension of A, and so consider $0 \neq D \subseteq C$. Since $\mathcal{L}$ is well-ordered we can find a smallest $\alpha \in \mathcal{L}$ such that $D_\alpha \neq 0$. Then α is not a limit ordinal, since otherwise we get a contradiction $0 \neq D_\alpha \to \prod_{\gamma<\alpha} D_\gamma = 0$. For δ such that $\alpha = \delta + 1$, we have a commutative diagram

$$\begin{array}{ccccccc} & & D_\alpha & \longrightarrow & C_\alpha & = & \Pi_{\gamma \in W_\alpha} E(\operatorname{Ker} h_\gamma) & = & C_\delta \times E(\operatorname{Ker} h_\alpha) \\ & & \downarrow & & \downarrow & & & & \\ 0 & = & D_\delta & \longrightarrow & C_\delta & = & \Pi_{\gamma \in W_\delta} E(\operatorname{Ker} h_\gamma) & & \end{array}$$

where the horizontal arrows are inclusions, and so we conclude that $D_\alpha = 0 \times D_\alpha^-$ where $D_\alpha^- \subseteq E(\operatorname{Ker} h_\alpha)$. Hence there exists $0 \neq x \in \operatorname{Ker} h_\alpha$ such that $(0, x) \in D_\alpha$. Now $\operatorname{Im} \tau_\alpha = \operatorname{Im}\left(\tau_{\alpha-1} h_\alpha \prod \bar{h}_\alpha\right)$ and $(0, x) \in D_\alpha$ where $x \in \operatorname{Ker} h_\alpha$ implies $(0, x)$ belongs to $\operatorname{Im}(\tau_{\alpha-1} h_\alpha \prod h_\alpha) = \operatorname{Im} \tau_\alpha$. Hence $\tau_\alpha(A_\alpha) \cap D_\alpha \neq 0$ which means $\tau(A) \cap D \neq 0$. Thus $\tau \colon A \to C$ is an essential monomorphism. Also C is an injective group and therefore C is the injective hull of A.

Applied to the special case $\mathcal{L} = 3$, Proposition 2.3 leads to the following. ∎

COROLLARY 2.4. *The injective hull of* $\begin{array}{c} A \\ \big\downarrow h \\ B \end{array}$ *is*

$$\begin{array}{ccc} A & \xrightarrow{v} & E(B)\times E(\operatorname{Ker} h) \\ h\big\downarrow & & \big\downarrow \\ B & \xrightarrow{u} & E(B) \end{array}$$

where u *embeds* B *into its injective hull and* $v = (uh)\prod k$ *and* $k\colon A \to E(\operatorname{Ker} h)$ *extends the natural embedding* $\operatorname{Ker}(h) \to A$.

LEMMA 2.5. *For a boolean locale* $\mathcal{L}$, *if* E *is not compact, then there exists* $A, B \in$ *AbSh*$\mathcal{L}$ *such that* $A \subseteq B$ *is essential but* $AE \subseteq BE$ *is not essential in* Ab.

PROOF: Let $E = \bigvee_{i\in I} U_i$ where I is an infinite set. Then there exists a countable subset, say J of I, so that we can write $E = \left(\bigvee_{i\in J} U_i\right)\bigvee S$ where $S = \left(\bigvee_{i\in J} U_i\right)'$. Since $\mathcal{L}$ is boolean, we can find a sequence $\{U_n\}_{n\in\omega}$ such that $U_n \wedge U_m = 0$ and $E = \bigvee_{n\in\omega} U_n$. Define A_n, $B_n \in AbSh \downarrow U_n$ by $A_nU \doteq Z/p_nZ$, $B_nU \doteq Z/p_n^2Z$ for some prime p_n where $p_n \neq p_m$ if $n \neq m$. Then $B_n \supseteq Z_n$ is essential in $AbSh \downarrow U_n$, for if $0 \neq \phi \in B_nU$ but $\notin A_nU$ then $\phi(a) \neq 0$ for some $a \in Z/(p_n^2)$ where order $a = p_n^2$ and so $0 \neq p_n\phi \mid \phi(a) \in A(\phi(a))$ which shows $A_n \subseteq B_n$ is essential. If A, $B \in AbSh\mathcal{L}$ are defined by $A = \prod_{n\in\omega}(\alpha_n)_*A_n$, $B = \prod_{n\in\omega}(\alpha_n)_*B_n$, where $(\alpha_n)_*\colon AbSh \downarrow U_n \to AbSh\mathcal{L}$ corresponds to the morphism $\alpha_n\colon \mathcal{L} \to\downarrow U_n$, $U \rightsquigarrow U \wedge U_n(0.7)$, then $AU = \prod_{n\in\omega} A_n(U \wedge U_n)$ and $BU = \prod_{n\in\omega} B_n(U \wedge U_n)$, $U \in \mathcal{L}$. We claim $A \subseteq B$ is essential in $AbSh\mathcal{L}$. If $0 \neq \phi = (\phi_n) \in BU$, then for some $m \in \omega$, $0 \neq \phi_m \in B_m(U \wedge U_m)$, and so by the above argument for some $a \in Z/(p_m^2)$, $p_m\phi_m \mid \phi_m(a) \in A_m(\phi_m(a))$. Since $U_m \wedge U_n = 0$ for all $n \neq m$, we get $p_m\phi \mid \phi_m(a) = (p_m\phi_n \mid \phi_m(a))_{n\in\omega} \in AU$, since all components are zero except when $n = m$. Hence $A \subseteq B$ is essential. To show that $AE \subseteq BE$ is not essential, consider $\phi = (\phi_n) \in BE = \prod_{n\in\omega} B_nU_n$ where ϕ_n is of order p_n^2. If $AE \subseteq BE$ was essential then there exists $k \in Z$ such that $k\phi_n \in A_nU_n$ for all $n \in \omega$. This means $p_n \mid k$ for all $n \in \omega$, hence $k = 0$. Hence result. ∎

LEMMA 2.6. *The global functor* $\Gamma\colon AbSh\mathcal{L} \to Ab$ *preserves injective hulls if and only if it preserves essential extensions.*

PROOF: $(\Leftarrow)$ Clear, by the hypothesis and the fact that the functor Γ preserves injectives (since the functor $-_{\mathcal{L}}$ is an exact left adjoint of Γ (0.7)).

$(\Rightarrow)$ Let $A \mapsto B$ be an essential monomorphism in $AbSh\mathcal{L}$, and $A \mapsto A'$ be the natural embedding of A into its injective hull A'. Then there exists $f\colon B \to A'$ such that $fi = j$, which is actually a monomorphism (since i is essential). By hypothesis

$AE \rightarrowtail A'E = AE \longrightarrow BE \longrightarrow A'E$ is an essential monomorphism in Ab, hence $i_E\colon AE \rightarrowtail BE$ is an essential monomorphism. ∎

PROPOSITION 2.7. *The functor* $\Gamma\colon AbSh\mathcal{L} \to Ab$ *preserves injective hulls if and only if* $\mathcal{L}$ *is a finite boolean locale.*

PROOF: ($\Leftarrow$) By Lemma 2.6, it is enough to show that Γ preserves essential extensions. If $\mathcal{L}$ is finite boolean then $\mathcal{L} \cong \mathcal{O}(X)$ for a finite discrete space X, therefore $AbSh\mathcal{L} \cong Ab^{|X|}$. So $A \subseteq B$ essential in $AbSh\mathcal{L}$ implies $A\{x\} \subseteq B\{x\}$ is essential in Ab for all $x \in |X|$. Therefore $\prod_{x\in|X|} A\{x\} = \Gamma A \subseteq \Gamma B = \prod_{x\in|X|} B\{x\}$ is essential in Ab, since finite product in Ab preserve essential extension.

($\Rightarrow$) By Lemma 2.6 it follows that the functor $\Gamma\colon AbSh\mathcal{L} \to Ab$ preserves essential extensions. We first show that $\mathcal{L}$ is boolean. If not, then there exists a $W \in \mathcal{L}$ such that W is dense. Let $AU \subseteq Q_{\mathcal{L}}U$ be the subgroup consisting of all $\phi \in Q_{\mathcal{L}}U$ such that $\bigvee_{0\neq a\in Q} \phi(a) \leqslant U \wedge W$. Then A is a subgroup of $Q_{\mathcal{L}}$ [2]. Define B in $AbSh\mathcal{L}$ by $BU = A(U \wedge W)$, with the restrictions as given by A. Then $h\colon A \to B$ given by the restriction map of A is a monomorphism, since W is dense in $\mathcal{L}$. Moreover this monomorphism is essential for if $0 \neq \phi \in BU = A(U \wedge W)$ then clearly $\phi \mid (U \wedge W) = h_{U\wedge W}(\phi) = \phi \neq 0$. By hypothesis $AE \to BE = AW = Q_{\mathcal{L}}W$ is an essential monomorphism. Consider $\phi \in Q_{\mathcal{L}}W$ with $\phi(1) = W$. By essentialness, there exists a ψ in AE such that $0 \neq h_E(\psi) = \psi \mid W = m\phi$ for some $m \in Z$. Then $(m\phi)(m) = W$, so $\psi(m) \wedge W = W$, that is, $W \leqslant \psi(m)$, which means $\psi(m)$ is dense in $\mathcal{L}$. So if $k \neq m$ then $\psi(m) \wedge \psi(k) = 0$ implies $\psi(k) = 0$, therefore $\psi(m) = E$. But $\psi \in AE$, so $\bigvee_{k\neq 0} \psi(k) \leqslant W$, and $m \neq 0$ implies $\psi(m) \leqslant W$, that is $E = W$, hence $\mathcal{L}$ is boolean. By Lemma 2.5 it follows that E is compact. But $\mathcal{L}$ boolean implies each $U \in \mathcal{L}$ is compact, hence $\mathcal{L}$ is spatial. Therefore $\mathcal{L} = \mathcal{O}(X)$ for some discrete space X, which by compactness of E means X is a finite discrete space. Hence we have the result. ∎

REMARK 2.8: If $\mathcal{L}$ is finite boolean then so are all $\downarrow U$, and hence all functors Γ_U preserve injective hulls whenever $\Gamma = \Gamma_E$ does.

3. CHARACTERISING INJECTIVES FOR SOME SPECIAL LOCALES

We have seen in our previous discussion that an injective $A \in AbSh\mathcal{L}$ has the following two properties:

(a) For all $U \in \mathcal{L}$, each AU is an injective abelian group in Ab.

(b) Whenever $V \leqslant U$ in $\mathcal{L}$, then the restriction $AU \to AV$ is a split epimorphism in Ab.

Hence it is reasonable to ask if the properties (a) and (b) characterise injectives in $AbSh\mathcal{L}$. The answer is yes for some special locales which we shall discuss although the question still remains open for an arbitary $\mathcal{L}$. Recall that for $\mathcal{L} = 3$, Banaschewski has shown that injectives in $AbSh3$ are exactly those groups which satisfy the conditions (a) and (b) [2]. This fact is crucial in the following proofs.

PROPOSITION 3.1. *If $\mathcal{L}$ satisfies the descending chain condition then $A \in AbSh\mathcal{L}$ is injective if and only if it satisfies conditions (a) and (b).*

PROOF: To prove the remaining implication, consider any essential extension $B \supseteq A$. If $A \subset B$, then $\mathcal{L}$ has DCC, we can find a minimal $S \in \mathcal{L}$ such that $AS \subset BS$. Clearly, for all $U < S$, $AU = BU$. If $W = \bigvee U(U < S)$ then $AW = BW$, since for any $b \in BW$, $b \mid U \in BU = AU$ for $U < S$ implies $b \in AW$, hence $W < S$.

Consider the commutative diagram,

$$\begin{array}{ccc} AS & \xrightarrow{\subset} & BS \\ \downarrow & & \downarrow \\ AW & = & BW \end{array}$$

in $AbSh3$. If $0 \neq b \in BS$, then by essentialness there exist $V \leqslant S$, and $m \in Z$ such that $0 \neq mb \mid V \in AV$. Now either $V = S$ which means $0 \neq mb \in AS$, or $V < S$ and then $V \leqslant W$ so $0 \neq mb \mid V = (mb \mid W) \mid V$ implies $0 \neq mb \mid W \in BW = AW$. Thus $BS \to BW$ is an essential extension of $AS \to AW$ in $AbSh3$. But by the given hypothesis $AS \to AW$ is injective in $AbSh3$ [2] and hence $AS = BS$. Thus $A = B$, which means A is injective in $AbSh\mathcal{L}$. ∎

COROLLARY 3.2. *If $\mathcal{L}$ is finite or well-ordered then the conditions (a) and (b) characterise injectives in $AbSh\mathcal{L}$.*

PROOF: Clear, since these locales have descending chain conditions. ∎

PROPOSITION 3.3. *For any inversely well-ordered $\mathcal{L}$, $A \in AbSh\mathcal{L}$ is injective if and only if it satisfies conditions (a) and (b).*

PROOF: If $\mathcal{L}$ is inversely well-ordered then the elements of $\mathcal{L}$ may be arranged in the form $E = U_0 > U_1 > U_2 \cdots > U_\lambda = 0$, so that $L^{opp} \cong \lambda + 1$ for some ordinal λ. Since each non-empty subset of $\mathcal{L}$ has a largest element it follows that every element in $\mathcal{L}$ has only trivial covers, hence every presheaf on $\mathcal{L}$ is also a sheaf on $\mathcal{L}$. In particular, $Z_{\mathcal{L}} U_\alpha = Z$ for all α. If $A \in AbSh\mathcal{L}$ satisfies conditions (a) and (b), then we claim that A is injective. The proof will use the Baer criterion (1.1), so consider a diagram,

$$\begin{array}{ccc} C & \longrightarrow & Z_{\mathcal{L}} \\ {\scriptstyle h}\downarrow & & \\ A & & \end{array}$$

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where the horizontal arrow is the inclusion. Our aim is to extend h to all of $Z_{\mathcal{L}}$. If $C = 0$, then we are done. If $C \neq 0$, then we can pick the first α_0 such that $CU_{\alpha_0} \neq 0$. If $U_\alpha > U_\beta$, then the commutativity of the diagram

$$\begin{array}{ccc} CU_\alpha & \longrightarrow & Z_{\mathcal{L}}U_\alpha = Z \\ \downarrow & & \| \\ CU_\beta & \longrightarrow & Z_{\mathcal{L}}U_\beta = Z \end{array}$$

where the horizontal arrows are inclusions, implies $CU_\alpha \subseteq CU_\beta$. Let U_{α_1} be the first element in $\mathcal{L}$ such that $CU_{\alpha_0} \subset CU_{\alpha_1}$. Proceeding in the same fashion we obtain a strictly ascending chain of subgroups of Z given by $0 \neq CU_{\alpha_0} \subset CU_{\alpha_1} \subset CU_{\alpha_2} \subset \dots$. Since Z is noetherian, this chain must terminate after a finite number of steps and so for some $n, CU_{\alpha_n} = CU_\alpha$ for all $\alpha \geqslant \alpha_n$.

If we consider the finite chain $F = U_{\alpha_0} > U_{\alpha_1} > \dots > U_{\alpha_n}$ (which has only trivial covers), then the presheaf $AU_{\alpha_0} \to AU_{\alpha_1} \to \dots \to AU_{\alpha_n}$ satisfies condition (a) and (b) and so by our last result it is an injective group in $AbShF$. Hence there exist morphisms $g_{U_{\alpha_i}}\colon Z \to AU_{\alpha_i}$, such that $g_{U_{\alpha_i}} \mid C_{U_{\alpha_i}} = h_{U_{\alpha_i}}$, and $g_{U_{\alpha_i+1}} = g_{U_{\alpha_i}} \mid U_{\alpha_{i+1}}$ for all $i = 0, 1, \dots, n$. For any $U_\alpha \in \mathcal{L}$, where $\alpha \neq \alpha_0, \alpha_1, \dots, \alpha_n$ define g_{U_α} as follows: $g_{U_\alpha} = i_\alpha g_{U_{\alpha_0}}$ if $0 \leqslant \alpha \leqslant \alpha_0$ where $i_\alpha\colon AU_{\alpha_0} \to AU_\alpha$ is the inclusion into the product $AU_{\alpha_0} \to AU_0$ followed by the restriction may $AU_0 \to AU_\alpha$

$$\begin{aligned} g_{U_\alpha} &= g_{U_{\alpha_0}} \mid U_\alpha && \text{if } \alpha_0 \leqslant \alpha < \alpha_1 \\ g_{U_\alpha} &= g_{U_{\alpha_1}} \mid U_\alpha && \text{if } \alpha_1 \leqslant \alpha < \alpha_2 \\ &\vdots \\ g_{U_\alpha} &= g_{U_{\alpha_n}} \mid U_\alpha && \text{if } \alpha \geqslant \alpha_n \end{aligned}$$

Since $g_{U_{\alpha_{i+1}}} = g_{U_{\alpha_i}} \mid U_{\alpha_{i+1}}$, it follows for all $U_\alpha \leqslant U_{\alpha_0}$, we have $g_{U_\alpha} = g_{U_{\alpha_0}} \mid U_\alpha$. It remains to show that g extends h. Let $U\alpha$, U_β be arbitary elements of $\mathcal{L}$ such that $U_\alpha \leqslant U_\beta$. Then there are three cases:

(i) $U_{\alpha_0} \geqslant U_\beta \geqslant U_\alpha$
(ii) $U_\beta \geqslant U_{\alpha_0} \geqslant U_\alpha$
(iii) $U_\beta \geqslant U_\alpha \geqslant U_{\alpha_0}$.

Case (i). In this case $g_{U_\beta} = g_{U_{\alpha_0}} \mid U_\beta$, so

$g_{U_\beta} \mid U_\alpha = \left(g_{U_{\alpha_0}} \mid U_\beta\right) \mid U_\alpha = g_{U_{\alpha_0}} \mid U_\alpha = g_{U_\alpha}$ hence the diagram

$$\begin{array}{ccc} Z & = & Z \\ {\scriptstyle g_{U_\beta}}\downarrow & & \downarrow{\scriptstyle g_{U_\alpha}} \\ AU_\beta & \longrightarrow & AU_\alpha \end{array}$$

commutes.

Case (ii). $g_{U_\alpha} = i_\beta g_{U_{\alpha_0}}$ hence

$$g_{U_\beta} \mid U_\alpha = \left(g_{U_\beta} \mid U_{\alpha_0}\right) \mid U_\alpha = \left(\left(i_\beta g_{U_{\alpha_0}}\right) \mid U_{\alpha_0}\right) \mid U_\alpha = g_{U_{\alpha_0}} \mid U_\alpha = g_{U_\alpha}.$$

Case (iii). $g_{U_\beta} = i_\beta g_{U_{\alpha_0}}$, therefore

$$g_{U_\beta} \mid U_\alpha = \left(i_\beta g_{U_{\alpha_0}}\right) \mid U_\alpha = i_\alpha g_{U_{\alpha_0}} = g_{U_\alpha}.$$

Hence, we conclude that g is indeed a morphism of sheaves. Now to check that g extends h, we consider any $U_\alpha \in \mathcal{L}$. If $U_\alpha > U_{\alpha_0}$, then $CU_\alpha = 0$, so $g_{U_\alpha} \mid C_{U_\alpha} = 0 = h_{U_\alpha}$. So let us suppose that $U_{\alpha_0} \geqslant U_\alpha$. Then $g_{U_\alpha} \mid CU_\alpha = g_{U_\alpha} \mid CU_{\alpha_0}(\text{if } CU_\alpha = CU_{\alpha_0}) = \left(g_{U_{\alpha_0}} \mid CU_{\alpha_0}\right) \mid U_\alpha = h_{U_{\alpha_0}} \mid U_\alpha = h_{U_\alpha}$. If $CU_\alpha \neq CU_{\alpha_0}$, then $\alpha_1 \leqslant \alpha$. If $CU_\alpha = CU_{\alpha_1}$, then again we are done by the same argument with α_1 in place of α_0 since $g_{U_{\alpha_1}} \mid CU_{\alpha_1} = h_{U_{\alpha_1}}$. Otherwise, $CU\alpha \supset CU_{\alpha_1}$ and in that case $\alpha_2 \leqslant \alpha$ and one can proceed as before. Continuing in the same way one sees that $g_{U_\alpha} \mid CU_\alpha = h_{U_\alpha}$ for all α, that is g extends h. This shows A is injective. ∎

COROLLARY 3.4. *In $AbSh\mathcal{L}$ where $\mathcal{L}$ is inversely well-ordered, the direct sum of injectives is injective.*

PROOF: Let $A = \oplus_{i\in I} A_i$, where A_i is an injective group in $AbSh\mathcal{L}$. Then each A_iU is divisible in Ab, for all $U \in \mathcal{L}$. Therefore $AU = \oplus A_iU$ is divisible in Ab. For any $V \leqslant U$ in $\mathcal{L}$, each $A_iU \to A_iV$ is a split epimorphism, $\oplus_I A_iU \to \oplus_I A_iV$, that is $AU \to AV$ is a split epimorphism in Ab. By Proposition 3.3, A is injective, hence the result. ∎

COUNTEREXAMPLE 3.5: Here we show that the direct sum of injectives in $AbSh\mathcal{L}$ is not always injective for an arbitary $\mathcal{L}$. Consider an infinite space X, with the topology given by $U \in \mathcal{O}X$ if and only if $U = X, \emptyset$, or $x \notin U$ where x is a fixed point of X.

Then $\{y\} \in \mathcal{O}X$ if and only if $y \neq x$. For all $z \in |X|$, define $A_z = \phi_*(Q)$ where $\phi\colon \mathcal{L} \to 2$ is the locale lattice homomorphism corresponding to the point $z \in |X|$ (0.7). Then $A_zU \subseteq B : U \to Q^U$ consists of all $a \in BU$, with support contained in $\{z\}$. Let

$A = \oplus_{x \in |X|} A_x$. We claim A is not injective, although each A_x is an injective group (0.7). Note that A can be taken as a subgroup of B, and $f \in BX$ belongs to AX if and only if there exists a cover $X = \bigcup_{i \in I} U_i$ such that $f \mid U_i$ is of finite support for all $i \in I$. Since X has only trivial covers it follows that $X = U_i$ for some $i \in I$. Hence AX consists of all f in Q^X of finite support and so $AX \subset BX$. Note that for $X \neq U$, $AU = BU$. Now let $0 \neq a \in BX$. If $a(y) \neq 0$ for any $y \neq x$ then $0 \neq a \mid \{y\} \in A\{y\}$, and otherwise $a(y) = 0$ for all $y \neq x$ so that $a \in AX$. Hence B is an essential extension of A and therefore A is not injective. Since B is also injective (0.7), if follows that B is the injective hull of A. ∎

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