The Elementary Divisors, Associated with 0, of a Singular M-matrix

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1. Many investigations have been concerned with a square matrix P with non-negative coefficients (elements). It is remarkable that many interesting properties of P are determined by the set Σ of index pairs of positive (*i.e.* non-zero) coefficients of P, the actual values of these coefficients being irrelevant. Thus, for example, the number of characteristic roots equal in absolute value to the largest non-negative characteristic root ρ depends on Σ alone, if P is irreducible. If P is reducible, then Σ determines the standard forms of P (cf. § 3). The multiplicity of ρ depends on Σ, and on the set S of indices of those submatrices in the diagonal in a standard form of P which have ρ as a characteristic root. It has apparently not been considered before whether Σ and S also determine the elementary divisors associated with ρ. We shall show that, in general, the elementary divisors do not depend on these sets alone, but that necessary and sufficient conditions may be found in terms of Σ and S (*a*) for the elementary divisors associated with ρ to be simple, and (*b*) that there is only one elementary divisor associated with ρ.

The square matrix $A = [a_{ij}]$ is called an M-matrix [1] if (1) $a_{ii} \geqq 0$ for all i; (2) $a_{ij} \leqq 0$ when $i \neq j$; and (3) all non-zero characteristic roots of A have positive real part. If $P = [p_{ij}]$ is a square matrix with non-negative coefficients and ρ is its greatest non-negative characteristic root, then $\rho \geqq p_{ii}$, for all i (O. Taussky [7]). Hence $\rho I - P$ is a singular M-matrix. Conversely, if A is a singular M-matrix and $\rho \geqq m_{ii}$ for all i, then $\rho I - A$ is a matrix with non-negative elements. Thus it is equivalent, and rather more convenient, to study the elementary divisors associated with the characteristic root 0 of a singular M-matrix.

2. We shall now explain our notation and terminology, which differ in some respects from the usual ones. We introduce a partial ordering on a set of conformable matrices with real coefficients by setting A ($\geqq B$ if $a_{ij} \geqq b_{ij}$ for all i, j, where $A = [a_{ij}]$ and $B = [b_{ij}]$. A second partial ordering is introduced by setting $A \geqq B$, if *either*

[1] The term M-determinant was used by A. Ostrowski [4], [5]. It has been proved [6], p. 19, that our definition is equivalent to Ostrowski's.

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$a_{ij} > b_{ij}$ for all i, j, *or* $A = B$. Expressions such as $A\,(> B$, and $A <)\, B$ then have their natural meanings. If $A > 0$, we call A "strictly positive," if $A\,(> 0$ we call A "weakly positive" or just "positive." We shall similarly call A negative if $A <)\, 0$.

The notation used by previous authors (cf. Frobenius [2], Wielandt [8], and others) is less convenient and a little less satisfactory logically. It obscures the fact that we are continually dealing with *two* partial orderings. While $A > B$ has the same meaning in both notations, these authors use $A \geqq B$ in place of $A\,(\geqq B$. Where we may write "$A\,(> 0$," they would have to write "$A \geqq 0$ but $A \neq 0$." It is surely unfortunate, also, that in their notation "$A \geqq B$" is not equivalent to "either $A > B$ or $A = B$."

We note that if $A = [a]$ is 1×1, then $A > 0$ is equivalent to $A\,(> 0$, and if A is identified with a then $A > 0$ has its usual meaning.

Column and row vectors may be regarded as matrices, and the same notation will be employed there.

3. Our principal results will be enumerated in terms of the numbers $R_{ij}(A, P)$ defined below. Let A be a square matrix [1] and let P be the diagonally symmetric partition $[A_{ij}]$, $i, j = 1, \ldots, k$. For $i, j = 1, \ldots, k$ we set

$$r_{ij}(A, P) = 0 \text{ if } i \neq j \text{ and } A_{ij} = 0,$$

and

$$r_{ij}(A, P) = 1 \text{ if } i = j, \text{ or if } A_{ij} \neq 0.$$

Where no confusion can arise we shall write r_{ij} for $r_{ij}(A, P)$. Next we set

$$R_{ij}(A, P) = \max r_{ih}\, r_{hl} \ldots r_{nj},$$

the maximum being taken over all sequences $(i, h, \ldots, n, j)$. Again we shall generally write R_{ij} for $R_{ij}(A, P)$. For future reference we note that

$$\text{either } R_{ij} = 0 \text{ or } R_{ij} = 1;$$

$$R_{ii} = 1 \text{ for } i = 1, \ldots, k;$$

$$\sum_{h=1}^{k} R_{ih} R_{hj} \geqq R_{ij} \geqq R_{il} R_{lj}, \qquad 1 \leqq l \leqq k; \tag{1}$$

$$\sum_{h=1,\, h \neq i}^{k} r_{ih} R_{hj} \geqq R_{ij} \geqq \max_{h \neq i} r_{ih} R_{hj} \qquad \text{if } i \neq j. \tag{2}$$

[1] The field of the coefficients of A is here immaterial. But in the remaining sections we shall assume that all matrices occurring have real coefficients.

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If A is a square matrix, then there exists a permutation matrix, T, for which $A^* = T^{-1} AT$ can be partitioned $[A^*_{ij}]$, $i, j = 1, \ldots, k$ such that (1) $A^*_{ij} = 0$ if $i < j$, (2) A^*_{ii}, $i = 1, \ldots, k$, is irreducible. We shall call A^* a standard form of A, and we shall say that A^* is in standard form. In general, the standard form of A is not unique. If A^0 is a standard form of A, and A^0_{ii}, $i = 1, \ldots, l$, are the irreducible matrices in its diagonal, then $l = k$, and $A^0_{\sigma(i)\sigma(j)} = T^{-1}_i A^*_{ij} T_j$, where σ is a permutation of $(1, \ldots, k)$ and $T_1, \ldots, T_k$ are permutation matrices. Thus there exists a one-one correspondence between the irreducible submatrices in the diagonal of any two standard forms such that corresponding submatrices have the same characteristic roots. In particular, all standard forms have the same number of singular irreducible submatrices in the diagonal.

In view of what is to follow we shall examine the connection between the $R_{ij}(A, P)$ and a standard form of A.

LEMMA 1. *Let P be the partition $[A_{ij}]$, $i, j = 1, \ldots, k$ of the square matrix A such that the A_{ii} in the diagonal are irreducible. Then A is in standard form if and only if $R_{ij} = 0$ whenever $i < j$.*

Proof. We must show that '$R_{ij} = 0$ whenever $i < j$' is equivalent to '$r_{ij} = 0$ whenever $i < j$'. Clearly '$R_{ij} = 0$ whenever $i < j$' implies '$r_{ij} = 0$ whenever $i < j$'. To prove the converse we note that if $i < j$, then any sequence $(i, h, \ldots, n, j)$ contains two consecutive members l, m such that $l < m$. The lemma follows from the definition of R_{ij}.

THEOREM 1. *Let P be the partition $[A_{ij}]$, $i, j = 1, \ldots, k$ of A, where the A_{ii} are irreducible, and let A be in standard form. Let $1 \leqslant \alpha, \beta \leqslant k$. There exists a permutation σ of $(1, \ldots, k)$ for which $A^* = [A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form and $\sigma(\beta) < \sigma(\alpha)$ if and only if $R_{\beta\alpha}(A, P) = 0$.*

Proof. Let P^* be the partition $[A_{\sigma(i)\sigma(j)}]$ of A^* and put $R^*_{ij} = R_{ij}(A^*, P^*)$. We have $R^*_{\sigma(i)\sigma(j)} = R_{ij} = R_{ij}(A, P), i, j = 1, \ldots, k$. Hence by Lemma 1, if A^* is in standard form and $\sigma(\beta) < \sigma(\alpha)$, then $R_{\beta\alpha} = R^*_{\sigma(\beta)\sigma(\alpha)} = 0$.

Conversely let $R_{\beta\alpha} = 0$. Since by (1) $R_{\beta\alpha} \geqslant R_{\beta i} R_{i\alpha}$ it follows that $R_{\beta i} R_{i\alpha} = 0$ for $i = 1, \ldots, k$. Hence we may partition $(1, \ldots, k)$ into three sets E_1, E_2, E_3 so that $i \in E_1$ if $R_{\beta i} = 1$ and $R_{i\alpha} = 0$; $i \in E_2$ if $R_{\beta i} = R_{i\alpha} = 0$; and $i \in E_3$ if $R_{\beta i} = 0$ and $R_{i\alpha} = 1$. Let σ be the permutation of $(1, \ldots, k)$ for which $\sigma(i) < \sigma(j)$ if $i < j$ and $i \in E_\lambda$, $j \in E_\mu$, with

$\lambda < \mu$; while $\sigma(i) < \sigma(j)$ if $i, j \epsilon E_\lambda$ and $i < j$, where $\lambda = 1, 2, 3$. Let P^* be the partition $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, of A^*. Let $R^*_{ij} = R_{ij}(A^*, P^*)$. Let $i \epsilon E_\lambda$, $j \epsilon E_\mu$, and suppose that $\sigma(i) < \sigma(j)$. Then $\lambda \leqq \mu$. If $\lambda = \mu$, then $i < j$ so that by Lemma 1, $R^*_{\sigma(i)\sigma(j)} = R_{ij} = 0$. If, on the other hand, $\lambda < \mu$, then either $\lambda = 1$ or $\mu = 3$. If $\lambda = 1$, then $R_{\beta i} = 1$, whence $R_{ij} = R_{\beta i} R_{ij} \leqq R_{\beta j} = 0$, since $\mu \geqq 2$. If $\mu = 3$, then $R_{ja} = 1$, whence $R_{ij} = R_{ij} R_{ja} \leqq R_{ia} = 0$, since $\lambda \leqq 2$. We conclude that $R^*_{\sigma(i)\sigma(j)} = R_{ij} = 0$ whenever $\sigma(i) < \sigma(j)$. Thus, by Lemma 1, A^* is in standard form. We need now only prove that $\sigma(\beta) < \sigma(a)$. But $R_{\beta\beta} = 1$, $R_{\beta a} = 0$, $R_{aa} = 1$ imply that $\beta \epsilon E_1$, $a \epsilon E_3$, and the result follows.

4. We now turn to the consideration of M-matrices. If the matrix A is partitioned $[A_{ij}]$, $i, j, = 1, \ldots, k$, we shall assume any column vector x to be conformably partitioned into $(x_1, \ldots, x_k)$.

LEMMA 2. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$ be an M-matrix in standard form. Let $x = (x_1, \ldots, x_k)$ and let*

$$\left.\begin{array}{l} x_i = 0 \text{ when } R_{ia} = 0 \\ x_i > 0 \text{ when } R_{ia} = 1, \end{array}\right\} \tag{3}$$

for $i = 1, \ldots, h-1$, where $h > a$. If

$$\left.\begin{array}{l} y_1 = 0 \\ y_i = -\sum_{j=1}^{i-1} A_{ij} x_j,\ i = 2, \ldots, k \end{array}\right\} \tag{4}$$

then

$$\left.\begin{array}{l} y_h = 0 \quad \text{if } R_{ha} = 0 \\ y_h (> 0 \quad \text{if } R_{ha} = 1 \end{array}\right\}. \tag{5}$$

Proof. Clearly $y_h (\geqq 0$; and $y_h = 0$ if and only if $A_{hj} x_j = 0$ for $j = 1, \ldots, h-1$, since $A_{hj} x_j \leqq) 0$ for $j = 1, \ldots, h-1$. Hence, by the assumptions about the x_j, $y_h = 0$ if and only if

$$r_{hj} R_{ja} = 0, j = 1, \ldots, h-1. \tag{6}$$

Since $r_{hj} = 0$ when $h < j$ we have $\sum_{j=1}^{h-1} r_{hj} R_{ja} = \sum_{j=1, j \neq h}^{k} r_{hj} R_{ja}$ and $\max_{j<h} r_{hj} R_{ja} = \max_{j \neq h} r_{hj} R_{ja}$. Since $h \neq a$, it can now easily be shown from (2) that (6) holds if and only if $R_{ha} = 0$. The lemma follows.

THEOREM 2. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$ be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. If $a \epsilon S$, and*

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$R_{\beta\alpha} = 0$ *whenever* $\beta \in S$, $\beta \neq \alpha$, *then there exists a positive characteristic column vector* x *of* A *associated with* 0 *satisfying* (3) *for* $i = 1, \ldots, k$.

Proof. Let x be any column vector, and let y satisfy (4). Then $Ax = 0$ if and only if

$$A_{ii}x_i = y_i \tag{7}$$

for $i = 1, \ldots, k$.

Now let $x_i = 0$ when $i < \alpha$. The singular irreducible M-matrix $A_{\alpha\alpha}$ has a strictly positive characteristic vector x_α associated with 0, cf. [2], [4]. As A is in standard form, $R_{i\alpha} = 0$ when $i < \alpha$, $R_{\alpha\alpha} = 1$, and therefore $x_1, \ldots, x_\alpha$ satisfy (3).

Let us suppose inductively that $x_1, \ldots, x_{h-1}$, $h > \alpha$, satisfy (3). If $y_1, \ldots, y_h$ satisfy (4), then y_h also satisfies (5) by Lemma 2. Thus if $R_{h\alpha} = 0$ then $y_h = 0$; and so if $x_h = 0$ then x_h satisfies (7) for $i = h$. If $R_{h\alpha} = 1$, then $y_h (> 0$, and by assumption A_{hh} is non-singular. It is known that the inverse of a non-singular irreducible M-matrix is strictly positive ([2], [4]). Hence if $x_h = A_{hh}^{-1} y_h$, then $x_h > 0$, and x_h satisfies (7). We have thus constructed a vector x_h satisfying (3) and (7), for $i = h$. The theorem follows by induction.

For the sake of completeness we shall prove the well-known Corollary 1.

COROLLARY 1. *A singular M-matrix has a positive characteristic vector associated with* 0.

Proof. Let α be the largest member of S. Then $R_{\beta\alpha} = 0$, whenever $\beta \in S$, $\beta \neq \alpha$, and the corollary follows from Theorem 2.

It is also convenient to state Corollary 2 at this point.

COROLLARY 2. *Let* $\gamma_1, \ldots, \gamma_s$ *be the members of* S. *If* $R_{\beta\alpha} = 0$ *whenever* $\alpha, \beta \in S$, $\alpha \neq \beta$, *then* A *has* s *linearly independent characteristic column vectors* $x^1, \ldots, x^s$ *associated with* 0, *where* x^j *satisfies* (3) *with* $\alpha = \gamma_j$.

Proof. Theorem 2 shows the existence of the characteristic vectors $x^j, j = 1, \ldots, s$, satisfying (3) with $\alpha = \gamma_j$. Suppose that $\sum_{h=1}^{s} \lambda_h x^h = 0$. Then, for $i = 1, \ldots, k$, we have $\sum_{h=1}^{s} \lambda_h x_i^h = 0$. Let $\alpha = \gamma_j$. Since $R_{\beta\alpha} = 0$ when $\beta = \gamma_h$, $h \neq j$, it follows that $x_\alpha^h = 0$ if $h \neq j$. Hence $\lambda_j x_\alpha^j = 0$; and $x^j \neq 0$ now implies $\lambda_j = 0$. The linear independence of $x^1, \ldots, x^s$ follows.

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5. If $x = (x_1, .., x_k)$ we shall call x_i the ith vector component of x.

LEMMA 3. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $\gamma_1, \ldots, \gamma_s$, where $\gamma_{j-1} < \gamma_j$, be the members of the set S of indices of singular A_{ii}. If there exist m linearly independent characteristic vectors of A associated with 0, then for each integer n, $n \leqq m$, there are at least n of these vectors such that the ith vector component is non-zero for some $i \leqq \gamma_{n+s-m}$.*

Proof. If $\gamma_{n+s-m} = k$ there is nothing to prove. So let $\gamma_{n+s-m} < k$ and suppose that $x^1, \ldots, x^m$ are linearly independent characteristic vectors associated with 0, such that $x^j_i = 0$, for $i = 1, \ldots, \gamma_{n+s-m}$ and $j = n, \ldots, m$. If $\mu = \gamma_{n+s-m} + 1$, the vectors $(x^j_\mu, \ldots, x^j_k)$, $j = n, \ldots, m$, form $n - m + 1$ linearly independent characteristic vectors associated with 0 of the matrix $B = [A_{ij}]$, $i, j = \mu, \ldots, k$. But the multiplicity of 0 in B equals the number of singular A_{ii} in B, and so equals $m - n$. This yields a contradiction, and the lemma follows.

LEMMA 4. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $\gamma_1, \ldots, \gamma_s$, where $\gamma_{j-1} < \gamma_j$, be the members of S. If A has s linearly independent characteristic column vectors associated with 0, then there exists a set $x^1, \ldots, x^s$ of such vectors for which*

$$\left.\begin{aligned} x^j_i &= 0 \quad \text{if } i < \gamma_j \\ \text{but} \quad x^j_i &\neq 0 \quad \text{if } i = \gamma_j \end{aligned}\right\} \tag{8}$$

for $j = 1, \ldots, s$.

Proof. *Let* $z^1, \ldots, z^s$ be linearly independent characteristic vectors associated with 0. If $z^j_i = 0$ if $i < \delta_j$ but $z^j_i \neq 0$ if $i = \delta_j$, then $A_{ii} z^j_i = 0$ for $i = \delta_j$. Hence $\delta_j \epsilon S$. Thus $z^j_i = 0$ if $i < \gamma_1$, for $j = 1, \ldots, s$. It also follows from Lemma 3, with $m = s$, $n = 1$, that for some j we have $z^j_i \neq 0$, if $i = \gamma_1$. We may therefore assume inductively that we have linearly independent characteristic vectors $x^1, \ldots, x^n, z^{n+1}, \ldots, z^s$, associated with 0, such that (a) (8) holds for $j = 1, \ldots, n$ and (b) $z^j_i = 0$ if $i < \gamma_n$, for $j = n + 1, \ldots, s$. Let $\alpha = \gamma_n$. Then $A_{\alpha\alpha} x^n_\alpha = A_{\alpha\alpha} z^j_\alpha = 0$, $j = n + 1, \ldots, s$. Since an irreducible singular

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M-matrix has only one linearly independent characteristic column vector associated with 0, it follows that $z^j_\alpha = \lambda_j x^n_\alpha$, $j = n+1, \ldots, s$. Let $x^j = z^j - \lambda_j x^n, j = n+1, \ldots, s$. The vectors $x^1, \ldots, x^s$ are linearly independent characteristic vectors associated with 0, and, for $j = n+1, \ldots, s$, $x^j_i = 0$ if $i \leqq \gamma_n$. Hence by the remark at the beginning of the proof, $x^j_i = 0$ if $i < \gamma_{n+1}$, for $j = n+1, \ldots, s$. It follows from Lemma 3 that there is a $j \geqq n+1$ such that $x^j_i \neq 0$, if $i = \gamma_{n+1}$. Suppose this $j = n+1$. Then (*a*) (8) holds for $j=1, \ldots, n+1$, and (*b*) $x^j_i = 0$ if $i < \gamma_{n+1}$, for $j = n+2, \ldots, s$. The lemma follows by induction.

6. The following lemma is of some interest in itself. It is related to a theorem of Collatz [1], and other results on positive irreducible matrices.

LEMMA 5. *Let A be an irreducible singular M-matrix and let Ax ($\geqq$ 0 or 0) $\geqq Ax$. Then $Ax = 0$.*

Proof. Let Ax ($\geqq$ 0 or $Ax \leqq$) 0, and let u' be the strictly positive characteristic row vector of A, associated with 0. If either z (> 0 or z <) 0, then either $u'z > 0$ or $u'z < 0$. Hence $Ax = 0$.

We now come to one of our main theorems.

THEOREM 3. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. The elementary divisors associated with the characteristic root 0 are all linear if and only if $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$.*

Proof. Let S have the s members $\gamma_1, \ldots, \gamma_s$ where $\gamma_{j-1} < \gamma_j$. The elementary divisors associated with 0 are all linear if and only if that characteristic root has s linearly independent characteristic vectors associated with it.

If $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$ then by Corollary 2 to Theorem 2, 0 has s linearly independent characteristic vectors associated with it. Suppose, conversely, that 0 has the s linearly independent characteristic vectors $x^1, \ldots, x^s$ associated with it. By Lemma 3, we may assume that $x^1, \ldots, x^s$ satisfy (S). Let us assume that for some $\alpha, \beta \in S$, $\alpha \neq \beta$, we have $R_{\beta\alpha} = 1$. We shall obtain a contradiction. We may choose α, β so that $R_{\beta\alpha} = 1, \beta > \alpha$ and $\beta - \alpha \leqq \beta' - \alpha'$ for all $\alpha', \beta' \in S$, $\alpha' \neq \beta'$, for which $R_{\beta'\alpha'} = 1$. If $\alpha \leqslant \gamma \leqq \delta \leqq \beta$ and $\gamma, \delta \in S$ then $R_{\delta\gamma} = 0$ unless $\gamma = \alpha$ and $\delta = \beta$. Let

B be the matrix $[A_{ij}]$, $i, j = \alpha, \ldots, \beta - 1$. Let $\delta_1, \delta_2, \ldots, \delta_r$ be the indices of the singular A_{ii} of B in ascending order of magnitude. Thus $\delta_1 = \alpha = \gamma_j$, say. We deduce from Corollary 2 to Theorem 2 that B has r linearly independent characteristic vectors $(z^h_\alpha, \ldots, z^h_{\beta-1})$, $h = 1, \ldots, r$, associated with 0, where z^h_i satisfies (3) (provided we replace α by δ_h there), for $i = \alpha, \ldots, \beta - 1$. Since the multiplicity of 0 in B is r, any characteristic vector of B associated with 0 is a linear combination of these. Since $\alpha = \gamma_j$, $(x^j_\alpha, \ldots, x^j_{\beta-1})$ is a characteristic vector of B associated with 0. Hence $x^j_i = \sum_{h=1}^{r} \lambda_h z^h_i$, for $i = \alpha, \ldots, \beta - 1$. Further, $\lambda_1 \neq 0$ since $x^j_\alpha \neq 0$ but $z^h_\alpha = 0$ if $h = 2, \ldots, r$. It follows that $A_{\beta\beta} x^j_\beta = \sum_{h=1}^{r} \lambda_h y^h_\beta$ where $y^h_\beta = -\sum_{i=\alpha}^{\beta-1} A_{\beta i} z^h_i$, $h = 1, \ldots, r$. On putting $z^h_i = 0$ when $i = 1, \ldots, \alpha - 1$, we obtain from Lemma 2 that $y^h_\beta \geq 0$ if $R_{\beta\gamma} = 1$, but $y^h_\beta = 0$ if $R_{\beta\gamma} = 0$, where $\gamma = \delta_h$. Hence $y^1_\beta \geq 0$, but $y^h_\beta = 0$, for $h = 2, \ldots, r$. Thus $A_{\beta\beta} x^j_\beta = \lambda_1 y'_\beta$, and so either $A_{\beta\beta} x^j_\beta \geq 0$ or $0 \geq A_{\beta\beta} x^j_\beta$. But this is not possible, by Lemma 5. It follows that $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$. The theorem is proved.

In view of Theorem 1 we obtain immediately

COROLLARY 1. *The elementary divisors associated with* 0 *are all linear if and only if for each* $\alpha \in S$ *there exists a permutation* σ *of* $(1, \ldots, k)$ *such that* $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, *is in standard form, and* $\sigma(\beta) \leqq \sigma(\alpha)$ *for any* $\beta \in S$.

The square matrix $A = [a_{ij}]$ is called *Minkowskian* if (1) $a_{ii} \geqq 0$ for all i, (2) $a_{ij} \leqq 0$, when $i \neq j$, and (3) $\sum_j a_{ij} \geqq 0$ for all i. A Minkowskian matrix is an M-matrix: cf. [4], [6].

COROLLARY 2. *Let* A *be a singular Minkowskian matrix. The elementary divisors associated with* 0 *are all linear.*

Proof. We may assume that $A = [A_{ij}]$, $i, j = 1, \ldots, k$, is in standard form. Let S be as above. If $C = [c_{ij}]$ is an irreducible Minkowskian matrix, then C is singular if and only if $\sum_j c_{ij} = 0$ for all i, [4]. Hence if $\alpha \in S$, then $A_{\alpha j} = 0$, provided that $j \neq \alpha$. It follows that $R_{\alpha j} = 0$, if $j \neq \alpha$. The corollary now follows from Theorem 2.

In the case when $\sum_j a_{ij} = 0$ for all i, this result has already been proved by Ledermann [3].

7. Results similar to those we have found for characteristic column vectors may be stated for characteristic row vectors. Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be in standard form. The transposed matrix $A' = [A'_{ij}]$, $i, j, = 1, \ldots, k$ (where $A'_{ij} = (A_{ji})'$) is not necessarily in standard form. However, if σ is the permutation for which $\sigma(i) = k + 1 - i$, $i = 1, \ldots, k$, then $B = [A'_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form. Let P, P', and Q be the partition described above of A, A' and B respectively. Then $R_{\sigma(j)\sigma(i)}(B, Q) = R_{ji}(A', P') = R_{ij}(A, P)$. To any characteristic row vector $(u'_1, \ldots, u'_k)$ of A, associated with 0, there corresponds the characteristic column vector $(u_{\sigma(1)}, \ldots, u_{\sigma(k)})$ of B associated with 0. We may deduce Theorem 2*a* and Corollary 2*a*, from Theorem 2 and Corollary 2.

THEOREM 2*a*. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. If $\alpha \in S$ and $R_{\alpha\beta} = 0$ whenever $\beta \in S$* and *$\beta \neq \alpha$, then there exists a positive characteristic row vector $u' = (u_1', \ldots, u'_k)$ associated with* 0, *satisfying*

$$\left.\begin{array}{ll} u_i = 0 & \text{when } R_{\alpha i} = 0 \\ u_i > 0 & \text{when } R_{\alpha i} = 1 \end{array}\right\}, \qquad (9)$$

for $i = 1, \ldots, k$.

COROLLARY 2*a*. Let *$\gamma_1, \ldots, \gamma_s$ be the members of S. If $R_{\alpha\beta} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$ then A has s linearly independent characteristic row vectors $u'^1, \ldots, u'^s$ associated with* 0, *where u'^j satisfies* (9) *with $\alpha = \gamma_j$.*

THEOREM 4. *Let A be a singular M-matrix. If the elementary divisors associated with the characteristic root* 0 *are all linear, then the principal idempotent element associated with* 0 *is positive.*

Proof. Let C be a matrix whose characteristic root ω, of multiplicity s, has only linear elementary divisors associated with it. There exist linearly independent characteristic column vectors $x^1, \ldots, x^s$ and linearly independent characteristic row vectors $u'^1, \ldots, u'^s$ associated with ω, such that $u'^h x^j = \delta_{hj}$, $h, j = 1, \ldots, s$, the Kronecker delta. The principal idempotent element associated with ω is the matrix $\sum_{h=1}^{s} x^h u'^h$.

Let P^* be the partition $A^* = [A^*_{ij}]$, $i, j = 1, \ldots, k$, of a standard form of the singular M-matrix A with only linear elementary divisors associated with 0, and let $S = (\gamma_1, \ldots, \gamma_s)$ be the set of indices of singular A^*_{ii}. By Theorem 3, $R^*_{\beta a} = R_{\beta a}(A^*, P^*) = 0$, when $a, \beta \epsilon S$, $a \neq \beta$. Hence by Corollary 2 to Theorem 2 there exist linearly independent characteristic column vectors $x^1, \ldots, x^s$ associated with 0 such that x^j satisfies (3) with $a = \gamma_j$. Similarly, by Corollary 2a to Theorem 2a there exist linearly independent characteristic row vectors $u'^1, \ldots, u'^s$, associated with 0, such that u'^j satisfies (9) with $a = \gamma_j$. Let $a = \gamma_j$ and $\beta = \gamma_h$. Since $u'^h x^j = \sum_{i=1}^{k} u'^h_i \; x^j_i$ and $u'^h_i x^j_i = 0$ if and only if $R^*_{\beta i} R^*_{ia} = 0$ it follows that $u'^h x^j \geqslant 0$, and that $u'^h x^j = 0$ if and only if $\sum_{i=1}^{k} R^*_{\beta i} R^*_{ia} = 0$. But we may deduce from (1) that $\sum_{i=1}^{k} R^*_{\beta i} R^*_{ia} = 0$ if and only if $R^*_{\beta a} = 0$. Hence $u'^h x^j = 0$ if $h \neq j$, but $u'^j x^j > 0$, for $h, j = 1, \ldots, s$. We may clearly assume that $u'^1, \ldots, u'^s$ have been multipled by positive factors so that $u'^h x^j = \delta_{hj}$. Then the idempotent element of A^* associated with 0 is $E^* = \sum_{j=1}^{s} x^j u'^j$. Thus $E^* (\geqq 0$. If E is partitioned conformably with A, then $E_{ii} = \sum_{h=1}^{s} x^h_i \, u'^h_i$, $i = 1, \ldots, k$. Since $u'^j_a > 0$, and $x^j_a > 0$, but $u'^h_a = 0$, and $x^h_a = 0$, when $h \neq j$, it follows that $E^*_{aa} = x^h_a \, u'^h_a > 0$. Hence $E^* (> 0$.

The principal idempotent element E of A associated with 0 is obtained from that of A^* by means of a transformation by a permutation matrix. Hence E, too, is positive.

8. We have already remarked that the elementary divisors of a matrix A, associated with the characteristic root 0 of multiplicity s, are all linear if and only if there are s linearly independent characteristic vectors associated with 0. In the next two sections we shall discuss the other extreme case when there is only one elementary divisor associated with the characteristic root 0 of a singular M-matrix. Equivalent conditions are (a) that 0 has only one linearly independent characteristic vector associated with it; or (b) that there exists a set $x^1, \ldots, x^s$ of column vectors such that

$$Ax^j = x^{j+1}, j = 1, \ldots, s-1; \; Ax^s = 0, \text{ and } x^s \neq 0, \tag{10}$$

where s is again the multiplicity of 0.

LEMMA 6. *Let* $A = [A_{ij}]$, $i, j = 1, \ldots, k$, *be a singular M-matrix in standard form. Let* $S = (\gamma_1, \ldots, \gamma_s)$, *where* $\gamma_{j-1} < \gamma_j$, *be the set of indices of singular* A_{ii}. *If* $x^1, \ldots, x^s$ *is a set of column vectors satisfying* (10), *then* x^j, $j = 1, \ldots, s$, *satisfies* (8).

Proof. For $j = 1, \ldots, s$, let $x^j_i = 0$ if $i < \delta_j$, but $x^j_i \neq 0$ if $i = \delta_j$. Since $Ax^s = 0$, it follows, as in the proof of Lemma 4, that $\delta_s \epsilon S$. Let us assume inductively that (*a*) $\delta_j \epsilon S$ for $j = h, \ldots, s$; and that (*b*) $\delta_{j-1} < \delta_j$ for $j = h+1, \ldots, s$. These assumptions hold for $h = s$. If y^j_i satisfies (4), when $x_i = x^j_i$, then $Ax^j = x^{j+1}$ if and only if

$$A_{ii}x^j_i = y^j_i + x^{j+1}_i, \tag{11}$$

for $i = 1, \ldots, k$. Let $\beta = \delta_h$. Then $y^h_\beta = 0$; and we may deduce from (*b*) that $x^{h+1}_\beta = 0$. Hence (11) holds for $i = \beta$ and $j = h$ if and only if $A_{\beta\beta}x^h_\beta = 0$. Hence either $x^h_\beta > 0$ or $x^h_\beta < 0$. But $A_{\beta\beta}x^{h-1}_\beta = y^{h-1}_\beta + x^h_\beta$. It follows by Lemma 5 that (*a*) and (*b*) imply that $y^{h-1}_\beta \neq 0$. Thus $\delta_{h-1} < \beta = \delta_h$. We deduce that if $\alpha = \delta_{h-1}$ then $x^h_\alpha = y^{h-1}_\alpha = 0$. Hence by (11) $A_{\alpha\alpha}x^{h-1}_\alpha = 0$, and so $\delta_{h-1}\epsilon S$. By induction we obtain that $\delta_j \epsilon S, j = 1, \ldots, s$, and that $\delta_{j-1} < \delta_j, j = 2, \ldots, s$. Hence $\delta_j = \gamma_j$, $j = 1, \ldots, s$ and the lemma is proved.

9. LEMMA 7. *Let A be an irreducible M-matrix. Let z and y, where* $y \,(> 0$, *be column vectors conformable with A. Then there exist a real* λ *and a column vector x such that* $Ax = \lambda y + z$.

Proof. If A is non-singular, then there exists such an x for any λ. Suppose A singular. Since 0 is a simple characteristic root of A, the nullity of A is 1. By Lemma 5, y is linearly independent of the columns of A. Hence any column vector is a linear combination of the columns of A and y. The lemma follows.

THEOREM 5. *Let* $A = [A_{ij}]$, $i, j = 1, \ldots, k$, *be a singular M-matrix in standard form. Let S be the set of indices of singular* A_{ii}. *There is only one elementary divisor associated with the characteristic root* 0 *of A if and only if* $R_{\beta\alpha} = 1$, *whenever* $\alpha, \beta \epsilon S$ *and* $\beta > \alpha$.

Proof. Suppose that there is only one elementary divisor associated with 0. Let $x^1, \ldots, x^s$ be a set of column vectors satisfying (10). By Lemma 6, x^j satisfies (8), $j = 1, \ldots, s$. Let $\alpha = \gamma_{h-1}$, $\beta = \gamma_h$. The conditions (a) and (b) of the proof of Lemma 6 are clearly satisfied, and so $y_\beta^{h-1} = -\sum_{i=1}^{\beta-1} A_{ii}x_i^{h-1} = -\sum_{i=a}^{\beta-1} A_{ii}x_i^{h-1} \neq 0$. But $(x_a^{h-1}, \ldots, x_{\beta-1}^{h-1})$ is the one linearly independent characteristic column vector associated with 0 of $B = [A_{ij}]$, $i, j = a, \ldots, \beta - 1$, since $x_i^h = 0$, for $i = a, \ldots, \beta - 1$. It therefore follows from Theorem 2 that x_i^h, $i = a, \ldots, \beta - 1$, satisfies (3), provided that $x^1, \ldots, x^s$ have been multiplied by -1 if necessary. Therefore x_i^h, $i = 1, \ldots, \beta - 1$, satisfy (3). Hence, by Lemma 2, it follows from $y_\beta^{h-1} \neq 0$ that $R_{\beta a} = 1$, when $a = \gamma_{h-1}$, $\beta = \gamma_h$. This is a particular case of the required result. To deduce the general case, let $a = \gamma_j$, $\beta = \gamma_h$, where $h > j$. Then $R_{\beta a} \geqslant R_{\gamma_h \gamma_{h-1}} \ldots R_{\gamma_{j+1}\gamma_j} = 1$. Hence $R_{\beta a} = 1$, whenever a, $\beta \epsilon S$ and $\beta > a$.

Conversely, let us suppose that $R_{\beta a} = 1$ whenever a, $\beta \epsilon S$ and $\beta > a$. By Theorem 2, Corollary 1, there exists a characteristic column vector x^s, associated with 0, which satisfies (8). Let us suppose that there exist column vectors x^j satisfying (8), $j = h, \ldots, s$, such that

$$Ax^j = x^{j+1},\ j = h, \ldots, s-1,\ Ax^s = 0,\ x^s \neq 0. \tag{12}$$

We shall construct a vector x^{h-1} satisfying (8) and $Ax^{h-1} = x^h$. Let $a = \gamma_{h-1}$, $\beta = \gamma_h$. Let $x_i^{h-1} = 0$, $i = 1, \ldots, a - 1$, and let $(x_a^{h-1}, \ldots, x_{\beta-1}^{h-1})$ be the one linearly independent characteristic vector, associated with 0, of $B = [A_{ij}]$, $i, j = a, \ldots, \beta - 1$. Then x_i^{h-1}, $i = 1, \ldots, \beta - 1$, satisfies (3) provided that the x_i^{h-1} have all been multiplied by -1, if necessary. If, for $i = 1, \ldots, \beta$, y_i^{h-1} is chosen to satisfy (4) with x replaced there by x^{h-1}, then it follows by Lemma 2 that $y_\beta^{h-1} (> 0$. Hence by Lemma 7, there exist a λ and an x_β^{h-1} such that $A_{\beta\beta}x_\beta^{h-1} = \lambda y_\beta^{h-1} + x_\beta^h$. We now write x_i^{h-1} for λx_i^{h-1}, $i = 1, \ldots, \beta - 1$, λy_i^{h-1} for y_i^{h-1}, $i = 1, \ldots, \beta$, and leave x_β^{h-1} unchanged. Since $x h_i = 0$, when $i < \beta$, x_i^{h-1}, $i = 1, \ldots, \beta$ satisfies (11) with $j = h - 1$.

Let us assume inductively that x_i^{h-1}, $i = 1, \ldots, l - 1$, satisfies (11), where $l > \beta$. We must consider two cases: $l \notin S$ and $l \epsilon S$. If $l \notin S$, then (11) is satisfied when $x_l^{h-1} = A_{ll}^{-1}(y_l^{h-1} + x_l^h)$. If $l \epsilon S$, say $l = \gamma_{h+m}$ (where clearly $m > 0$) then there exist, by Lemma 7, a λ and an x_l^{h-1} for which $A_{ll}x_l^{h-1} = y_l^{h-1} + x_l^h + \lambda x_l^{h+m}$, since either $x_l^{h+m} > 0$ or $x_l^{h+m} < 0$. We now replace x^j by $x^j + \lambda x^{j+m}$, $j = h, \ldots, s$, where, by convention, $x^{j+m} = 0$ if $j + m > s$. Then $x^h, \ldots, x^s$ again satisfy (12) and since the original $x_i^{h+m} = 0$ when $i < l$, it follows that $x_1^{h-1}, \ldots, x_l^{h-1}$

satisfy (11) with $j = h - 1$. By induction we obtain a vector $x^{h-1} = (x_1^{h-1}, \ldots, x_k^{h-1})$ satisfying (11) with $j = h - 1$. Thus in addition to (12) we have $Ax^{h-1} = x^h$, where x^{h-1} satisfies (8). Using induction again we obtain a set of vectors satisfying (10) and the theorem is proved.

In virtue of Theorem 1 we obtain the following corollary.

COROLLARY. *Let γ be the largest member of S. There is only one elementary divisor associated with the characteristic root 0 of A if and only if for any permutation σ of $(1, \ldots, k)$ for which $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form, we have $\sigma(\gamma) \geqslant \sigma(a)$ whenever $a \in S$.*

An argument along the lines of the second half of the proof of Theorem 5 would lead to the following result, which we shall enunciate as a theorem, though we shall omit the proof. First we should have to generalise Lemma 2.

THEOREM 6. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $S = (\gamma_1, \ldots, \gamma_s)$, $\gamma_{j-1} < \gamma_j$, be the set of indices of singular A_{ii}. If there is only one elementary divisor associated with 0 then there exist positive column vectors $z^1, \ldots, z^s$ such that $x^1, \ldots, x^s$ satisfy (10) if $x^j = (-1)^j z^j$, $j = 1, \ldots, s$. In fact*

$$z_i^h = 0 \text{ when } \sum_{j=h}^{s} R_{i\gamma_j} = 0,$$

and

$$z_i^h > 0 \text{ when } \sum_{j=h}^{s} R_{i\gamma_j} > 0.$$

The standard forms of A depend only on the set Σ of index pairs of non-zero coefficients of A. It is clearly decided by Σ and S whether (*a*) $R_{\beta a} = 0$, whenever $a, \beta \in S$, and $a \neq \beta$; or (*b*) $R_{\beta a} = 1$, whenever $a, \beta \in S$ and $\beta > a$. Let P be a positive matrix, with largest characteristic root ρ. The degrees of the elementary divisors of P associated with ρ are the same as those of the elementary divisors, associated with 0, of the M-matrix $\rho I - P$. Hence Theorems 3 and 5 fulfil the claims of § 1.

10. Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $S = (\gamma_1, \ldots, \gamma_s)$, $\gamma_{j-1} < \gamma_j$, be as usual, the set of indices of singular A_{ii}. If $s = 1$, then there is clearly only one elementary divisor, associated with 0, and it is linear. If $s = 2$, then there are two elementary divisors of degree 1, or one of degree 2, associated with 0, according as $R_{\beta a} = 0$ or $R_{\beta a} = 1$, where $a = \gamma_1$, and $\beta = \gamma_2$. Suppose that $s = 3$ and let $\gamma_1 = a$, $\gamma_2 = \beta$ and $\gamma_3 = \gamma$. Since

$R_{\alpha\beta} = R_{\alpha\gamma} = R_{\beta\gamma} = 0$, and since $R_{\gamma\beta} = R_{\beta\alpha} = 1$ implies that $R_{\gamma\alpha} = 1$, we may consider seven cases according as $R_{\gamma\beta}$, $R_{\gamma\alpha}$, $R_{\beta\alpha}$ are 1 or 0. The cases $R_{\gamma\beta} = R_{\gamma\alpha} = R_{\beta\alpha} = 0$, and $R_{\gamma\beta} = R_{\gamma\alpha} = R_{\beta\alpha} = 1$, are covered by Theorems 3 and 5. We deduce that in the five other cases 0 must have associated with it one elementary divisor of degree 1 and one of degree 2. This means that A has two linearly independent column vectors z^1, z^2 associated with 0, and that there exists a vector x satisfying either $Ax = z^1$ or $Ax = z^2$. By considering each of the five cases separately it is possible to demonstrate the existence of these vectors without any appeal to Theorems 3 and 5.

As an example we shall consider the case $R_{\gamma\beta} = R_{\gamma\alpha} = 1$, $R_{\beta\alpha} = 0$. Though we shall use the theorems proved previously it would be possible to use special cases of these results which could be proved more simply. It follows from Theorem 5, applied to $[A_{ij}]$, $i, j = \beta, \ldots, k$, that A has a characteristic vector x^3, associated with 0, for which $x^3_i = 0$ when $i < \gamma$, and that there exists an x^2 satisfying $Ax^2 = x^3$. Let $(z^j_1, \ldots, z^j_{\gamma-1})$, $j = 1, 2$, be the two linearly independent characteristic column vectors, associated with 0, of $[A_{ij}]$, $i, j = 1, \ldots, \gamma - 1$. The existence of these vectors is shown by Theorem 3. By Theorem 1, Corollary 2, we may assume that $z^j_i = 0$ if $R_{i\gamma_j} = 0$ and $z^j_i > 0$ if $R_{i\gamma_j} = 1$, for $j = 1, 2$. Hence, by Lemma 2, $w^j_\gamma (> 0, j = 1, 2$, where $w^j_\gamma = -\sum_{h=1}^{\gamma-1} A_{\gamma h} z^j_h$. By Lemma 7, there exist a λ and an x^1_γ satisfying

$$A_{\gamma\gamma}\, x^1_\gamma = w^1_\gamma + \lambda\, w^2_\gamma \,. \tag{13}$$

Let $x^1_i = z^1_i + \lambda z^2_i$, $i = 1, \ldots, \gamma - 1$. Then $(x^1_1, \ldots, x^1_\gamma)$ is non-zero and, for $i = 1, \ldots, \gamma$, x^1_i satisfies (7) provided y^1_i is chosen to satisfy (4). Since A_{ii} is non-singular if $i > \gamma$, it is easy to establish the existence of a vector $x^1 = (x^1_1, \ldots, x^1_k)$ where x^1_i satisfies (7) for $i = 1, \ldots, k$. Thus x^1 and x^3 are characteristic column vectors associated with 0, which are linearly independent as $x^1_\alpha = z^1_\alpha + \lambda z^2_\alpha = z^1_\alpha > 0$, while $x^3_\alpha = 0$; and $Ax^2 = x^3$.

It follows from Lemma 5 that $\lambda < 0$ in (13). Hence $x^1_\beta = z^1_\beta + \lambda z^2_\beta = \lambda z^2_\beta < 0$, but $x^1_\alpha > 0$, as already noted. Hence x^1 is neither positive nor negative. It is easily established that this property is shared by one of any two linearly independent characteristic vectors, associated with 0, of the matrix A we have been considering. It is due to this that S and Σ do not necessarily completely determine the elementary divisors associated with 0 of a singular M-matrix when $s > 3$. Thus, in both B and C below, $k = 4$, $S = (1, 2, 3, 4)$, $s = 4$ and $R_{43} = R_{21} = 0$, $R_{42} = R_{41} = R_{32} = R_{31} = 1$.

$$B = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \end{bmatrix} \qquad C = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{bmatrix}.$$

But B is of rank 1, C of rank 2. Hence the elementary divisors associated with 0 differ for the two matrices. We also note that the principal idempotent element, associated with 0, of both B and C is the unit matrix, which is, of course, positive. It follows that the converse of Theorem 4 does not hold.

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