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A REMARK ON THE DERIVATIVE OF THE ONE-DIMENSIONAL HARDY–LITTLEWOOD MAXIMAL FUNCTION

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Dedicated to Professor Kôzô Yabuta on the occasion of his 60th birthday

J. Kinnunen proved that if $p > 1$, $d \leqslant 1$ and f is a function in the Sobolev space $W^{1,p}(\mathbf{R}^d)$, then the first order weak partial derivatives of the Hardy–Littlewood maximal function $\mathcal{M}f$ belong to $L^p(\mathbf{R}^d)$. We shall show that, when $d = 1$, Kinnunen's result can be extended to the case where $p = 1$.

1. RESULT

The derivative of the maximal function has been studied in, for example, Kinnunen [3], Kinnunen and Lindqvist [4] and Buckley [1].

For a locally integrable function f on $\mathbf{R}^d$, where $d \geqslant 1$, the Hardy–Littlewood maximal function $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \bigl|f(y)\bigr|\,dy, \tag{1}$$

where the supremum is taken over all cubes Q containing $x \in \mathbf{R}^d$. Here, $|Q|$ denotes the volume of the cube Q. The well-known theorem of Hardy, Littlewood and Wiener asserts the following. If $f \in L^p(\mathbf{R}^d)$, where $1 < p \leqslant \infty$, then $\mathcal{M}f \in L^p(\mathbf{R}^d)$ and

$$\|\mathcal{M}f\|_p \leqslant A_p \|f\|_p, \tag{2}$$

where the constant A_p depends only on p and the dimension d. If $f \in L^1(\mathbf{R}^d)$, then for every $\lambda > 0$

$$\Bigl|\bigl\{x \in \mathbf{R}^d : \mathcal{M}f(x) > \lambda\bigr\}\Bigr| \leqslant \frac{A}{\lambda}\|f\|_1,$$

where the constant A depends only on d. Recall that when $1 \leqslant p \leqslant \infty$, the Sobolev space $W^{1,p}(\mathbf{R}^d)$ consists of functions f in $L^p(\mathbf{R}^d)$ whose first order weak partial derivatives $D_i f$ belong to $L^p(\mathbf{R}^d)$, when $i = 1, 2, \ldots, d$.

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In [**3**], Kinnunen showed that if $f \in W^{1,p}\left(\mathbf{R}^d\right)$, where $1 < p < \infty$ and $d \geqslant 1$, then $\mathcal{M}f \in W^{1,p}\left(\mathbf{R}^d\right)$ and

$$\left|(D_i\mathcal{M}f)(x)\right| \leqslant (\mathcal{M}D_if)(x), \quad i = 1, 2, \ldots, d, \tag{3}$$

for almost every $x \in \mathbf{R}^d$. Equations (2) and (3) imply that

$$\|D_i\mathcal{M}f\|_p \leqslant A_p\|D_if\|_p \quad i = 1, 2, \ldots, d. \tag{4}$$

Kinnunen's method to prove (3) cannot be applied to the case where $p = 1$, since it depends on the L^p-boundedness of $\mathcal{M}$.

The purpose of this paper is to extend (4) to the case where $p = d = 1$. Notice that if $f \in W^{1,1}(\mathbf{R})$, then $\mathcal{M}f$ is a bounded function and hence is differentiable in the sense of distributions.

THEOREM 1. *If $f \in W^{1,1}(\mathbf{R})$, then the derivative of $\mathcal{M}f$ is an integrable function, and*

$$\left\|(\mathcal{M}f)'\right\|_1 \leqslant 2\|f'\|_1.$$

Kinnunen proved his results for the maximal function which is defined as the supremum taken over all balls centred at x. If one reads [**3**] carefully, then one sees that the corresponding results hold for the maximal function which is defined as (1).

2. Proof

A crucial point in our argument is to consider one-sided maximal functions. For a locally integrable function f on the line, define the one-sided maximal functions $\mathcal{M}_lf$ and $\mathcal{M}_rf$ by

$$\mathcal{M}_lf(x) = \sup_{s>0} \frac{1}{s} \int_{x-s}^{x} \left|f(y)\right| dy,$$
$$\mathcal{M}_rf(x) = \sup_{t>0} \frac{1}{t} \int_{x}^{x+t} \left|f(y)\right| dy.$$

The following relation is obvious,

$$\mathcal{M}f(x) = \max\left\{\mathcal{M}_lf(x), \mathcal{M}_rf(x)\right\}. \tag{5}$$

In the rest of this paper, we assume that $f \in W^{1,1}(\mathbf{R})$, and we shall state the results only for $\mathcal{M}_l$, but the corresponding results hold for $\mathcal{M}_r$ as well. Notice that if $f \in W^{1,1}(\mathbf{R})$, then (after adjusting on a set of measure zero) f may be taken to be continuous—and then f vanishes at infinity, for it is uniformly continuous and integrable. Notice further that then $\mathcal{M}_lf$ is continuous and vanishes at infinity (see the proof of Theorem 4.1 in [**3**]). Therefore, the set

$$E = \left\{x \in \mathbf{R} : \mathcal{M}_lf(x) > \left|f(x)\right|\right\}$$

is open and hence E can be written as

$$E = \bigcup_j I_j = \bigcup_j (\alpha_j,\ \beta_j),$$

where $(\alpha_j,\ \beta_j)$ are disjoint open intervals.

Lemma 2. *With the definitions above, the following hold.*

(a) *$\mathcal{M}_l f$ is a nonincreasing function on each I_j.*

(b) *$\mathcal{M}_l f$ is a locally Lipschitz function on each I_j. In particular, $\mathcal{M}_l f$ is an absolutely continuous function on each compact subinterval of I_j.*

Proof: (a) Take $K = [\alpha,\ \beta] \subset I_j$. It suffices to prove that $\mathcal{M}_l f$ is nonincreasing on K. By the continuity of $|f|$ and $\mathcal{M}_l f$ we have

$$\varepsilon \equiv \inf_{x \in K} \mathcal{M}_l f(x) - \left|f(x)\right| > 0.$$

By the uniform continuity of $|f|$ there exists $\delta > 0$ such that

$$\left|f(y)\right| < \left|f(x)\right| + \frac{\varepsilon}{2}, \quad x \in K,\ |y - x| \leqslant \delta. \tag{6}$$

The definition of ε and (6) imply that

$$\mathcal{M}_l f(x) = \sup_{s > \delta} \frac{1}{s} \int_{x-s}^{x} \left|f(y)\right| dy, \quad x \in K. \tag{7}$$

We shall see that

$$\mathcal{M}_l f(x - h) \geqslant \mathcal{M}_l f(x), \quad x - h, x \in K,\ 0 < h \leqslant \delta. \tag{8}$$

Suppose that $s > \delta$. Then, from (6),

$$\begin{aligned} \frac{1}{s} \int_{x-s}^{x} \left|f(y)\right| dy &= \frac{s-h}{s} \cdot \frac{1}{s-h} \int_{x-s}^{x-h} \left|f(y)\right| dy + \frac{h}{s} \cdot \frac{1}{h} \int_{x-h}^{x} \left|f(y)\right| dy \\ &\leqslant \max\left\{\mathcal{M}_l f(x - h),\ \left|f(x)\right| + \frac{\varepsilon}{2}\right\}. \end{aligned} \tag{9}$$

Taking the supremum on the left-hand side of (9) when $s > \delta$, we have

$$\mathcal{M}_l f(x) \leqslant \max\left\{\mathcal{M}_l f(x - h),\ \left|f(x)\right| + \frac{\varepsilon}{2}\right\}$$

by (7). By the definition of ε we also have $\mathcal{M}_l f(x) \geqslant \left|f(x)\right| + \varepsilon$. Thus, we obtain (8).

(b) Let K and δ be as in the proof of (a). Suppose that $x, x + h \in K$, $h > 0$, and $s > \delta$. Then it follows from (a) that

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$$\begin{aligned}(10)\qquad \frac{1}{s}\int_{x-s}^{x}\left|f(y)\right|dy-\mathcal{M}_l f(x+h) &\leqslant \frac{1}{s}\int_{x-s}^{x}\left|f(y)\right|dy-\frac{1}{s+h}\int_{x-s}^{x+h}\left|f(y)\right|dy\\ &\leqslant \frac{1}{s}\int_{x-s}^{x}\left|f(y)\right|dy-\frac{1}{s+h}\int_{x-s}^{x}\left|f(y)\right|dy\\ &=\frac{1}{s+h}\cdot\frac{1}{s}\int_{x-s}^{x}\left|f(y)\right|dy\cdot h\\ &\leqslant \frac{\mathcal{M}_l f(x)}{\delta}\cdot h\\ &\leqslant \frac{\mathcal{M}_l f(\alpha)}{\delta}\cdot h.\end{aligned}$$

Taking the supremum on the left-hand side of (10) when $s > \delta$, we obtain

$$0\leqslant \mathcal{M}_l f(x)-\mathcal{M}_l f(x+h)\leqslant Ch$$

by (7) and (a). ∎

Proposition 3. *If $f \in W^{1,1}(\mathbf{R})$, then the distributional derivatives of $\mathcal{M}_l f$ and $\mathcal{M}_r f$ are integrable functions, and*

$$(11)\qquad \left\|(\mathcal{M}_l f)'\right\|_1 \leqslant \|f'\|_1,\quad \left\|(\mathcal{M}_r f)'\right\|_1 \leqslant \|f'\|_1.$$

Proof: We shall prove the proposition only for $\mathcal{M}_l f$. We note that if $f \in W^{1,1}(\mathbf{R})$, then $|f| \in W^{1,1}(\mathbf{R})$ and

$$(12)\qquad \left\||f|'\right\|_1 = \|f'\|_1$$

(see [**2**]).

Recall that

$$E=\bigcup_j I_j=\bigcup_j(\alpha_j,\ \beta_j).$$

Set $F=\mathbf{R}\setminus E$. From Lemma 2, $\mathcal{M}_l f$ is diferentiable almost everywhere on each I_j, and the derivative, v say, satisfies $v \leqslant 0$. We shall prove that the weak derivative of $\mathcal{M}_l f$ is given by

$$(13)\qquad (\mathcal{M}_l f)'=\chi_E v+\chi_F|f|',$$

where χ_E and χ_F denote the indicator functions of the sets E and F.

For a test function $\phi \in \mathcal{D}(\mathbf{R})$ we see that

$$(14)\qquad \int_{I_j}\mathcal{M}_l f(y)\phi'(y)\,dy=\left[\left|f(\beta_j)\right|\phi(\beta_j)-\left|f(\alpha_j)\right|\phi(\alpha_j)\right]-\int_{I_j}v(y)\phi(y)\,dy$$

by the continuity of $\mathcal{M}_l f$ and a limiting argument. (Here, and later, if $\alpha_j = -\infty$ or if $\beta_j = +\infty$, then $f(\alpha_j) = 0$ and $f(\beta_j) = 0$; similar remarks apply to $\mathcal{M}_l f(\alpha_j)$ and

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$\mathcal{M}_l f(\beta_j)$.) It follows from (14) that

$$\begin{aligned}
&\int_{\mathbf{R}} \mathcal{M}_l f(y)\phi'(y)\,dy \\
&\qquad = \int_{E\cup F} \mathcal{M}_l f(y)\phi'(y)\,dy \\
&\qquad = \sum_j \Big[\big|f(\beta_j)\big|\phi(\beta_j) - \big|f(\alpha_j)\big|\phi(\alpha_j)\Big] - \int_E v(y)\phi(y)\,dy + \int_F \big|f(y)\big|\phi'(y)\,dy \\
&\qquad = \int_E \big|f(y)\big|\phi'(y)\,dy + \int_E |f|'(y)\phi(y)\,dy - \int_E v(y)\phi(y)\,dy + \int_F \big|f(y)\big|\phi'(y)\,dy \\
&\qquad = \int_{\mathbf{R}} \big|f(y)\big|\phi'(y)\,dy + \int_E |f|'(y)\phi(y)\,dy - \int_E v(y)\phi(y)\,dy \\
&\qquad = -\int_{\mathbf{R}} \big(\chi_E(y)v(y) + \chi_F(y)|f|'(y)\big)\phi(y)\,dy.
\end{aligned}$$

This relation implies (13).

Now, we shall prove (11). For each interval I_j, since $v \leqslant 0$, we have

$$\begin{aligned}
\int_{I_j} \big|v(y)\big|\,dy &= \mathcal{M}_l f(\alpha_j) - \mathcal{M}_l f(\beta_j) \\
&= \big|f(\alpha_j)\big| - \big|f(\beta_j)\big| \\
&= -\int_{I_j} |f|'(y)\,dy \leqslant \int_{I_j} \big|\,|f|'(y)\big|\,dy.
\end{aligned} \tag{15}$$

From (15) and (12) we obtain

$$\big\|(\mathcal{M}_l f)'\big\|_1 = \int_E |v| + \int_F \big|\,|f|'\big| \leqslant \big\|\,|f|'\big\|_1 = \|f'\|_1. \qquad \square$$

We need one more lemma.

Lemma 4. *Let f and g be (real valued) integrable functions on the line, and set $F(x) = \int_{-\infty}^{x} f(y)\,dy$, $G(x) = \int_{-\infty}^{x} g(y)\,dy$, and $H(x) = \max\big\{F(x),\ G(x)\big\}$. Then the weak derivative of H is an integrable function, and*

$$\|H'\|_1 \leqslant \|f\|_1 + \|g\|_1.$$

This lemma can be proved easily (see [**2**, Lemma 7.6]).

The theorem now follows from (5), Lemma 4 and Proposition 3.

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