An Interpolation Series for Integral Functions

By Sheila Scott Macintyre

(*Received 7th October*, 1949. *Read 4th November*, 1949.)

1. The Gontcharoff interpolation series[1]

$$\sum_{n=0}^{\infty} F^{(n)}(a_n)\, G_n(z), \tag{1.0}$$

where

$$G_0(z) = 1, \quad G_n(z) = \int_{a_0}^{z} dz' \int_{a_1}^{z'} dz'' \dots \int_{a_{n-1}}^{z^{(n-1)}} dz^{(n)} \quad (n > 0),$$

has been studied in various special cases. For example, if $a_n = a_0$ (all n), (1.0) reduces to the Taylor expansion of $F(z)$. If $a_n = (-1)^n$, J. M. Whittaker[2] showed that the series (1.0) converges to $F(z)$ provided $F(z)$ is an integral function whose maximum modulus satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \tfrac{1}{4}\pi,$$

the constant $\frac{1}{4}\pi$ being the "best possible". In the case $|a_n| \leqslant 1$, I have shown[3] that the series converges to $F(z)$ provided $F(z)$ is an integral function whose maximum modulus satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \cdot 7259,$$

and[4] that while ·7259 is not the "best possible" constant here, it cannot be replaced by a number as great as ·7378.

In this paper, I consider a generalisation of Whittaker's result, namely the case in which $a_n = \omega^n$ where $|\omega| = 1$ $(\arg \omega \neq 0)$, and prove

Theorem I. *The series* $\sum_{n=0}^{\infty} F^{(n)}(\omega^n)\, P_n(z)$, *where* $|\omega| = 1$, $\arg \omega \neq 0$,

$$P_0(z) = 1, \quad P_n(z) = \int_{1}^{z} dz' \int_{\omega}^{z'} dz'' \int_{\omega^2}^{z''} dz''' \dots \int_{\omega^{n-1}}^{z^{(n-1)}} dz^{(n)} \quad (n > 0),$$

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[1] The notation used here differs from that adopted in **6** (Chapter III) in the omission of a factor $n\,!$ from $G_n(z)$.

[2] J. M. Whittaker, **5**, 458.

[3] S. S. Macintyre, **4**.

[4] S. S. Macintyre, **3**.

https://doi.org/10.1017/S001309150001395X Published online by Cambridge University Press

converges uniformly to $F(z)$ in any bounded region, provided $F(z)$ is an integral function whose maximum modulus satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \rho_1, \tag{1.1}$$

ρ_1 being the modulus of the smallest zero of the integral function $f(z, \omega)$ defined by the power series

$$f(z, \omega) = \sum_{n=0}^{\infty} \omega^{\frac{1}{2}n(n-1)} z^n/n!\,. \tag{1.2}$$

The constant ρ_1 is shown to be the "best possible" in this case, and it is evident that Whittaker's result follows as a special case, since

$$f(z, -1) = \sin z + \cos z.$$

It is possible to sharpen condition (1.1) of Theorem I, and we prove

THEOREM II. *If we define ω, $P_n(z)$, ρ_1 as in Theorem I, the series $\sum_{n=0}^{\infty} F^{(n)}(\omega^n)\, P_n(z)$ converges uniformly to $F(z)$ in any bounded region, provided $F(z)$ is an integral function satisfying*

$$F(z) = O\{e^{\rho_1|z|}\phi(\rho_1|z|)\}, \tag{1.3}$$

where $\phi(z)$ is a function of z such that $\sum_{k=1}^{\infty} \sqrt{k}\,\phi(k)$ is absolutely convergent.

2. Let the moduli of the zeros of $f(z, \omega)$ be arranged in a sequence ρ_n in ascending order of magnitude. Differentiating (1.2) we have

$$\frac{\partial f}{\partial z} = \sum_{1}^{\infty} \omega^{\frac{1}{2}n(n-1)} z^{n-1}/(n-1)\,!$$

$$= \sum_{0}^{\infty} \omega^{\frac{1}{2}n(n+1)} z^n/n\,! \tag{2.1}$$

$$= f(\omega z, \omega). \tag{2.2}$$

Consider
$$g(x, z) = f\left(xz, \frac{1}{\omega}\right) \Big/ f\left(z, \frac{1}{\omega}\right) \tag{2.3}$$

$$= \sum_{0}^{\infty} z^n Q_n(x) \quad (|z| < \rho_1). \tag{2.4}$$

Since
$$g(x, 0) \equiv 1, \quad g(1, z) \equiv 1,$$

it follows that

$$Q_0(x) = 1 \quad \text{and} \quad Q_n(1) = 0 (n > 0). \tag{2.5}$$

Now it follows from (2.4), (2.5) that

$$\frac{\partial g}{\partial x} = \sum_{1}^{\infty} z^n Q_n'(x) \quad (|z| < \rho_1), \tag{2.6}$$

and from (2.2), (2.3) that

$$\begin{aligned}\frac{\partial g}{\partial x} &= z\, g(x/\omega, z) \\ &= \sum_{0}^{\infty} z^{n+1} Q_n(x/\omega).\end{aligned} \tag{2.7}$$

Hence, using (2.6), (2.7), we have

$$Q_n'(x) = Q_{n-1}(x/\omega) \quad (n \geqslant 1). \tag{2.8}$$

It follows from (2.5), (2.8) that

$$Q_n(x) = \int_1^x dx' \int_1^{x'/\omega} dx'' \int_1^{x''/\omega} dx''' \dots \int_1^{x^{(n-1)}/\omega} dx^{(n)},$$

or, by the transformations $\zeta^{(k)} = \omega^{k-1} x^{(k)}$,

$$\omega^{\frac{1}{2}n(n-1)} Q_n(x) = \int_1^x d\zeta' \int_\omega^{\zeta'} d\zeta'' \dots \int_{\omega^{n-1}}^{\zeta^{(n-1)}} d\zeta^{(n)} = P_n(x). \tag{2.9}$$

Now, integrating

$$R_n(z) = \int_1^z dz' \int_\omega^{z'} dz'' \dots \int_{\omega^{n-1}}^{z^{(n-1)}} F^{(n)}(z^{(n)})\, dz^{(n)} \tag{2.10}$$

repeatedly by parts[1], we find

$$R_n(z) = F(z) - \sum_{r=0}^{n-1} F^{(r)}(\omega^r) P_r(z).$$

Hence
$$F(z) = \sum_{r=0}^{n-1} F^{(r)}(\omega^r) P_r(z) + R_n(z). \tag{2.11}$$

Let C, Γ be the circles $|z| = \frac{1}{2}\rho_1$, $|z| = \frac{1}{2}(\rho_1 + \rho_2)$ respectively. From (2.4) we have
$$Q_n(x) = \frac{1}{2\pi i} \int_C \frac{g(x, z)}{z^{n+1}}\, dz. \tag{2.12}$$

If $f\left(z, \dfrac{1}{\omega}\right)$ has p zeros (denoted by $z_1, z_2, \dots, z_p$) on $|z| = \rho_1$, then $g(x, z)$ has p poles (at most) between C and Γ, residues $A_1(x), A_2(x), \dots, A_p(x)$ respectively, these residues being bounded for x in any bounded region.

Now $|f(z, \omega)|$ has no zeros on Γ and thus has a positive minimum on Γ which will be denoted by m. Since $|f(z, \omega)| = \left| f\left(z, \dfrac{1}{\omega}\right)\right|$, we have, using

[1] See J. M. Whittaker, 6, 39, for a detailed argument of this nature.

(1.2) and (2.3),

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{g(x,\, z)}{z^{n+1}}dz\right| \leqslant e^{\frac{1}{2}(\rho_1+\rho_2)|x|}\Big/ m\left(\frac{\rho_1+\rho_2}{2}\right)^n \tag{2.13}$$

$$\leqslant B(x)\Big/\left(\frac{\rho_1+\rho_2}{2}\right)^n, \tag{2.14}$$

where $B(x)$ is bounded for x in any bounded region. Moreover the residue of $g(x,\, z)/z^{n+1}$ at $z = z_s$ $(s = 1,\, 2,\, \ldots,\, p)$ is $A_s(x)/z_s^{n+1}$ and this is of absolute magnitude $|A_s(x)|\,\rho_1^{n+1}$. Now

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(x,\, z)}{z^{n+1}}dz - \frac{1}{2\pi i}\int_C \frac{g(x,\, z)}{z^{n+1}}dz$$

is equal to the sum of the residues of $g(x,\, z)/z^{n+1}$ at the points $z_1,\, z_2,\, \ldots,\, z_p$ and it follows from (2.12), (2.14) that

$$|Q_n(x)| \leqslant \sum_{s=1}^{p} \frac{|A_s(x)|}{\rho_1^{n+1}} + B(x)\Big/\left(\frac{\rho_1+\rho_2}{2}\right)^n$$

$$\leqslant \frac{A(x)}{\rho_1^{n+1}} + B(x)\Big/\left(\frac{\rho_1+\rho_2}{2}\right)^n, \tag{2.15}$$

where $A(x)$ and $B(x)$ are bounded for x in any bounded region. On integrating both sides of the equation from 1 to x/ω we can show by induction that, for any integer L,

$$\frac{x^L}{L\,!\,\omega^{\frac{1}{2}L(L+1)}} = \sum_{r=0}^{L} \frac{Q_{L-r}(x/\omega)}{r\,!\,\omega^{\frac{1}{2}r(r-1)}} \tag{2.16}$$

and thus

$$S_{n,k}\left(\frac{x}{\omega}\right) = \int_1^{x/\omega} dx' \int_1^{x'/\omega} dx'' \ldots \int_1^{x^{(n-1)}/\omega} dx^{(n)} \int_0^{x^{(n)}/\omega} dx^{(n+1)} \ldots \int_0^{x^{(k-1)}/\omega} dx^{(k)},$$

$$= \frac{1}{\omega^{\frac{1}{2}(k-n)(k-n+1)}} \int_1^{x/\omega} dx' \int_1^{x'/\omega} dx'' \ldots \int_1^{x^{(n-1)}/\omega} \frac{x^{k-n}}{(k-n)\,!}\, dx$$

$$= \sum_{r=0}^{k-n} \frac{Q_{k-r}(x/\omega)}{r\,!\;\omega^{\frac{1}{2}r(r-1)}} \tag{2.17}$$

from (2.16), making use of (2.5) and (2.8). Also, from (2.15) and (2.17) it follows that

$$\left|S_{n,k}\left(\frac{x}{\omega}\right)\right| \leqslant A\,(x) \sum_{r=0}^{k-n} \frac{1}{r\,!\,\rho_1^{k-r+1}} + B(x) \sum_{r=0}^{k-n} \frac{1}{r\,!}\left(\frac{2}{\rho_1+\rho_2}\right)^{k-n}$$

$$\leqslant A\,(x)\, e^{\rho_1}/\rho_1^{k+1} + B(x)\, e^{\frac{1}{2}(\rho_1+\rho_2)}\Big/\left(\frac{\rho_1+\rho_2}{2}\right)^k. \tag{2.18}$$

Again, if we use the transformations (2.9), the formula

$$\omega^{\frac{1}{2}k(k-1)} S_{n,k}(x) = \int_1^x d\zeta' \int_\omega^\zeta d\zeta'' \dots \int_{\omega^{n-1}}^{\zeta^{(n-1)}} d\zeta^{(n)} \int_0^{\zeta^{(n)}} d\zeta^{(n+1)} \dots \int_0^{\zeta^{(k-1)}} d\zeta^{(k)} \quad (2.19)$$

arises from the definition of $S_{n,k}\ (x/\omega)$ in (2.17).

3. From (2.10), (2.19), on expanding $F^{(n)}(z)$ in its Taylor series, we get

$$R_n(z) = \int_1^z dz' \int_\omega^{z'} dz'' \dots \int_{\omega^{n-1}}^{z^{(n-1)}} \sum_{k=n}^{\infty} F^{(k)}(0) \frac{z^{(k-n)}}{(k-n)!} dz$$

$$= \sum_{k=n}^{\infty} F^{(k)}(0)\, \omega^{\frac{1}{2}k(k-1)} S_{n,k}(z), \quad (3.1)$$

as follows from (2.17) and (2.19). If $F(z)$ is an integral function, we have

$$F^{(k)}(0) = \frac{k!}{2\pi i} \int_{|\zeta| = k/\rho_1} \frac{F(\zeta)}{\zeta^{k+1}} d\zeta,$$

and using Stirling's approximation for $k!$, if $F(z)$ satisfies (1.3), we have

$$F^{(k)}(0) = O\{\rho_1{}^k \sqrt{k}\, \phi(k)\}.$$

Hence, from (2.18) and (3.1), $R_n(z)$ is less in modulus than the sum of the remainders of two convergent series and thus tends to zero as n tends to infinity. From (2.11) it then follows that the interpolation series

$$\sum_{r=0}^{\infty} F^{(r)}(\omega^r)\, P_r(z) \quad (3.3)$$

converges uniformly to $F(z)$ in any bounded region provided $F(z)$ is an integral function satisfying (1.3). This completes the proof of Theorem II and hence of Theorem I.

Let z_1, where $|z_1| = \rho_1$, be the zero of smallest modulus of $f(z, 1/\omega)$. That the constant ρ_1 of Theorems I and II is the "best possible" is seen by taking $F(z) = f(zz_1, 1/\omega)$ for which the maximum modulus $M(r)$ clearly satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} = |z_1| = \rho_1.$$

Then, by (2.2), $F^{(n)}(\omega^n) = z_1{}^n f(z_1, 1/\omega)$ for all n. Thus for this function all the terms of the series (3.3) are identically zero. It should be noted that $\bar{z}_1$ is the zero of smallest modulus of $f(z, \omega)$.

The numerical value of ρ_1 has been calculated[1] for $\arg \omega = \frac{2}{3}\pi,\ \frac{4}{5}\pi,\ \frac{3}{4}\pi$

[1] See R. P. Boas, **1** and **2**; S. S. Macintyre, **3**.

and the equivalent in radians of 136°, 137°, the values of ρ_1 in these cases being approximately ·746, ·7398, ·7379, ·7378 and ·7378 respectively.

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THE UNIVERSITY,
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