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ON TOPOLOGICAL INVARIANTS ASSOCIATED WITH A POLYNOMIAL WITH ISOLATED CRITICAL POINTS

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Abstract. We consider a polynomial $f:\mathbb{R}^n \to \mathbb{R}$ with isolated critical points and we relate $\chi(f^{-1}(0))$ and $\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})$ to the topological degrees of polynomial maps defined in terms of f.

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1. Introduction. Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping and let $W = F^{-1}(0)$. Let $G_1, \ldots, G_l$ be polynomials. An interesting problem is the computation of $\chi(W)$ and $\chi\,(W \cap \{G_1 \geq 0, \ldots, G_l \geq 0\})$ in terms of the polynomials F_i and G_j.

When W is compact, Szafraniec [**17**] and Bruce [**4**] proved that there exists a polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ with an algebraically isolated critical point at the origin such that

$$\chi(W) = \frac{1}{2}((-1)^n - \deg_0 \nabla P),$$

where $\deg_0 \nabla P$ is the topological degree at the origin of the gradient of P. The study of the case of W non-compact has been done in [**6**, **18**, **19**], but only when $1 \leq k < n$ and W is a smooth manifold of dimension $n-k$. In [**18**], Szafraniec constructs a polynomial map $H : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$. He proves that $H^{-1}(0) \subset B_R^{n+k}$, where B_R^{n+k} is a ball in $\mathbb{R}^{n+k}$ centered at the origin with sufficiently big radius R, and that $\chi(W) = (-1)^k \deg h$, where $h = H/||H|| : S_R^{n+k-1} \to S^{n+k-1}$ and $S_R^{n+k-1} = \partial B_R^{n+k}$. In [**6**], the authors consider a polynomial algebra A and they prove, assuming $\dim_{\mathbb{R}} A < +\infty$, that

$$\chi(W) \equiv \dim_{\mathbb{R}} A \bmod 2. \tag{1}$$

This latter formula is refined in [**19**], where it is proved that there exist two bilinear symmetric forms Φ and Φ_M on A such that

$$\begin{aligned} &\text{if } n-k \text{ is odd} \quad \chi(W) = (-1)^k \text{signature } \Phi, \\ &\text{if } n-k \text{ is even} \quad \chi(W) = \text{signature } \Phi_M. \end{aligned} \tag{2}$$

In [**8**], we started the investigation of the case in which W admits a finite number of singularities. We generalize first formula (1) above and we obtain

$$\chi(W) + \Sigma_\mu \equiv \dim_{\mathbb{R}} A \bmod 2,$$

where Σ_μ is the sum of the Milnor numbers at the singularities of W. Then we generalize formulae (2) but only in the cases of curves ($k = n - 1$) and of odd-dimensional hypersurfaces ($k = 1$ and n is even).

The first aim of this paper is to solve the case of even-dimensional hypersurfaces with isolated singularities. Actually we give a new method that works for both parities. We consider a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ with a finite number of critical points, some of them possibly lying in the fibre $f^{-1}(0)$. We make the additional assumption that $f(0) > 0$. Taking $(x, \lambda) = (x_1, \ldots, x_n, \lambda)$ as a coordinate system for $\mathbb{R}^{n+1}$, we define four polynomial mappings H, K, L_1 and L_2 in the following way : $H(x, \lambda) = (\lambda x + \nabla f, f)$, $K(x, \lambda) = (\lambda x + \nabla f, \lambda f)$, $L_1(x, \lambda) = (\nabla f, \lambda f - 1)$ and $L_2(x, \lambda) = (\nabla f, \lambda f^2 - 1)$. Here ∇f denotes the gradient vector of f. We prove, in our Theorem 5.10, that the zero sets of these applications are compact and that, if n is even, then

$$\chi(f^{-1}(0)) = \deg H + \deg \nabla f - \deg L_2,$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg K - \deg L_1,$$

and if n is odd, then

$$\chi(f^{-1}(0)) = \deg K - \deg L_1,$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg H - \deg \nabla f + \deg L_2.$$

By $\deg H$, which we call the *total degree* of H, we mean the topological degree of the map $\frac{H}{\|H\|} : S^n_{R'} \to S^n$, where $S^n_{R'} = \partial B^{n+1}_{R'}$ and $H^{-1}(0) \subsetneq B^{n+1}_{R'}$.

These formulae are global polynomial versions of a result due to Khimshiasvili on the Euler characteristic of the real Milnor fibre. It states that, if $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is an analytic function-germ with an isolated critical point at the origin, then

$$\chi\left(g^{-1}(\delta) \cap B^n_\varepsilon\right) = 1 - \text{sign}\,(-\delta)^n \deg_0 \nabla g,$$

for any regular value δ of g, $0 < |\delta| \ll \varepsilon \ll 1$. Here $\deg_0 \nabla g$ is the topological degree of $\frac{\nabla g}{\|\nabla g\|} : S^{n-1}_\varepsilon \to S^{n-1}$. A proof of this can be found in **[1]**, **[10]**, **[14]** or **[21]**.

The proof of our main theorem is based on Morse theory for manifolds with corners. Putting $\omega(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$, we study the critical points of Morse perturbations of $\omega_{|f^{-1}(\delta)\cap B^n_R}$, $\omega_{|\{f\geq\delta\}\cap B^n_R}$ and $\omega_{|\{f\leq\delta\}\cap B^n_R}$, where δ is a regular value of f close to 0. These critical points are in bijection with non-degenerate zeros of $\tilde{H}_\delta$ and $\tilde{K}_\delta$, two appropriate perturbations of H and K, and their Morse indices are related to the local degree of $\tilde{H}_\delta$ and $\tilde{K}_\delta$ at those zeros. This gives a link between $\chi(f^{-1}(\delta) \cap B^n_R)$ and $\chi(\{f \geq \delta\} \cap B^n_R) - \chi(\{f \leq \delta\} \cap B^n_R)$ and the topological degrees of H and K. Then we relate $\chi(f^{-1}(0))$ (respectively $\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})$) to $\chi(f^{-1}(\delta) \cap B^n_R)$ (respectively $\chi(\{f \geq \delta\} \cap B^n_R) - \chi(\{f \leq \delta\} \cap B^n_R)$).

In Section 2, we recall some facts about Morse theory for manifolds with corners. In Section 3, we give methods for the computation of the total degree of a polynomial mapping. These methods will be useful in the application of our theorems to concrete examples. Section 4 is devoted to some technical lemmas : we relate a Morse index to a local topological degree. Finally we prove our degree formulae in Section 5.

Some computations are given at the end of the paper. They have been done with a program written by Andrzej Lecki. The author is very grateful to him and Zbigniew Szafraniec for giving him this program.

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2. Morse theory for manifolds with corners. We generalize the notion of correct critical points and Morse correct functions, defined for manifolds with boundary in [**13**], to the case of manifolds with corners. Then we relate the Euler characteristic of a manifold with corners to the indices of correct critical points.

Let us start with some basic facts on manifolds with corners. Our reference is [**5**]. A manifold with corners M is defined by an atlas of charts modelled on open subsets of $\mathbb{R}^n_+$. We write ∂M for its boundary. We shall make the additional assumption that the boundary is partitioned into pieces $\partial_i M$, themselves manifolds with corners, such that in each chart, the intersections with the coordinate hyperplanes $x_j = 0$ correspond to distinct pieces $\partial_i M$ of the boundary. For any set I of suffices, we write $\partial_I M = \cap_{i\in I}\partial_i M$ and we make the convention that $\partial_\emptyset M = M \setminus \partial M$.

Any n-manifold M with corners can be embedded in an n-manifold M^+ without boundary so that the pieces $\partial_i M$ extend to submanifolds $\partial_i M^+$ of codimension 1 in M^+. We shall assume that M^+ is provided with a Riemannian metric.

Let M be a manifold with corners and $\omega : M^+ \to \mathbb{R}$ a smooth map. We consider the points P that are critical points of $\omega_{|\partial_I M^+}$.

Definition 2.1. A critical point P is *correct* (respectively *Morse correct*) if, taking $I(P) := \{i\,|P \in \partial_i M\}$, P is a critical (respectively Morse critical) point of $\omega_{|\partial_{I(P)}M^+}$, and is not a critical point of $\omega_{|\partial_J M^+}$ for any proper subset J of $I(P)$.

Note that a 0-dimensional corner point P is always a critical point because in this case $\partial_{I(P)}M^+ = \{P\}$, which is a 0-dimensional manifold.

Definition 2.2. The maps ω with all critical points Morse correct are called *Morse correct*.

Proposition 2.3. *The set of Morse correct functions is dense and open in the space of all maps $M^+ \to \mathbb{R}$.*

Proof. This is clear from classical Morse theory, because there is a finite number of pieces $\partial_I M^+$. □

The index $\lambda(P)$ of ω at a Morse correct point P is defined to be that of $\omega_{|\partial_{I(P)}M^+}$. If P is a correct critical point of ω, $i \in I(P)$, and J is formed from $I(P)$ by deleting i, then in a chart at P with $\partial_J M$ mapping to $\mathbb{R}^p_+$ and $\partial_{I(P)}M$ to the subset $x_1 = 0$, the function ω is non-critical, but its restriction to $x_1 = 0$ is. Hence $\partial\omega/\partial x_1 \neq 0$.

Definition 2.4. We say that ω is *inward at P* if, for each $i \in I(P)$, we have $\partial\omega/\partial x_1 > 0$.

Remark 2.5. By our convention, if $I(P) = \emptyset$, then ω is inward at P.

Theorem 2.6. *If M is compact and ω is Morse correct, then*

$$\chi(M) = \sum \left\{(-1)^{\lambda(P)} \mid P \textit{ is an inward Morse critical point}\right\}.$$

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Proof. This is a consequence of stratified Morse theory [**11**, **12**]. A good summary of the results we use can be found in [**3**, Section 2].

The manifold with corners M is a compact Whitney stratified set of M^+, with stratum the $\partial_I M$. The function $\omega : M \to \mathbb{R}$ is easily seen to be a Morse function in the sense of [**11**] and so

$$\chi(M) = \sum\{\alpha(\omega, P) \mid P \text{ correct critical point}\},$$

where

$$\alpha(\omega, P) = 1 - \chi(\omega^{-1}(\omega(P) - \delta) \cap B(P, \varepsilon)),$$

with $0 < \delta \ll \varepsilon \ll 1$. Here $B(P, \varepsilon)$ is the ball centered at P of radius ε in the Riemannian manifold M^+. If P belongs to $\partial_\emptyset M$ then $\alpha(\omega, P)$ is exactly $(-1)^{\lambda(P)}$. If P belongs to $\partial_I M$, $I \neq \emptyset$, then $\alpha(\omega, P) = (-1)^{\lambda(P)}.\alpha_{nor}(\omega, P)$, where $\alpha_{nor}(\omega, P)$ is the normal index of ω at P. It is defined as follows. Choose a normal slice V at P; that is, a closed submanifold of M^+ of dimension $n - \dim \partial_I M$, that intersects $\partial_I M$ in P orthogonally. We obtain

$$\alpha_{nor}(\omega, P) = 1 - \chi(\omega^{-1}(\omega(P) - \delta) \cap B(P, \varepsilon) \cap V).$$

Let us compute this normal index. We can assume that $\omega(P) = 0$. Also we can choose a local chart $(x_1, \dots, x_n)$ centered at P such that $\partial_I M$ is given by $\{x_1 = \dots = x_k = 0\}$ and V is given by $\{x_{k+1} = \dots = x_n = 0\}$, $k < n$. Locally M is the set $\{x_1 \geq 0, \dots, x_k \geq 0\}$. Furthermore, since P is a correct point, $\partial\omega/\partial x_j(P) \neq 0$ for each $j \in \{1, \dots, k\}$ and, by an appropriate change of coordinates, the restriction of ω to V is just the linear form

$$\sum_{j=1}^{k} \frac{\partial \omega}{\partial x_j}(P) x_j.$$

It is then straightforward to see that $\alpha_{nor}(\omega, P) = 1$ if $\partial\omega/\partial x_j(P) > 0$, for all $j \in \{1, \dots, k\}$, and $\alpha_{nor}(\omega, P) = 0$ otherwise. This proves the theorem. □

3. Total degree of a polynomial mapping. We study the topological degree on a big sphere of a polynomial mapping. Let $(x_1, \dots, x_N)$ be a coordinate system in $\mathbb{R}^N$. Let $F = (F_1, \dots, F_N) : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial mapping such that $F^{-1}(0)$ is compact. There is $R \gg 0$ such that $F^{-1}(0) \subsetneq B_R^N$. Recall that $\deg F$ stands for the topological degree of $\frac{F}{\|F\|} : S_R^{N-1} \to S^{N-1}$. We give two methods due to Szafraniec for computing $\deg H$. The first one [**18**] enables us to reduce this computation to the computation of a local degree at the origin. Let $I : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$ be the inversion defined by $I(x) = x/\|x^2\|$, let d_i denote the degree of the polynomial F_i for each $i \in \{1, \dots, N\}$ and let

$$F'(x) = \left(\|x\|^{2d_1} \cdot F_1 \circ I(x), \dots, \|x\|^{2d_N} \cdot F_N \circ I(x)\right) \text{ for } x \neq 0.$$

Then F' can be extended to a polynomial map $\mathbb{R}^N \to \mathbb{R}^N$ such that 0 is isolated in $F'^{-1}(0)$. Let $r = 1/R$; the map

$$\begin{aligned} S_r^{N-1} &\to S_R^{N-1} \\ x &\mapsto I(x) \end{aligned}$$

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is of degree +1. Clearly, the maps $F' : S_r \to \mathbb{R}^N \setminus \{0\}$ and $F \circ I : S_r \to \mathbb{R}^N \setminus \{0\}$ are homotopic, and so, if r is small and if $\deg_0 F'$ is the degree of $\frac{F'}{\|F'\|}$ around S_r^{N-1}, then $\deg F = \deg_0 F'$.

LEMMA 3.1. $\deg F = \deg_0 F'$.

Using the Eisenbud-Levine-Khimshiashvili's formula [**9**, **14**], the computation of deg H reduces to the problem of calculating a signature of an appropriate bilinear symmetric form. Unfortunately the formula of the above lemma is difficult to implement because it involves polynomials with a large number of monomials. However, if we add the assumption that the polynomial factor algebra $A_F = \frac{\mathbb{R}[x_1,\dots,x_N]}{(F_1,\dots,F_N)}$ is finite dimensional as a vector space over $\mathbb{R}$, then we can use the following more effective method. Let $\phi : A_F \to \mathbb{R}$ be the Kronecker symbol or global residue on A_F. A description of this residue can be found in [**2**, **7**, **16**, **19**, **20**]. It is a linear functional with which we can define the following bilinear symmetric form Φ :

$$\Phi : A_F \times A_F \to \mathbb{R}, \;\; \Phi(f, g) = \phi(fg).$$

THEOREM 3.2. *The form* Φ *is non-degenerate and*

$$\deg F = \text{signature } \Phi.$$

Proof. See [**20**, Theorem 1.5]. □

Now we assume that $F^{-1}(0)$ is a finite set, which is realized if $\dim_{\mathbb{R}} A_F < +\infty$. Let $q_1, \dots, q_t$ be the zeros of F and for all $i \in \{1, \dots, t\}$, let $\deg_{q_i} F$ be the degree of $\frac{F}{\|F\|}$ around a small sphere centered at q_i. Let $P : \mathbb{R}^N \to \mathbb{R}$ be a polynomial. We wish to compute

$$\sum_{i=1}^{t} \text{sign } P(q_i) \cdot \deg_{q_i} F.$$

We write $(x, \lambda) = (x_1, \dots, x_n, \lambda)$ for a coordinate system in $\mathbb{R}^{N+1}$ and we define $G : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by $G(x, \lambda) = (F, \lambda P - 1)$.

LEMMA 3.3. *The set* $G^{-1}(0)$ *is finite and*

$$\sum_i \text{sign } P(q_i) \cdot \deg_{q_i} F = \deg G.$$

Proof. A point (x, λ) belongs to $G^{-1}(0)$ if and only if $F(x) = 0$ and $P(x) \neq 0$. Hence

$$G^{-1}(0) = \left\{ \left(q_i, \frac{1}{P(q_i)} \right) \middle| \; P(q_i) \neq 0 \right\}$$

and

$$\deg G = \sum_{i \mid P(q_i) \neq 0} \deg_{(q_i, \frac{1}{P(q_i)})} G.$$

Changing F if necessary, we can assume that q_i is a non-degenerate zero of F. It is then a simple determinant computation to see that $(q_i, \frac{1}{P(q_i)})$ is a non-degenerate zero of G

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and that

$$\deg_{(q_i, \frac{1}{P(q_i)})} G = \text{sign } P(q_i) \cdot \deg_{q_i} F. \qquad \square$$

4. An index computation. We characterize a Morse correct critical point of an analytic function defined on an analytic manifold with boundary. We relate its Morse index to a local topological degree.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic function and let $p \in f^{-1}(0)$ be such that $\nabla f(p) \neq 0$. From the implicit function theorem, $f^{-1}(0)$ is a smooth $(n-1)$-manifold in the neighborhood of p. Let $\omega : (\mathbb{R}^n, p) \to (\mathbb{R}, \omega(p))$ be an analytic function defined around p. Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be given by

$$H(x, \lambda) = (\lambda \nabla \omega(x) + \nabla f(x), f(x)).$$

We shall study the situation at the point p.

LEMMA 4.1. *The function $\omega_{|\{f*0\}}$ ($*$ is either $\leq$ or $\geq$) admits a correct critical point at p if and only if there exists $\lambda \neq 0$ such that $H(p, \lambda) = 0$. Furthermore λ is uniquely determined.*

Proof. A point $p \in f^{-1}(0)$ is a critical point of $\omega_{|\{f*0\}}$ if and only if there exists μ such that $\nabla\omega(p) + \mu\nabla f(p) = 0$. Moreover it is correct if and only if $\mu \neq 0$. The number λ sought is thus $1/\mu$. If there is $\lambda' \neq \lambda$ with $H(p, \lambda') = 0$ then $\nabla\omega(p) = 0$, which contradicts the fact that p is correct. $\square$

LEMMA 4.2. *The function $\omega_{|\{f*0\}}$ admits a Morse correct critical point at p if and only if there exists $\lambda \neq 0$ such that $H(p, \lambda) = 0$ and $JH(p, \lambda) \neq 0$, JH being the Jacobian determinant of H. Furthermore, if s is the Morse index of $\omega_{|f^{-1}(0)}$ at p then*

$$(-1)^s = \text{sign } \lambda^n \times \text{sign } JH(p, \lambda).$$

Proof. Let $\bar{H} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined by

$$\bar{H}(x, \lambda) = (\nabla\omega(x) + \lambda\nabla f(x), f(x)).$$

In [**18**], Szafraniec proves in Lemma 1.4 that $\omega_{|f^{-1}(0)}$ has a Morse critical point at p if and only if there is a unique μ such that $\bar{H}(p, \mu) = 0$ and $J\bar{H}(p, \mu) \neq 0$. In this case, $(-1)^{s+1} = \text{sign } J\bar{H}(p, \mu)$. Now

$$J\bar{H}(p, \mu) = \det(\bar{a}_{i,j})_{1 \leq i,j \leq n+1},$$

where

$$\begin{aligned}
\bar{a}_{i,j} &= \frac{\partial^2 \omega}{\partial x_i \partial x_j}(p) + \mu \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \quad \text{for } (i, j) \in \{1, \dots, n\}^2, \\
\bar{a}_{i,n+1} &= \bar{a}_{n+1,i} = \frac{\partial f}{\partial x_i}(p) \quad \text{for } i \in \{1, \dots, n\}, \\
\bar{a}_{n+1,n+1} &= 0.
\end{aligned}$$

Then

$$J\bar{H}(p, \mu) = \mu^{n-1} \times \det(a_{i,j})_{1 \leq i,j \leq n+1},$$

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where

$$a_{i,j} = \frac{1}{\mu}\frac{\partial^2 \omega}{\partial x_i \partial x_j}(p) + \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \quad \text{for } (i,j) \in \{1, \dots, n\}^2,$$

and $a_{i,j} = \bar{a}_{i,j}$ otherwise. Putting $\lambda = 1/\mu$ and using the fact that $-\lambda \frac{\partial \omega}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p)$ for all $i \in \{1, \dots, n\}$, we see that $JH(p, \lambda) = -\lambda^{n-2} J\bar{H}(p, \mu)$. □

5. Degree formulas. Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial with isolated critical points and that $f(0) > 0$. Let $\omega(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. The polynomials H, K, L_1 and L_2 are defined this way : $H(x, \lambda) = (\lambda x + \nabla f, f)$, $K(x, \lambda) = (\lambda x + \nabla f, \lambda f)$, $L_1(x, \lambda) = (\nabla f, \lambda f - 1)$ and $L_2(x, \lambda) = (\nabla f, \lambda f^2 - 1)$.

By Lemma 3.3, we already know that $L_1^{-1}(0)$ and $L_2^{-1}(0)$ are finite. We shall describe the set $H^{-1}(0)$ and $K^{-1}(0)$. We define $\Sigma_f := \{\nabla f = 0\}$, $\Sigma_0 := \Sigma_f \cap f^{-1}(0)$ and $M := f^{-1}(0) \setminus \Sigma_0$. It is clear that M is either empty or a smooth manifold of dimension $n - 1$. The polynomial function $\omega_{|M}$ has a finite number of critical values [**15**, Corollary 2.8] which implies that the set C of critical points of $\omega_{|M}$ is bounded.

LEMMA 5.1. *A point p belongs to C if and only if there exists $\lambda \neq 0$ such that $H(p, \lambda) = 0$. Furthermore λ is uniquely determined.*

Proof. Since $f(0) > 0$, each critical point of $\omega_{|M}$ is a correct critical point. The lemma is a consequence of Lemma 4.1. □

LEMMA 5.2. *A point p belongs to Σ_0 if and only if $H(p, 0) = 0$.*

Proof. This is clear. □

Let $\Pi_x : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection on the n first components.

COROLLARY 5.3. *The set $\Pi_x(H^{-1}(0))$ is $C \sqcup \Sigma_0$.*

Proof. This follows from the two previous lemmas. □

LEMMA 5.4. *The set $H^{-1}(0)$ is compact.*

Proof. We know that $\Pi_x(H^{-1}(0))$ is bounded because C is and Σ_0 is finite. Moreover it is closed because it is the algebraic set defined by the vanishing of f and all the 2×2 minors of the jacobian matrix of the map (f, ω). Hence it is compact. For all $p \in \Pi_x(H^{-1}(0))$, there exists a unique $\lambda(p)$ such that

$$\lambda(p) \cdot p + \nabla f(p) = 0.$$

Since $p \neq 0$, the map of $\Pi_x(H^{-1}(0))$ given by $p \mapsto \lambda(p)$is continuous and so $H^{-1}(0) = \{(p, \lambda(p)) \mid p \in \Pi_x(H^{-1}(0))\}$ is compact. □

LEMMA 5.5. *A point p belongs to C if and only if there exists $\lambda \neq 0$ such that $K(p, \lambda) = 0$. Furthermore λ is uniquely determined.*

LEMMA 5.6. *A point p belongs to Σ_f if and only if $K(p, 0) = 0$.*

COROLLARY 5.7. *The set $\Pi_x(K^{-1}(0))$ equals $C \sqcup \Sigma_f$.*

LEMMA 5.8. *The set $K^{-1}(0)$ is compact.*

Proof. It is the union of $H^{-1}(0)$ and $\{(x, 0) \mid x \in \Sigma_f\}$. □

The expressions deg H and deg K do make sense. We choose $R > 0$ such that $C \cup \Sigma_f \subset B_R^n$. This implies that $f^{-1}(0) \cap B_R^n$ (respectively $\{f * 0\} \cap B_R^n$, $* \in \{\leq, \geq\}$) is a deformation retract of $f^{-1}(0)$ (respectively $\{f * 0\}$). Let us write $\Sigma_f = \{q_1, \dots, q_t\}$ with $\Sigma_0 = \{q_1, \dots, q_r\}$ $(r \leq t)$. We need the following lemma.

LEMMA 5.9. *If δ is a small regular value of f, then*

$$\chi\left(f^{-1}(\delta) \cap B_R^n\right) = \chi(f^{-1}(0)) - \text{sign}\,(-\delta)^n \sum_{i=1}^{r} \deg_{q_i} \nabla f,$$

$$\chi\left(\{f \geq \delta\} \cap B_R^n\right) - \chi\left(\{f \leq \delta\} \cap B_R^n\right)$$
$$= \chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) + \text{sign}\,(-\delta)^{n+1} \sum_{i=1}^{r} \deg_{q_i} \nabla f.$$

Proof. The first item is proved in exactly the same way as Khimshiasvili's formula mentioned in the introduction. We refer to [**1**, **10**, **14**, **21**] for a proof.

In order to prove the second equation, for $\delta > 0$, we use the facts that

$$\chi\left(\{f \geq 0\} \cap B_R^n\right) = \chi\left(\{f \geq \delta\} \cap B_R^n\right) + \chi\left(\{0 \leq f \leq \delta\} \cap B_R^n\right) - \chi\left(f^{-1}(\delta) \cap B_R^n\right),$$

and

$$\chi\left(\{f \leq \delta\} \cap B_R^n\right) = \chi\left(\{f \leq 0\} \cap B_R^n\right) + \chi\left(\{0 \leq f \leq \delta\} \cap B_R^n\right) - \chi\left(f^{-1}(0) \cap B_R^n\right),$$

and that $\{0 \leq f \leq \delta\} \cap B_R^n$ retracts to $f^{-1}(0) \cap B_R^n$. Applying this to $-f$ gives the result for $\delta < 0$. □

THEOREM 5.10. *If n is even, then*

$$\chi(f^{-1}(0)) = \deg H + \deg \nabla f - \deg L_2,$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg K - \deg L_1.$$

If n is odd, then

$$\chi(f^{-1}(0)) = \deg K - \deg L_1,$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - \deg H - \deg \nabla f + \deg L_2.$$

Proof. Let us choose $R' > 0$ such that $H^{-1}(0) \subsetneq B_{R'}^{n+1}$ and $K^{-1}(0) \subsetneq B_{R'}^{n+1}$. Since $\Pi_x(K^{-1}(0)) \subset C \cup \Sigma_f$, we can choose $R' \geq R$. Let $\delta \neq 0$ be a small regular value of f. We construct two appropriate deformations H_δ and K_δ of H and K in the following way:

$$H_\delta(x, \lambda) = (\lambda x + \nabla f(x), f(x) - \delta),$$
$$K_\delta(x, \lambda) = (\lambda x + \nabla f(x), \lambda(f(x) - \delta)).$$

We study first the topological degree of $\frac{K_\delta}{\|K_\delta\|}$ around $S_{R'}^n$. Let

$$m = \min\left\{\|K(x, \lambda)\| \mid (x, \lambda) \in S_{R'}^n\right\}.$$

On $S_{R'}^n$, $\|K - K_\delta\| = \lambda\delta$ and, if we take δ such that $|\delta R'| < \frac{m}{2}$, then $\|K_\delta\| > \frac{m}{2}$ on $S_{R'}^n$. This implies that this degree is well defined. We denote it by $\deg(K_\delta, R')$.

If there is a point $(p,\lambda)\in S^n_{R'}$ such that $K(p,\lambda)$ and $K_\delta(p,\lambda)$ point in opposite directions, then $\lambda p+\nabla f(p)=0$, for K and K_δ have the same n first components. Hence $\lambda f(p)$ and $\lambda(f(p)-\delta)$ have opposite signs. This can happen only if $|f(p)|<|\delta|$. But in this case $\|K(p,\lambda)\|<|\delta R'|<\frac{m}{2}$, a contradiction. We have proved that $\deg(K_\delta,R')=\deg K$. Similarly, $\deg(H_\delta,R')=\deg H$.

Let $(p,\lambda)\in H_\delta^{-1}(0)\cap B^{n+1}_{R'}$. By Lemma 4.1, p is a critical point of $\omega_{|f^{-1}(\delta)}$ and $\|p\|\le R'$. Since on $\{R\le\|x\|\le R'\}$, $\omega_{|f^{-1}(0)}$ does not admit critical points, $\omega_{|f^{-1}(\delta)}$ does not admit critical points on $\{R\le\|x\|\le R'\}$, for δ sufficiently small. Hence $\|p\|\le R$. Conversely, if p is a critical point of $\omega_{|f^{-1}(\delta)\cap B^n_R}$, then there exists λ such that $H_\delta(p,\lambda)=0$. Taking δ small enough, p is close to $C\cup\Sigma_0$ and so, by continuity, (p,λ) is close to $H^{-1}(0)$. Hence $(p,\lambda)\in B^{n+1}_{R'}$. We have proved that $\Pi_x(H_\delta^{-1}(0)\cap B^{n+1}_{R'})$ is exactly the set of critical points of $\omega_{|f^{-1}(\delta)\cap B^n_R}$ that we denote by C_δ. Similarly $\Pi_x(K_\delta^{-1}(0)\cap B^{n+1}_{R'})=C_\delta\sqcup\Sigma_f$.

Let us compute $\deg(H,R')$. We choose a function $\tilde\omega:\mathbb{R}^n\to\mathbb{R}$ that uniformly approximates ω in the Whitney C^2-topology and such that $\tilde\omega_{|f^{-1}(\delta)\cap B^n_R}$ is Morse correct. One notices that, since the gradient of ω is outward pointing along $f^{-1}(0)\cap S^{n-1}_R$, $\tilde\omega_{|f^{-1}(\delta)\cap B^n_R}$ is not inward at any critical point lying in $f^{-1}(\delta)\cap S^{n-1}_R$. Let $\{p_1,\dots,p_m\}$ be the set of critical points of $\tilde\omega_{|f^{-1}(\delta)\cap B^n_R}$ lying in $\{\|x\|<R\}$ and let $\{s_1,\dots,s_m\}$ be the set of their respective indices. Since $f(0)>0$, $\omega_{|\{f\ge\delta\}}$ and $\omega_{|\{f\le\delta\}}$ are correct and so are $\tilde\omega_{|\{f\ge\delta\}}$ and $\tilde\omega_{|\{f\le\delta\}}$.

By Lemma 4.1, for all $j\in\{1,\dots,m\}$ there exists $\lambda_j\ne0$ such that $\lambda_j\nabla\tilde\omega(p_j)+\nabla f(p_j)=0$. By Lemma 4.2, each (p_j,λ_j) is a non-degenerate zero of $\tilde H_\delta$, that is defined by

$$\tilde H_\delta(x,\lambda)=(\lambda\nabla\tilde\omega(x)+\nabla f(x),f-\delta),$$

and

$$(-1)^{s_j}=\operatorname{sign}\lambda_j^n\times\operatorname{sign}J\tilde H_\delta(p_j,\lambda_j).$$

Summing over all the points p_j and using the fact that $\tilde H_\delta$ is close to H_δ, we obtain

$$\deg(H_\delta,R')=\sum_{j=1}^m\operatorname{sign}\lambda_j^n\times(-1)^{s_j}.$$

We have to compute $\deg(K_\delta,R')$. First we see that, putting

$$\tilde K_\delta(x,\lambda)=(\lambda\nabla\tilde\omega(x)+\nabla f(x),\lambda(f-\delta)),$$

the points (p_j,λ_j) are non-degenerate zeros of $\tilde K_\delta$ and

$$J\tilde K_\delta(p_j,\lambda_j)=\lambda_jJ\tilde H_\delta(p_j,\lambda_j).$$

Hence

$$(-1)^{s_j}=\operatorname{sign}\lambda_j^{n-1}\times\operatorname{sign}J\tilde K_\delta(p_j,\lambda_j).$$

The points $(q_i,0)$ are the other zeros of $\tilde K_\delta$. Taking a Morse approximation of f around a point q_i, if necessary, which gives us an approximation of $\tilde K_\delta$ near $(q_i,0)$, we prove that

$$\deg_{(q_i,0)}\tilde{K}_\delta = \text{sign}(f(q_i)-\delta)\times \deg_{q_i}\nabla f.$$

Finally we get that

$$\deg(K_\delta, R') = \sum_{j=1}^{m} \text{sign}\,\lambda_j^{n-1}\times(-1)^{s_j} + \sum_{i=1}^{t}\text{sign}(f(q_i)-\delta)\times \deg_{q_i}\nabla f.$$

Now we relate these two degrees to Euler characteristics. By Theorem 2.6, we have

$$\chi\left(f^{-1}(\delta)\cap B_R^n\right) = \sum_{j=1}^{m}(-1)^{s_j},$$
$$\chi\left(\{f\geq\delta\}\cap B_R^n\right) = 1+\sum_{j|\lambda_j<0}(-1)^{s_j},$$
$$\chi\left(\{f\leq\delta\}\cap B_R^n\right) = \sum_{j|\lambda_j>0}(-1)^{s_j}.$$

The term 1 that appears in the second formula is the contribution of the point 0, which is a Morse critical point of $\omega_{|\{f\geq\delta\}\cap B_R^n}$. Note also that, since $\nabla\omega$ is outward pointing along S_R^{n-1}, no inward critical point lies on this sphere. From the two latter formulae, we deduce that

$$\chi\left(\{f\geq\delta\}\cap B_R^n\right)-\chi\left(\{f\leq\delta\}\cap B_R^n\right) = 1-\sum_{j=1}^{m}\text{sign}\,\lambda_j\times(-1)^{s_j}.$$

Collecting all this information, we have, if n is even,

$$\chi\left(f^{-1}(\delta)\cap B_R^n\right) = \deg H, \qquad \textbf{(A)}$$

$$\chi\left(\{f\geq\delta\}\cap B_R^n\right)-\chi\left(\{f\leq\delta\}\cap B_R^n\right)-\sum_{i=1}^{t}\text{sign}(f(q_i)-\delta)\cdot\deg_{q_i}\nabla f = 1-\deg K. \qquad \textbf{(B)}$$

If n is odd, then

$$\chi\left(f^{-1}(\delta)\cap B_R^n\right)+\sum_{i=1}^{t}\text{sign}(f(q_i)-\delta)\cdot\deg_{q_i}\nabla f = \deg K, \qquad \textbf{(C)}$$

$$\chi\left(\{f\geq\delta\}\cap B_R^n\right)-\chi\left(\{f\leq\delta\}\cap B_R^n\right) = 1-\deg H. \qquad \textbf{(D)}$$

For $i\in\{1,\ldots,r\}$, $\text{sign}(f(q_i)-\delta) = -\text{sign}\,(\delta)$ and for $i\in\{r+1,\ldots,t\}$, we have $\text{sign}(f(q_i)-\delta) = \text{sign}\,f(q_i)$. Combining this with Lemma 5.9 yields, if n is even,

$$\chi(f^{-1}(0))-\sum_{i=1}^{r}\deg_{q_i}\nabla f = \deg H,$$

$$\chi(\{f\geq 0\})-\chi(\{f\leq 0\})-\sum_{i=r+1}^{t}\text{sign}(f(q_i))\cdot\deg_{q_i}\nabla f = 1-\deg K.$$

If n is odd, then

$$\chi(f^{-1}(0)) + \sum_{i=r+1}^{t} \operatorname{sign}(f(q_i)) \cdot \deg_{q_i} \nabla f = \deg K,$$

$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) + \sum_{i=1}^{r} \deg_{q_i} \nabla f = 1 - \deg H.$$

Finally, by Lemma 3.3,

$$\sum_{i=1}^{r} \deg_{q_i} \nabla f = \deg \nabla f - \deg L_2$$

and

$$\sum_{i=r+1}^{t} \operatorname{sign} f(q_i) \cdot \deg_{q_i} \nabla f = \deg L_1. \qquad \square$$

EXAMPLES. (1) Let $f(x_1, x_2) = -x_1^2 x_2^5 + x_1^4 x_2^3 + 5x_2^3 - 5x_1^2 x_2 - 4x_2^2 + 4x_1^2$. The computer gives that $\dim \mathbb{R}[x_1, x_2]/(f, f_{x_1}, f_{x_2}) = 13$ so that $f^{-1}(0)$ may admit singularities. Let us consider H and $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$H(x_1, x_2, x_3) = (x_1 x_3 + f_{x_1}, x_3(x_2 - 1) + f_{x_2}, f),$$

$$K(x_1, x_2, x_3) = (x_1 x_3 + f_{x_1}, x_3(x_2 - 1) + f_{x_2}, f x_3).$$

Here we use the distance function $\omega(x_1, x_2) = \frac{1}{2}(x_1^2 + (x_2 - 1)^2)$. Since $f(0, 1) = 1 > 0$, we can apply the previous theorems. Using methods of Section 3, we find $\deg H = 5$, $\deg K = 1$, $\deg \nabla f = -4$, $\deg L_1 = -1$ and $\deg L_2 = 1$. By our theorem, we have

$$\chi(f^{-1}(0)) = 5 + (-4) - 1 = 0,$$
$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 + (-1) + (-1) = -1.$$

(2) Let $f(x_1, x_2, x_3) = x_1^2 x_2^3 + x_1 x_2^4 - 2x_1^3 - 2x_1^2 x_2 - x_2^3 - x_1 x_2 + 2x_1 x_3 + x_3^2 + 2x_1 + 1$. First $\dim \mathbb{R}[x_1, x_2, x_3]/(f, f_{x_1}, f_{x_2}, f_{x_3}) = 6$, so that $f^{-1}(0)$ may have singularities. We find that $\deg H = 3$, $\deg K = 5$, $\deg \nabla f = -2$ and $\deg L_1 = \deg L_2 = 0$. Hence

$$\chi(f^{-1}(0)) = 5 - 0 = 5,$$

$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - 3 - (-2) = 0.$$

REMARK 5.11. Formulas given in Theorem 5.10 are still true with the weaker hypothesis that the set of critical points of f is compact and the proof is similar to the one we presented above. However, in that case, the maps H, K, L_1 and L_2 can not admit a finite number of zeros and so their total degrees are more difficult to compute.

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