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FINITE COMPLEXES WHOSE SELF-HOMOTOPY EQUIVALENCE GROUPS REALIZE THE INFINITE CYCLIC GROUP

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ABSTRACT. Examples of finite complexes are given whose self-homotopy equivalences group is isomorphic to the group of integers.

We denote by $\mathcal{E}(X)$ the group of (based) homotopy classes of self-homotopy equivalences of a space X. The group $\mathcal{E}(X)$ has been studied for various classes of spaces, and general properties such as finiteness properties, finite presentability *etc.*, have been obtained. One can find a review of these results in [1].

In this note we deal with one of the most fundamental questions, that is, the realizability problem: For a given group Π when does there exist a space X with $\mathcal{E}(X) \cong \Pi$? A large family of finite cyclic groups can be realized by the groups of self-homotopy equivalences of finite complexes by S. Oka [5]. In case $\Pi = Z$, the group of integers, D. W. Kahn [3] first constructed a space X for which $\mathcal{E}(X) \cong Z$. Kahn's space X is not finite dimensional, so he asked if there is a finite connected complex with the same property. We give an affirmative answer to this question.

Now we state our results. Recall that the homotopy group $\pi_7(S^4) \cong Z \oplus Z_{12}$ is generated by the Hopf map ν and suspension elements $E\nu'$ and $E\alpha$ of order 4 and of order 3 respectively, [6]. For an integer q, we define the map $f_q: S^7 \vee S^7 \longrightarrow S^4$ by $f_q|_{S^7_1} = q\nu$, $f_q|_{S^7_2} = E\nu'$, where $f_q|_{S^7_i}$ denotes the restriction of f_q to the i-th sphere. Let us denote by C_{f_q} the mapping cone of f_q.

THEOREM. *If q is an integer prime to* 6, *then* $\mathcal{E}(C_{f_q}) \cong Z$.

By definition, $\pi_7(C_{f_q}) = Z_q \oplus Z_3$ for an arbitrary integer q, so C_{f_q} and $C_{f_{q'}}$ are not homotopy equivalent whenever $q' \neq \pm q$. Thus we obtain the following.

COROLLARY. *The spaces C_{f_q} with q prime to* 6 *provide an infinite family of finite complexes each of whose self-homotopy equivalence group is isomorphic to Z.*

The proof of the theorem is based on an exact sequences obtained in [4] which can be regarded as a variant of the Barcus-Barratt exact sequence (Theorem 6.1, [2]). Let A and B be spaces and $f: A \longrightarrow B$ an arbitrary map. A map

$$\ell: C_f \longrightarrow C_f \vee SA$$

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is defined by shrinking the equator $A \times 1/2$ of CA, the reduced cone of A. Using ℓ, we define the map $\lambda\colon [SA, C_f] \longrightarrow [C_f, C_f]$ by $\lambda(\alpha) = \nabla(1 \vee \alpha)\ell$, where ∇ is the folding map of C_f.

We let $i\colon B \longrightarrow C_f$ and $p\colon C_f \longrightarrow SA$ denote the natural inclusion and the projection respectively. In the case that $i_*\colon [B,B] \longrightarrow [B, C_f]$, $p^*\colon [SA, SA] \longrightarrow [C_f, SA]$ and the suspension map $\Sigma\colon [A,A] \longrightarrow [SA, SA]$ are bijections, we define two maps

$$\Phi\colon [C_f, C_f] \longrightarrow [B,B]$$

and

$$\Psi\colon [C_f, C_f] \longrightarrow [A,A]$$

by $\Phi = i_*^{-1} i^*$, $\Psi = \Sigma^{-1} p^{*-1} p_*$ respectively. In particular, when we take A to be a wedge sum of spheres $S^{m-1} \vee S^{m-1}$ we have:

PROPOSITION (THEOREM 2.6 [4]). *If B is a simply connected CW complex of $\dim B \leq m-2$ $(m > 2)$ and $f\colon S^{m-1} \vee S^{m-1} \longrightarrow B$ is an arbitrary map, then there is an exact sequence of groups as follows.*

$$[S^m \vee S^m, B] \xrightarrow{\lambda i_*} \mathcal{E}(C_f) \xrightarrow{\Phi\times\Psi} G \longrightarrow 1.$$

Here $G = \{(\delta, \epsilon) \in \mathcal{E}(B) \times \mathcal{E}(S^{m-1} \vee S^{m-1}) \mid \delta f = f\epsilon\}$.

PROOF OF THE THEOREM. Apply the proposition to C_{f_q}, we obtain

$$(1) \qquad [S^8 \vee S^8, S^4] \xrightarrow{\lambda i_*} \mathcal{E}(C_{f_q}) \longrightarrow G \longrightarrow 1.$$

It is easy to see that $\mathcal{E}(S^4) \cong Z_2 = \{\pm\iota_4\}$, $\mathcal{E}(S^7 \vee S^7) \cong \mathrm{GL}(2, Z)$. Let ϵ be an element of $\mathcal{E}(S^7 \vee S^7)$ and (a_{ij}) be the corresponding matrix of ϵ, that is, a_{ij} $(1 \leq i,j \leq 2)$ are integers such that for the natural generators ι_i^7 $(i = 1,2)$ of $H_7(S^7 \vee S^7)$ $\epsilon_*(\iota_i^7) = \sum_{j=1,2} a_{ij}\iota_j^7$.

$$(2) \qquad f_q\epsilon|_{S_i^7} = qa_{i1}\nu + a_{i2}E\nu' \quad (i = 1,2).$$

It is well known that $[\iota_4, \iota_4] = 2\nu - (E\nu' + E\alpha)$, and hence $(-\iota_4)\nu = \nu - (E\nu' + E\alpha)$. Thus, if $f_q\epsilon = (-\iota_4)f_q$, then $f_q\epsilon|_{S_1^7} = q(\nu - E\nu' - E\alpha)$. By the formula (2), this is impossible when q is prime to 3. Therefore, the group G is isomorphic to the subgroup of $\mathcal{E}(S^7 \vee S^7)$ consisting of elements ϵ, with, $f_q\epsilon = f_q$. From the last equality one can easily show that $a_{11} = 1$, $a_{21} = 0$ and $a_{12} = 0 \bmod 4$, $a_{22} \equiv 1 \bmod 4$. Moreover, the determinant of the matrix (a_{ij}) is ± 1, and hence $a_{22} = 1$.

$$G \cong \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \equiv 0 \bmod 4 \right\} \subset \mathrm{GL}(2, Z).$$

Since $\pi_8(S^7 \vee S^7) \cong Z_2 \oplus Z_2$ with generators η_7 and η_7, $\pi_8(S^4) \cong Z_2 \oplus Z_2$ with generators $E\nu'\eta_7$ and $\nu\eta_7$ (see [6]), we obtain that $f_{q*}([S^8 \vee S^8, S^7 \vee S^7]) = [S^8 \vee S^8, S^4]$ (q is an odd integer). Thus $\lambda i_*([S^8 \vee S^8, S^4]) = 0$. By (1), $\mathcal{E}(C_{f_q}) \cong G \cong Z$.

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