Can. J. Math., Vol. XXXIV, No. 1, 1982, pp. 17–22

COMMUTATIVITY CONDITIONS ON RINGS WITH INVOLUTION

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1. Let R be a ring with involution $*$. We denote by S, K and $Z = Z(R)$ the symmetric, the skew and the central elements of R respectively.

In [**4**] Herstein defined the hypercenter $T(R)$ of a ring R as

$$T(R) = \{a \in R \mid ax^n = x^n a, \text{ all } x \in R, \text{ for some } n = n(a, x) \geqq 1\}$$

and he proved that in case R is without non-zero nil ideals then $T(R) = Z(R)$.

In this paper we offer a partial extension of this result to rings with involution.

We focus our attention on the following subring of R:

$$H = H(R) = \{a \in R \mid as^n = s^n a, \text{ all } s \in S, \text{ for some } n = n(a) \geqq 1\}.$$

(We shall write $H(R)$ as H whenever there is no confusion as to the ring in question.)

Clearly H contains the central elements of R. Our aim is to show that in a semiprime ring R with involution which is 2 and 3-torsion free, the symmetric elements of H are central. This result cannot be strengthened as is easily seen by looking at the 2×2 matrices over a field with symplectic involution.

As a consequence we prove that, under the above hypotheses, the set

$$\{a \in S \mid (as)^n = s^n a^n, \text{ all } s \in S, \text{ for some } n = n(a) \geqq 1\}$$

is precisely $Z \cap S$.

We begin our study of H by recalling that an element $a \in R$ is *quasi-unitary* if $a + a^* + aa^* = a + a^* + a^*a = 0$; such an element induces the automorphism

$$\varphi\colon x \rightarrow x + ax + xa^* + axa^* = (1 + a)x(1 + a)^{-1}.$$

This automorphism preserves S and K, and leaves the elements of Z invariant and it preserves H. We write this as:

Remark 1. H is an invariant subring of R; that is, H is preserved by the automorphisms induced by the quasi-unitary elements of R.

Received July 19, 1980 and in revised form January 20, 1981.

https://doi.org/10.4153/CJM-1982-003-4 Published online by Cambridge University Press

We now recall a remark due to Herstein [**5**, Theorem 6.1.1].

Remark 2. Let R be 2-torsion free and let k be a skew quasi-regular element of R. If $a \in H$, then

$$(1 - k)^{-1} (ak - ka) (1 + k)^{-1} \in H.$$

The first question one would ask is what can be said about $H(R)$ when R is a simple artinian ring. This was done in [**6**] where we showed that under the above hypotheses $H \cap S = Z \cap S$.

We now quote a result which is essential to this note, namely:

THEOREM A. *Let R be a ring with* $*$, 2 *and* 3-*torsion free. If $a \in H \cap S$, $s \in S$ and $k \in K$ then $as - sa$ and $ak - ka$ are nilpotent.*

Proof (sketched). In [**6**] we have shown that for $a \in H \cap S$ and $s \in S$, then $as - sa$ is nilpotent. The proof runs as follows: we first consider the subring $\langle a, s \rangle$ of R generated by a and s. Factoring out its nil radical N we get a ring $\bar{A} = \langle a, s \rangle / N$ which is still 2 and 3-torsion free. We now concentrate our attention to the prime images of $\bar{A}$ and relate our question to the investigation of H in simple artinian rings.

The skew analog follows similarly.

2. Throughout the paper all rings are 2 and 3-torsion free. We begin with

LEMMA 1. *Let R be a prime ring with involution. Then H contains no symmetric nilpotent elements.*

Proof. Let $a \in H \cap S$ be such that $a^2 = 0$. For every $x \in R$, $ax + x^*a$ is a symmetric element; hence

$$a(ax + x^*a)^n = (ax + x^*a)^n a.$$

This implies $a(x^*a)^n = (ax)^n a$ which means $(ax)^n a \in S$.

Therefore, for all $x, y \in R$ we have:

$$((ax)^n(ay)^n)^n a = (((ax)^n(ay)^n)^n a)^* = a((y^*a)^n(x^*a)^n)^n =$$
$$((ay^*)^n(ax^*)^n)^n a = ((ay)^n(ax)^n)^n a.$$

We have shown that, for all $x, y \in R$,

$$((ax)^n(ay)^n)^n - ((ay)^n(ax)^n)^n \in l_R(a)$$

where $l_R(a) = \{r \in R \mid ra = 0\}$ is the left annihilator of a.

We now set $\bar{R} = aR/aR \cap l_R(a)$. Since R is a prime ring, $\bar{R}$ is also a prime ring. Moreover $\bar{R}$ satisfies the polynomial identity $(x^n y^n)^n = (y^n x^n)^n$.

Applying [**5**, Theorem 1.3.4] we get that $\bar{R}$ is an order in a simple algebra Q, finite dimensional over its center C, where C is the field of quotients of $Z(\bar{R})$. Moreover Q satisfies the same polynomial identities of $\bar{R}$; in particular Q satisfies $(x^n y^n)^n = (y^n x^n)^n$.

We now write Q as D_m where D is a finite dimensional central division algebra. If F is a maximal subfield of D, then $Q \otimes_C F \cong F_r$ where $r^2 = \dim_C Q$ and F_r satisfies the given polynomial identity.

Suppose $r > 1$ and let e_{ij} be the usual matrix units. Then it is enough to set $a = e_{11}$ and $b = e_{11} + e_{12}$ to get $(a^n b^n)^n = b$ and $(b^n a^n)^n = a$, a contradiction; thus $r = 1$.

It follows that Q is commutative and so is $\bar{R}$. This implies that R satisfies the generalized polynomial identity $axaya = ayaxa$.

By [**1**, Proposition 6], R contains a *-closed subring R_0 containing a which is an order in the 2×2 matrices over a field C. Since

$$a \in H(R) \cap S \cap R_0 \subset H(R_0) \cap S \subset H(C_2) \cap S$$

and, by [**6**], $H(C_2) \cap S \subset C$, it follows that $a = 0$.

By using the invariance of H, Lemma 1 and Theorem A imply the following:

LEMMA 2. *If R is prime then $H \cap S$ centralizes all skew nilpotent elements.*

Proof. Let $k \in K$ be such that $k^m = 0$. If $a \in H \cap S$, by Theorem A, $ak - ka \in N(\langle a, k\rangle)$ where $N(\langle a, k\rangle)$ is the nil radical of the subring generated by a and k. It follows that the element

$$\alpha = (1 + k)^{-1} (ak - ka) (1 - k)^{-1}$$

still belongs to $N(\langle a, k\rangle)$; hence α is nilpotent.

On the other hand, by Remark 2, α is a symmetric element belonging to H. We then quote Lemma 1 to get $\alpha = 0$, and so, $ak - ka = 0$.

We recall that if R is a prime ring with involution and $S = RC$ is the central closure of R, then S is endowed with an involution which extends the involution of R [**5**, Lemma 2.4.1].

We are now able to prove the main theorem of this paper. If $x, y \in R$, we use the notation $[x, y] = xy - yx$.

THEOREM 1. *Let R be a semiprime ring with involution which is 2 and 3-torsion free. Then $H \cap S = Z \cap S$.*

Proof. Suppose first that R is prime. Let $a \in H \cap S$. For all $x \in R$, then $x - x^* \in K$ and $x + x^* \in S$. Thus

$$[a, x + x^*] \in K \text{ and } [a, [a, x - x^*]] \in K.$$

By Lemma 2 we get

$$[a, [a, x + x^*]] = 0 \text{ and } [a, [a, [a, x - x^*]]] = 0,$$

https://doi.org/10.4153/CJM-1982-003-4 Published online by Cambridge University Press

and it is easy to deduce that

(1) $[a, [a, [a, x]]] = 0$ for all $x \in R$.

If x, y are elements of R, then from (1) it follows that

$$[a, [a, x]]\,[a, [a, y]] = 0$$

(see for instance [**5**, Lemma 1.1.9]). Thus for all $x, y \in R$ we have:

$$(a^2x - 2axa + xa^2)\,(a^2y - 2aya + ya^2) = 0.$$

By [**5**, Lemma 1.3.2] there exist λ, μ in the extended centroid C of R such that

(2) $a^2 - 2\lambda a + \mu = 0.$

Moreover since $a \in S$ and char $R \neq 2$, we may assume λ and μ to be symmetric elements of C.

Let now s be a symmetric element of R. Then, since $a^2 - 2\lambda a = -\mu \in C$, we have

$$(a^2 - 2\lambda a)s = s(a^2 - 2\lambda a);$$

hence $a^2s - sa^2 = 2\lambda(as - sa)$. Since $[a, [a, s]] = 0$, it follows that

$$2\lambda(as - sa) = a^2s - sa^2 = (as - sa)a + a(as - sa) = 2a(as - sa).$$

Therefore $2(a - \lambda)(as - sa) = 0$ and so, $(a - \lambda)\,(as - sa) = 0$.

Set $b = a - \lambda$. If $b \neq 0$, by the defining properties of the central closure of R, there exists a non-zero ideal $U = U^*$ of R such that $0 \neq bU \subset R$ (see [**5**, Chapter 1, § 3]).

Let now $x \in U$ and s be a symmetric element of R; then

$$k = bx(as - sa) + (as - sa)x^*b$$

is a skew element of R. Since $[a, [a, s^2]] = 0$ implies $[a, s]^2 = 0$, it follows that

$$k^2 = (as - sa)x^*b^2x(as - sa)$$

and so $k^3 = 0$.

By Lemma 2, $ak = ka$ and recalling that $b = a - \lambda$, we get $bk = kb$. We then deduce $b^2k = bkb$ and so $b^3x(as - sa) = 0$, for all $x \in U$, $s \in S$. We have proved that $b^3U(as - sa) = 0$. Since R is prime it is immediate that $as - sa = 0$ or $b^3U = 0$.

If $as - sa = 0$ for all $s \in S$ then, since a is also in S, we easily get $a \in Z$ (see [**5**, Theorem 2.1.5]).

If $b^3 = 0$, that is $(a - \lambda)^3 = 0$, from (2) we get $(a - \lambda)^2 = 0$.

Let now $s \in S$ be in R; from $[a, [a, s]] = 0$ we get $[b, [b, s]] = 0$. The last equality together with $b^2 = 0$ implies $-bsb = bsb$; that is $2bsb = 0$, and so $bSb = 0$. Since $b^* = b$ and $bSb = 0$ and R is prime then $b = 0$ and so $a = \lambda$.

Let now R be any semiprime ring 2 and 3-torsion free. Then R has a subdirect representation in prime rings whose characteristic is still different from 2 and 3.

Let P be a prime ideal of R. If $P^* \not\subset P$, then $P + P^*/P$ is a non-zero ideal of R/P and every element of such an ideal can be written as $x + P = x + x^* + P$ where $x \in P^*$. Let $a \in H(R) \cap S$; then

$$(a + P)(x + P)^n = (a + P)(x + x^* + P)^n = a(x + x^*)^n + P$$
$$= (x + x^*)^n a + P = (x + x^* + P)^n(a + P) = (x + P)^n(a + P).$$

Hence, for all $x + P \in P + P^*/P$,

$$(a + P)(x + P)^n = (x + P)^n(a + P)$$

where $n \geqq 1$ is a fixed integer. Since the exponent n is bounded, the primeness of R enables us to apply a result of [**2**, Theorem 2.1], and we conclude that $a + P$ centralizes $P + P^*/P$. Therefore $a + P \in Z(R/P)$.

If $P^* \subset P$, then R/P has the involution induced by the one in R. Moreover if $u \in R/P$ is a symmetric element, u^2 is a symmetric element of R/P which is the image of a symmetric element of R. An easy computation ensures that if $a \in H(R) \cap S$ then $a + P \in H(R/P) \cap S$. By the prime case studied above we can conclude that $a + P \in Z(R/P)$.

3. We consider the following set:

$$V = \{a \in R \mid (as)^n = s^n a^n, \text{ all } s \in S, \text{ for some } n = n(a) \geqq 1\}.$$

In [**2**] Felzenszwalb characterized the center of a semiprime ring by proving that the set

$$\{a \in R \mid (ax)^n = x^n a^n, \text{ all } x \in R, \text{ for some } n = n(a) \geqq 1\}$$

is exactly the center of the ring. Here we shall extend this result to rings with involution. In fact, using Theorem 1, we prove the following:

THEOREM 2. *Let R be a semiprime ring with involution which is 2 and 3-torsion free. Then $V \cap S = Z \cap S$.*

Proof. Let $a \in V \cap S$ and suppose that $a^{n+1} = 0$. Then, for all $s \in S$, $(as)^n a = s^n a^{n+1} = 0$. We will show that from $(as)^n a = 0$ it follows that $(as)^{n-1} a = 0$.

Let x be an element of R; then for all $s \in S$,

$$(sa)^{n-1}x^* + x(as)^{n-1} \in S$$

and we have:

$$0 = (a((sa)^{n-1}x^* + x(as)^{n-1}))^n a = (a(sa)^{n-1}x^* + ax(as)^{n-1})^n a;$$

it follows that

$$0 = (as)^{n-1}(a(sa)^{n-1}x^* + ax(as)^{n-1})^n a = (as)^{n-1}(ax(as)^{n-1})^n a.$$

Hence $((as)^{n-1}ax)^{n+1} = 0$; this implies that $(as)^{n-1}aR$ is a nil left ideal of R of bounded exponent. By Levitzki's Theorem we get $(as)^{n-1}a = 0$. By a repeated application of this argument, we obtain $asa = 0$, for all $s \in S$.

Let now $x \in R$; then $a(x + x^*)a = 0$ and so, $axa = -ax^*a$. Thus

$$a(xax)a = -a(x^*ax^*)a = axax^*a = -axaxa,$$

that is $2(ax)^2a = 0$; hence $(ax)^3 = 0$ for all $x \in R$. By Levitzki, $a = 0$.

Hence we may assume that $a^{n+1} \neq 0$. Since $a \in V \cap S$ then $(as)^n = s^n a^n$ and so, taking *, $(sa)^n = a^n s^n$. Consequently

$$a^{n+1}s^n = a(sa)^n = (as)^n a = s^n a^{n+1} \text{ for all } s \in S.$$

This shows that $a^{n+1} \in H \cap S$. By Theorem 1 we get that $a^{n+1} \in Z$. On the other hand

$$(as)^{n+1} = a(sa)^n s = a^{n+1}s^{n+1} = s^{n+1}a^{n+1} \text{ for all } s \in S.$$

By the above argument it follows that $a^{n+2} \in Z$.

Let P be a prime ideal of R. Then if $\bar{a}$ is the image of a in R/P then $\bar{a}^{n+1}, \bar{a}^{n+2} \in Z(R/P)$. Therefore, for every $\bar{x} \in R/P$,

$$\bar{a}^{n+1}\bar{a}\bar{x} = \bar{a}^{n+2}\bar{x} = \bar{x}\bar{a}^{n+2} = \bar{a}^{n+1}\bar{x}\bar{a}$$

which implies $\bar{a}^{n+1}(\bar{a}\bar{x} - \bar{x}\bar{a}) = 0$. Since $\bar{a}^{n+1}$ is central and so regular we get $\bar{a}\bar{x} - \bar{x}\bar{a} = 0$.

We have shown that $\bar{a}$ is central in R/P, for every prime ideal P of R. Hence a is central in R.

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