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THE N-DIMENSIONAL APPROXIMATION CONSTANT*

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I describe recent advances in our understanding of the simultaneous approximation problem.

It is a nice occasion to speak at Kurt Mahler's 80th birthday. Originally I was to speak on Abel's functional equation under the somewhat slender excuse that Kurt was working recently on functional equations related to Abel but than I thought it better to change my subject.

Let me first remind you of the simultaneous approximation problem. Given a set of n non-zero real numbers which I write in the vector notation as

$$\underline{x} = (x_1, \ldots, x_n) .$$

We want to approximate $\underline{x}$ by rational fractions $\underline{p}/q = \{p_1/q, \ldots, p_n/q\}$ with common denominator q. Dirichlet's box principle tells us that we can always determine $\underline{p}/q$ in infinitely many ways so that

$$|x_i - p_i/q| < q^{-(n+1)/n}, \quad i=1, \ldots, n .$$

I shall use the number $\max\limits_i q|qx_i - p_i|^n$ as a measure of goodness of approximation. Let us define the approximation constant of $\underline{x}$, $C(\underline{x})$, to be the infimum of all $c > 0$ for which the inequality

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$$\max_i q\,|qx_i - p_i|^n < c$$

has infinitely many solutions in integers $q > 0$, $p_1, \ldots, p_n$. We know of course that $c \le 1$ but $c < 1$ is actually known since Minkowski, whatever $\underline{x}$. The n-dimensional approximation constant C_n is the supremum of $C(\underline{x})$ for all possible $\underline{x}$. So C_n is the smallest positive number for which the above inequality has infinitely many integer solutions for all $c = C_n + \varepsilon$ and all real $\underline{x}$. The principal problem then is to determine the exact value of C_n. We know from Hurwitz that $C_1 = 1/\sqrt{5}$, but for no $n > 1$ is the value of C_n known.

There is a curious bifurcation of the problem into two separate questions, which appears already in the case of $n = 1$ although not as blatantly as for $n > 1$. We may ask for C_n when <u>all</u> possible non-zero vectors $\underline{x}$ are taken into consideration, and ask for the value C_n^* when only those $\underline{x}$ are admitted for which $x_1, \ldots, x_n$ together with 1 form a rational basis of a real algebraic number field of degree $n+1$. We certainly must have $C_n^* \le C_n$, with strict inequality if the worst approximable vectors $\underline{x}$ are <u>not</u> the algebraic kind. For $n = 1$ we know of course that $C_1^* = C_1$; for $C(x)$ to be $= 1/\sqrt{5}$, x must be in the quadratic number field of $\sqrt{5}$.

Although the exact value of C_n is not known, there exist good estimates from below. In 1927 Furtwängler showed that $C_n \ge 1/\sqrt{\Delta}$ where Δ is the absolute discriminant of any real number field of degree $n+1$; so to obtain the strongest Furtwängler estimate one has to take the real number field with smallest possible discriminant. For $n = 3$ this happens to be 23 (namely the field of $x^3-x-1 = 0$) so according to Furtwängler $C_3 \ge 1/\sqrt{23}$. Davenport later significantly improved on Furtwängler. Using Kurt Mahler's methods from the geometry of numbers he showed that $\sqrt{\Delta}$ in the inequality can be replaced by D_n, the critical determinant of the (n+1)-dimensional region

$$\max_i |x_i|^n |x_{n+1}| \le 1 .$$

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Fortunately for those who are not familiar with critical determinants, Davenport's inequality has a more mundane form. In fact he shows, using a remark by Cassels, that

$$D_n \leq \sqrt{\Delta}_{n,s}/V_{n,s} \quad , \text{ hence } \quad C_n \geq V_{n,s}/\sqrt{\Delta}_{n,s}$$

where $\Delta_{n,s}$ is the smallest absolute value of discriminants of real number fields of degree $n+1$ which have s pairs of complex conjugate algebraic conjugates (so that $2s \leq n$) and $2^n V_{n,s}$ is the greatest volume of an origin-centred parallelotope inscribed in the region

$$\prod_{i=1}^{n-2s} |x_i| \prod_{i=n-2s+1}^{n-s} \tfrac{1}{2}(x_i^{\,2}+x_{s+i}^{\,2}) \leq 1 \ .$$

(Warning to readers of Davenport's 1955 paper: there is some mix-up with the factor 2^n). For instance in the case of $n=2,\ s=0$ the Davenport region is the double hyperbola $|x_1|.|x_2| \leq 1$, and in the case of $n=2, s=1$ it is the disk $x_1^{\,2} + x_2^{\,2} \leq 2$. Since the unit cube can always be inscribed in the Davenport region, $V_{n,s} \geq 1$, and Furtwängler is a corollary. Indeed no greater parallelogram can be inscribed in a circle with radius $\sqrt{2}$ and so $V_{2,1} = 1$, but clearly $V_{2,0} = 2$, and since 49 is the minimal discriminant of a totally real cubic field (namely of $x^3+2x^2-x-1=0$), we obtain the improved estimate $C_2 \geq 2/7$ $(>1/\sqrt{23})$. This estimate is originally due to Cassels.

The beautiful inequality of Davenport has several interesting features. Notice first that to compute $V_{n,s}/\sqrt{\Delta}_{n,s}$ we need to know two wholly unrelated quantities: one is the minimal absolute discriminant of real number fields of degree $n+1$ and various reality types. This at present is a problem for the computer and as computer techniques improve, so will (hopefully) our knowledge of minimal discriminants. For instance

$$\Delta_{3,1} = 275 \ , \quad \Delta_{3,0} = 725$$

$$\Delta_{4,2} = 1609, \quad \Delta_{4,1} = 4511 \ , \ \Delta_{4,0} = 14641 \text{ (Hunter 1957)},$$

and figures are now available up to $n=6$.

The other quantity which appears in Davenport's inequality, namely $V_{n,s}$, is a harmless looking quantity and one would think at first sight that its order of difficulty is that of a hard student competition problem (in fact in this year's ΣUMS* competition I did set a problem nearly equivalent to the calculation of $V_{4,2}$). On closer inspection it turns out to be much tougher than it looks and it is only comparatively recently (1980) that Tom Cusick was able to show that $V_{3,1} = 2$ and $V_{3,0} = \frac{1}{4}\sqrt{27}$. Indeed details for the second result were so bothersome that Cusick throws up his hands in despair and omits details. Quite recently Sam Krass (to whom I shall come back later) has obtained through simpler calculations the inequalities

$$V_{4,2} \geq 16/9 \ , \quad V_{4,1} \geq 2 \ , \quad V_{4,0} \geq 4 \ ,$$

and more general inequalities for all $V_{n,s}$. I commend this problem to your attention.

My other remark concerning Davenport is this: Although discriminants of number fields do appear in the inequality, from Davenport's proof it does not at all follow that the inequality still holds if we restrict $\underline{x}$ to bases of real number fields. For n=2 this was only quite recently settled by Cusick and Adams, and we now know that $C_2^* = 2/7$. It seems likely to me (though this could be a minority opinion) that also C_2 is equal to 2/7 and indeed that Davenport's inequality is a strict equality for all n, both for C_n and C_n^*, but this is certainly a very hazardous guess.

Now to a quite recent result of Sam Krass. Some twenty years ago I proposed a continued fraction-like n-dimensional approximation algorithm for the practical calculation of simultaneous approximation fractions for a given $\underline{x}$. Since that time there was a proliferation of such algorithms, some supplying all good approximations but only at the expense of arithmetic complexity. If you are not content with regarding such an algorithm merely as an object of interest for its own sake, like for instance Jacobi's, but wish to turn it into a useful mathematical tool

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then arithmetic or geometric simplicity is of overriding importance. Whether all good approximations are really supplied by the algorithm is not necessarily a decisive factor. Suppose that the algorithm supplies infinitely many really good approximations, then it is conceivable that it can be used for the purposes of the approximations constants C_n and C_n^*.

It is not necessary for me here to go into details of the algorithm itself. The essential point is that instead of approximating $\underline{x}$ by single rational points (in n-space) I use rational vertices of an n-simplex inside which the point is situated. The algorithm merely supplies a standard method for chopping up n-space into such simplexes. For instance when $n=1$, the chopping up corresponds to a Farey dissection of the real line and the vertices of the intervals are just consecutive approximation fractions (including intermediary fractions) of the continued fraction development of x.

Suppose that $m_i^{(j)}/m_0^{(j)}$ is the approximation fraction of x_i at the j-th vertex; then we can form the matrix $M = (m_i^{(j)})$, $i,j = 0,\ldots,n$, and because of the way the algorithm proceeds, we always have $\det M = \pm 1$. It follows that the volume of the simplex is just $(n!.m_0^{(0)}\ldots m_0^{(n)})^{-1}$. The approximation number of $\underline{x}$ with respect to the 0-vertex $\underline{m}^{(0)}/m_0^{(0)} = \underline{m}/m_0$ say, is

$$m_0.\operatorname*{Max}_i|m_0 m_i - m_i|^n = m_0^{n+1}.\operatorname*{Max}_i|x_i - m_i/m_0|^n .$$

Now as the denominators grow, the simplexes become quite small, and it is better to scale them up to a size where the volume is just $1/m!$. That is, we multiply the coordinates by the factor $\lambda = (m_0^{(0)}\ldots m_0^{(n)})^{1/n}$. Let us write $w_i = m_0^{(i)}/(m_0^{(0)}\ldots m_0^{(n)})^{1/(n+1)}$ so that $\prod_{i=0}^{n} w_i = 1$; also let us write

$$\xi_i = \lambda(x_i - m_i/m_0) .$$

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Then the approximation number is

$$w_0^{n+1} . \operatorname{Max}_i |\xi_i|^n \quad .$$

Here then is a "continuous" generalization of the approximation number. Given a simplex of volume 1 with n arbitrary real vertices and one vertex at the origin $\underline{0}$, n+1 "weights" w_i at the vertices with $\prod_{i=0}^{n} w_i = 1$, and a point $\underline{\xi}$ inside the simplex, we can assign an approximation number to his configuration consisting of a simplex, weights and a point. At each stage the algorithm supplies a unique new configuration and hence a corresponding approximation number, and it was my hope to show (perhaps with substantial computer help) that whatever the initial simplex configuration, eventually one will always land with a configuration of appropriately small approximation number, obtaining thereby an *upper* estimate for C_n.

This was also the original starting point of Sam Krass; but somewhat unexpectedly, success came from the opposite direction: an estimate from *below*, and referring to C_n^* . I shall confine myself to stating his main result; the computational details are quite heavy.

THEOREM OF KRASS. *Let* F *be a real algebraic number field of degree* n+1 *with* s *pairs of conjugate complex algebraic conjugates and absolute discriminant* $\Delta(F)$. *Then to every* $\varepsilon > 0$ *there exists a rational basis* $1, \xi_1, \ldots, \xi_n$ *of* F *such that*

$$C(\underline{\xi}) > V_{n,s}/\sqrt{\Delta(F)} - \varepsilon \ .$$

That is, Davenport's inequality is achieved within the relevant field.

This is a nice sharpening of Davenport's theorem. One of the interesting aspects of the method of proof is that Davenport's expression appears very naturally, without any reference to lattice constants and the geometry of numbers. This seems to indicate that the simplex method is in some sense complementary to the geometry of numbers. The algorithm itself does not appear in the argument, but weighted simplexes are extensively used, strengthening my belief that a simplex algorithm of this kind is the most effective extension of the ordinary continued fraction process.

I join everyone here in wishing Kurt Mahler many more years of activity.

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