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# ON NONREFLEXIVE BANACH SPACES WHICH CONTAIN NO $c_0$ OR $l_p$

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**Introduction.** The first infinite-dimensional reflexive Banach space $X$ such that no subspace of $X$ is isomorphic to $c_0$ or $l_p$, $1 \leqq p < \infty$, was constructed by Tsirelson [**8**]. In fact, he showed that there exists a Banach space with an unconditional basis which contains no subsymmetric basic sequence and which contains no superreflexive subspace. Subsequently, Figiel and Johnson [**4**] gave an analytical description of the conjugate space $T$ of Tsirelson's example and showed that there exists a reflexive Banach space with a symmetric basis which contains no superreflexive subspace; a uniformly convex space with a symmetric basis which contains no isomorphic copy of $l_p$, $1 < p < \infty$; and a uniformly convex space which contains no subsymmetric basic sequence and hence contains no isomorphic copy of $l_p$, $1 < p < \infty$. Recently, Altshuler [**2**] showed that there is a reflexive Banach space with a symmetric basis which has a unique symmetric basic sequence up to equivalence and which contains no isomorphic copy of $l_p$, $1 < p < \infty$. This space complements nicely an earlier theorem of Altshuler [**1**] which states that a Banach space $X$ with a symmetric basis $\{x_n, f_n\}$ is isomorphic to $c_0$ or $l_p$, $1 \leqq p < \infty$, if and only if both $X$ and the closed linear subspace $[f_n]_{n=1}^{\infty}$ in $X^*$ have unique symmetric basic sequences up to equivalence.

In this paper, we consider the problem of constructing nonreflexive Banach spaces which contain no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$. It is well-known that the nonreflexive James space $J$ contains no isomorphic copy of $c_0$ or $l_1$. However, it was proved [**5**] that every infinite dimensional subspace of $J$ contains an isomorphic copy of $l_2$. The construction of the space $J$ has been generalized to a large class of nonreflexive Banach spaces [**3**]. We show that in this class of generalized James spaces there exist quasi-reflexive spaces of order one which have a basis and contain no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$. We also show that there exist separable, non-quasi-reflexive spaces which contain no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$.

Let $E$ be a Banach space with monotone, normalized, symmetric

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basis $\{e_n\}$. The generalized James space is the Banach space

$$J(E) = \left\{x = (a_1, a_2, \ldots): \lim_n a_n = 0, \right.$$
$$\left. \|x\| = \sup \left\| \sum_{i=1}^{n} (a_{p_{i+1}} - a_{p_i})e_i \right\| < \infty \right\},$$

where the supremum is taken over all integers $1 \leqq p_1 < p_2 < \ldots < p_n$. It was proved [**3**], where $J(E)$ was defined using an equivalent norm, that the sequence of unit vectors $\{x_n\}$ is a basic sequence and that if $\{e_n\}$ is boundedly complete, then $\{x_n\}$ is a basis of $J(E)$. For the terminology used in this paper and the Banach spaces mentioned above which are constructed respectively by Altshuler, Figiel and Johnson, and Tsirelson, we refer to [**7**].

**1.** It is well-known that if $X$ is a Banach space with basis $\{x_n\}$ then every symmetric basic sequence in $X$ is equivalent to a block basic sequence of $\{x_n\}$. To consider subspaces of $X$ which are isomorphic to either $c_0$ or $l_p$, $1 \leqq p < \infty$, it suffices to consider the subspaces which are spanned by symmetric block basic sequences of $\{x_n\}$. For the rest of the paper, we shall assume that $E$ is a Banach space with a monotone, normalized symmetric basis $\{e_n\}$.

The following result follows easily from the definition of $J(E)$.

LEMMA 1. *If*

$$y_n = \sum_{i=p_n}^{q_n} a_i x_i, \; n = 1, 2, \ldots$$

*is a block basic sequence of* $\{x_n\}$ *in* $J(E)$ *with* $q_n + 2 < p_{n+1}$, $n = 1, 2, \ldots$ *then there exists a block basic sequence*

$$z_n = \sum_{i=h_n}^{k_n} b_i e_i, \quad n = 1, 2, \ldots$$

*in E such that* $\{y_n\}$ *dominates* $\{z_n\}$. *Furthermore, if* $\lim_n a_n = 0$ *then we may require that* $\lim_n b_n = 0$.

*Proof.* For each $n = 1, 2, \ldots$ there exist $i_1^n < i_2^n < \ldots < i_{m_n}^n$ such that

$$\|y_n\| = \left\| \sum_{j=1}^{m_n} (a_{i_{j+1}^n} - a_{i_j^n})e_j \right\|.$$

Without loss of generality, we may assume that $p_n - 1 \leqq i_1^n$ and $i_{m_n}^n \leqq q_n + 1$, for each $n = 1, 2, \ldots$. Define

$$z_n = \sum_{j=1}^{m_n} (a_{i_{j+1}^n} - a_{i_j^n})e_{i_j^n}, \quad n = 1, 2, \ldots.$$

Since $q_n + 2 < p_{n+1}$, $n = 1, 2, \ldots$, and $\{e_n\}$ is symmetric, $\{z_n\}$ is a block basic sequence in $E$ such that $\|z_n\| = \|y_n\|$, $n = 1, 2, \ldots$. It is clear that $\{y_n\}$ dominates $\{z_n\}$ and if $\lim_n a_n = 0$ then the coefficients of $\{z_n\}$ are approaching zero.

THEOREM 2. *If $\{e_n\}$ is a boundedly complete basis of $E$, then $J(E)$ does not contain a subspace isomorphic to $c_0$.*

*Proof.* Let

$$y_n = \sum_{i=p_n}^{q_n} a_i x_i, \quad n = 1, 2, \ldots$$

be a bounded block basic sequence in $J(E)$ which is equivalent to the unit vector basis of $c_0$. By taking a subsequence if necessary, we may assume that $q_n + 2 < p_{n+1}$, $n = 1, 2, \ldots$. By Lemma 1, $\{y_n\}$ dominates a bounded block basic sequence $\{z_n\}$ in $E$. However, $\{z_n\}$ dominates the unit vector basis of $c_0$. Hence $[z_n]_{n=1}^{\infty}$ is isomorphic to $c_0$, a contradiction.

**2.** Let $E$ be any reflexive Banach space with a monotone, normalized symmetric basis $\{e_n\}$. If $J(E)$ contains an isomorphic copy of $l_p$, $1 \leqq p < \infty$, we don't know, in general, whether this implies that $E$ contains an isomorphic copy of $l_p$, $1 \leqq p < \infty$. However, if $E$ is the reflexive space constructed either by Altshuler [**2**] or Figiel and Johnson [**4**], then we will show that $J(E)$ is a quasi-reflexive space of order one which contains no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$.

Let us recall the definition of the space $T$. Let $T_0$ be the space of all sequences of scalars which are eventually zero and let $\{t_n\}$ be the unit vector basis of $T_0$. For $x = \sum_n a_n t_n \in T_0$, define

$$\|x\|_0 = \max_n |a_n|$$

and

$$\|x\|_{n+1} = \max \left\{ \|x\|_n, \tfrac{1}{2} \max \sum_{j=1}^{k} \left\| \sum_{i=p_j+1}^{p_{j+1}} a_i t_i \right\|_n \right\}$$

where the inner max is taken over all $p_j$'s with $k \leqq p_1 < p_2 < \ldots < p_{k+1}$, $k = 1, 2, \ldots$. Let $\|x\| = \lim_n \|x\|_n$ for all $x$ in $T_0$. The space $T$ is the completion of $(T_0, \|\cdot\|)$. We need the following properties of $T$ (e.g., p. 96, [**7**]).

LEMMA 3. (i) *For any integer $k = 1, 2, \ldots$ and any normalized block*

$$v_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i, \quad n = 1, 2, \ldots, k$$

*with $k \leqq p_1 < p_2 < \ldots < p_{k+1}$, then*

$$\sum_{n=1}^{k} |\alpha_n| \geqq \left\| \sum_{n=1}^{k} \alpha_n v_n \right\| \geqq \tfrac{1}{2} \sum_{n=1}^{k} |\alpha_n|$$

*for all scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$.*

(ii) *Given* $1 \leqq p_0 < p_1 < \ldots$, *there exists an integer* $r$ *such that for any normalized block*

$$v_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i, \quad n = 0, 1, 2, \ldots, r$$

*in* $T$, *then*

$$\left\| v_0 + \frac{1}{r} \sum_{i=1}^{r} v_i \right\| \leqq \frac{7}{4}.$$

Now, for each $n = 1, 2, \ldots$ and $x = (a_1, a_2, \ldots) \in c_0$ let

(1) $$\|x\|_n = \sup_k \sum_{i=1}^{k} \hat{a}_i w_i / (2^n + 2^{-n} s_k)$$

where $\{\hat{a}_i\}$ is a rearrangement of $\{|a_i|\}$ in non-increasing order, $w_i = 1/i$, $i = 1, 2, \ldots$ and

$$s_k = \sum_{i=1}^{k} w_i, \quad k = 1, 2, \ldots.$$

Notice that

(2) $$2^{-n-1} \sup_i |a_i| \leqq \|x\|_n \leqq 2^n \sup_i |a_i|$$

for all $n = 1, 2, \ldots$ and $x = (a_1, a_2, \ldots) \in c_0$. Let

(3) $$E = \{x = (a_1, a_2, \ldots) \in c_0 : (\|x\|_1, \|x\|_2, \ldots, \|x\|_n, \ldots) \in T\}.$$

If $x \in E$, we define $\|x\| = \|\sum_n \|x\|_n t_n\|_T$. Altshuler [**2**] proved that $E$ is a reflexive space such that the unit vector basis $\{e_n\}$ is a symmetric basis of $E$ and $E$ contains no isomorphic copy of $l_p$, $1 \leqq p < \infty$.

If the sequence of norms in (1) are replaced by

$$\|x\|_n = \sup_k \sum_{i=1}^{k} |\hat{a}_i| / (2^n + 2^{-n} k), \quad n = 1, 2, \ldots.$$

then properties (2) also hold. The space $E$ obtained in (3) by this sequence of norms is the reflexive space constructed in [**4**]. They showed that the unit vectors $\{e_n\}$ form a symmetric basis of $E$ and $E$ contains no isomorphic copy of $l_p$, $1 < p < \infty$. For the rest of the paper, we shall let $E$ denote the reflexive space constructed above by either Altshuler or Figiel and Johnson. Using the properties (2) and induction as in the proof of Lemma 3.*b*.11 [**7**], we have the following result.

LEMMA 4. *Given* $\epsilon > 0$, $1 \leqq q_1 < q_2 < \ldots$ *and* $\delta_1 > \delta_2 > \ldots$ *with* $\lim_n \delta_n = 0$, *there exist integers* $1 \leqq m_1 < m_2 < \ldots$ *and* $1 = N_0 < N_1 < \ldots$ *such that for any block basic sequence* $\{z_i\}$ *of* $\{e_i\}$ *with*

$$z_i = \sum_{j=q_{m_i}+1}^{q_{m_{i+1}}} c_j e_j, \quad \|z_i\| \leqq 1 \quad \textit{and}$$

$$\sup_{q_{m_i} < j \leqq q_{m_{i+1}}} |c_j| \leqq \delta_{m_i}, \quad i = 1, 2, \ldots$$

*we have*

$$\left\|\sum_{i=1}^{\infty} \beta_i z_i\right\| \leqq \left\|\sum_{i=1}^{\infty} \beta_i v_i\right\|_T + \epsilon$$

*for all* $\{\beta_i\}$ *with* $\sup_i |\beta_i| \leqq 1$ *where*

$$v_i = \sum_{j=N_{i-i}}^{N_i - 1} \|z_j\|_j t_j, \quad i = 1, 2, \ldots.$$

*Moreover,* $\|v_i\| \leqq 1 + \epsilon$, $i = 1, 2, \ldots$.

*Thus for any block basic sequence*

$$z_m = \sum_{i=q_m+1}^{q_{m+1}} c_i e_i, \quad m = 1, 2, \ldots$$

*in* $E$ *such that* $\lim_i c_i = 0$, *there is a subsequence of* $\{z_m\}$ *which is equivalent to a block basic sequence of* $\{t_n\}$ *in* $T$.

*Remark.* The unit vectors in $E$ all have norm

$$A = \left\|\sum_n (2^n + 2^{-n})^{-1} t_n\right\| < 1.$$

Thus for the technical convenience of having a normalized basis, we renorm $E$ by using $A^{-1}$ times the original norm of $E$.

THEOREM 5. *The space* $J(E)$ *is quasi-reflexive of order one and no subspace of* $J(E)$ *is isomorphic either to* $c_0$ *or* $l_p$, $1 \leqq p < \infty$.

*Proof.* Since $E$ contains no isomorphic copy of $c_0$, Theorem 2 guarantees that $J(E)$ contains no subspace which is isomorphic to $c_0$.

Suppose that $J(E)$ contains an isomorphic copy of $l_p$, $1 \leqq p < \infty$. Since the unit vector basis of $l_p$, $1 \leqq p < \infty$, is equivalent to every bounded block basic sequence, we may assume that there exists a normalized block basic sequence

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i, \quad i = 1, 2, \ldots,$$

in $J(E)$ such that $\lim_n a_n = 0$ and $\{y_n\}$ is equivalent to the unit vector basis of $l_p$, $1 \leqq p < \infty$. By Lemma 1, there exists a block

$$z_n = \sum_{i=q_n+1}^{q_{n+1}} c_i e_i, \quad n = 1, 2, \ldots$$

in $E$ such that $\lim_n c_n = 0$ and $\{y_n\}$ dominates $\{z_n\}$. By Lemma 4, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ which is equivalent to a block $\{v_i\}$ of $\{t_n\}$ in $T$. Thus there are constants $K_1$, $K_2$, $K_3$ such that for each $k = 1, 2, \ldots$

$$K_1 k^{1/p} \geqq \left\|\sum_{i=1}^{k} y_{n_i}\right\| \geqq \left\|\sum_{i=1}^{k} z_{n_i}\right\| \geqq K_2 \left\|\sum_{i=1}^{k} v_i\right\| \geqq K_3 \left(\frac{k}{2}\right).$$

This is impossible when $1 < p < \infty$. Hence $J(E)$ contains no isomorphic copy of $l_p$, $1 < p < \infty$.

For the case $p = 1$, by a result of [**6**], let $1 \geqq \epsilon_0 > 7/8$ be such that for all scalars $\alpha_n$,

$$(*) \quad \epsilon_0 \sum_n |\alpha_n| \leqq \left\| \sum_n \alpha_n y_n \right\| \leqq \sum_n |\alpha_n|.$$

Let $\delta_n = 2(\sup_{p_n < i \leqq p_{n+1}} |a_i|)$, $n = 1, 2, \ldots$. Choose $\epsilon > 0$ such that

$$(1 + \epsilon)7/4 + 2\epsilon < 2\epsilon_0.$$

Since $\lim_n a_n = 0$, by taking a subsequence of $\{y_n\}$ if necessary, we may assume that $1 \geqq \delta_1 > \delta_2 > \ldots$ and $\sum_n \delta_n < \epsilon$.

Apply Lemma 4 to $\epsilon > 0$, $p_1 < p_2 < \ldots$ and $\{\delta_n\}$. Then there exist $\{m_i\}_{i=1}^\infty$ and $\{N_i\}_{i=0}^\infty$ such that the properties in Lemma 4 hold. For the sequence $\{N_i\}_{i=0}^\infty$, by Lemma 3, there exists an integer $r$ such that property (ii) in Lemma 3 holds. By $(*)$, we have

$$2\epsilon_0 \leqq \left\| y_{m_i} + \frac{1}{r} \sum_{i=2}^{r+1} y_{m_i} \right\|.$$

The norm on the right above is of the form

$$\left\| \sum_{i=1}^{k} (c_{q_{i+1}} - c_{q_i}) e_i \right\|$$

where for each $j = 1, 2, \ldots, r + 1$ there are at most two $i$'s such that $p_{m_j} < q_i \leqq p_{m_j+1}$ and $p_{m_{j+1}} < q_{i+1} \leqq p_{m_{j+1}+1}$. Notice that if $p_{m_j} < q_i \leqq p_{m_j+1}$ then

$$|a_{q_i}| \leqq \tfrac{1}{2}\delta_{m_j}.$$

Since $\{e_n\}$ is symmetric, for each $j = 1, 2, \ldots, r + 1$ there exist $q_i^{m_j}$, $i = 1, 2, \ldots, k_j$ such that

$$p_{m_j} < q_1^{m_j} < q_2^{m_j} < \ldots < q_{k_j}^{m_j} \leqq p_{m_j+1} \quad \text{and}$$

$$\begin{aligned} \left\| y_{m_1} + \frac{1}{r} \sum_{j=2}^{r+1} y_{m_j} \right\| &\leqq \left\| \sum_{i=1}^{k_1} (a_{q_{i+1}^{m_1}} - a_{q_i^{m_1}}) e_{q_i^{m_1}} \right. \\ &\quad \left. + \frac{1}{r} \sum_{j=2}^{r+1} \sum_{i=1}^{k_j} (a_{q_{i+1}^{m_j}} - a_{q_i^{m_j}}) e_{q_i^{m_j}} \right\| + \delta_{m_1} + \frac{1}{r} \sum_{j=2}^{r+1} \delta_{m_j} \\ &\leqq \left\| z_{m_1} + \frac{1}{r} \sum_{j=2}^{r+1} z_{m_j} \right\| + \epsilon \end{aligned}$$

where

$$z_j = \sum_{i=1}^{k_j} (a_{q_{i+1}^{m_j}} - a_{q_i^{m_j}}) e_{q_i^{m_j}}, \quad j = 1, 2, \ldots, r + 1.$$

Notice that $\|z_j\| \leqq \|y_{m_j}\| = 1$ and

$$\sup_i |a_{q_{i+1}m_j} - a_{q_i m_j}| \leqq \delta_{m_j}, \quad j = 1, 2, \ldots, r+1.$$

Hence by Lemma 4,

$$\left\| z_1 + \frac{1}{r} \sum_{j=2}^{r+1} z_j \right\| \leqq \left\| v_1 + \frac{1}{r} \sum_{j=2}^{r+1} v_j \right\|$$

where

$$v_i = \sum_{j=N_{i-1}}^{N_i - 1} \|z_j\|_j t_j \quad \text{and} \quad \|v_i\| \leqq 1 + \epsilon, \quad i = 1, 2, \ldots, r+1.$$

By Lemma 3,

$$2\epsilon_0 \leqq \left\| y_{m_1} + \frac{1}{r} \sum_{j=2}^{r+1} y_{m_j} \right\| \leqq \left\| v_1 + \frac{1}{r} \sum_{j=2}^{r+1} v_j \right\| + \epsilon < (1+\epsilon)\frac{7}{4} + 2\epsilon.$$

This contradicts the choice of $\epsilon$. This completes the proof that $J(E)$ contains no isomorphic copy of $l_1$. We conclude that $\{x_n\}$ is a shrinking basis of $J(E)$ and $J(E)$ is quasi-reflexive space of order one [**3**].

*Remark.* In [**3**], it is proved that if $\{e_n\}$ is block $p$-Hilbertian for some $1 < p < \infty$, then $J(E)$ is quasi-reflexive of order one. Hence if $E$ is a uniformly convex space with symmetric basis then $J(E)$ contains no subspace which is isomorphic to $l_1$.

**3.** Let $X$ be a quasi-reflexive space of positive order with basis $\{x_n\}$ such that $X$ contains no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$. For each $n = 1, 2, \ldots$, let $X_n = X$ and let

$$Y = \left(\sum_n \oplus X_n\right)_E = \{y = (y_1, y_2, \ldots): y_n \in X_n,$$
$$(\|y_1\|, \|y_2\|, \ldots, \|y_n\|, \ldots) \in E\},$$

where $E$ is Altshuler's space and $\|y\| = \|(\|y_1\|, \ldots, \|y_n\|, \ldots)\|_E$. Then clearly, $Y$ is a non-quasi-reflexive space with basis such that $Y^{**}$ is separable. For any block $\{y_n\}$ in $Y$, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that either $\{y_{n_i}\}$ is equivalent to a block in $X_n$ or a block in $E$ [**4**]. Since $X$ and $E$ contain no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$, it follows that $Y$ contains no isomorphic copy of $c_0$ or $l_p$, $1 \leqq p < \infty$.

Finally, it is obvious that if $X$ is a quasi-reflexive space of order one which contains no copy of $c_0$ or $l_p$, $1 \leqq p < \infty$ then the direct sum of $k$ copies of $X$ is a quasi-reflexive space of order $k$ which contains no $c_0$ or $l_p$, $1 \leqq p < \infty$.

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