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A GENERALISATION OF THE CLUNIE–SHEIL-SMALL THEOREM II

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Abstract

We study properties of the simply connected sets in the complex plane, which are finite unions of domains convex in the horizontal direction. These considerations allow us to state new univalence criteria for complex-valued local homeomorphisms. In particular, we apply our results to planar harmonic mappings obtaining generalisations of the shear construction theorem due to Clunie and Sheil-Small [‘Harmonic univalent functions’, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **9** (1984), 3–25].

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1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$. A function $f : \mathbb{D} \to \mathbb{C}$ is said to be harmonic in $\mathbb{D}$ if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are real harmonic, that is, they satisfy the Laplace equation. It is well known that, since $\mathbb{D}$ is simply connected, f can be written in the form

$$f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{D},$$

where h and g are analytic in $\mathbb{D}$. The Jacobian J_f of f in terms of h and g is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2, \quad z \in \mathbb{D}.$$

It is known that if f is harmonic in $\mathbb{D}$ and $J_f(z) \neq 0$ for all $z \in \mathbb{D}$, then f is locally one-to-one. Moreover, if $J_f > 0$ for all $z \in \mathbb{D}$, then f is locally one-to-one and sense-preserving. For more information about harmonic functions see, for example, [2].

Clunie and Sheil-Small in [1] gave the following theorem, known as the shear construction.

THEOREM 1.1. *A function $f = h + \overline{g}$ harmonic in $\mathbb{D}$ with positive Jacobian is a one-to-one sense-preserving mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is an analytic one-to-one mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

The shear construction has many applications as a univalence criterion and as a method of constructing harmonic mappings (see, for example, [3–11, 13–15]).

In this paper we generalise the theorem of Clunie and Sheil-Small and extend our previous results given in [12]. In Section 2 we prove some topological properties of simply connected sets. In Section 3 we use results from Section 2 to give new univalence conditions for complex-valued local homeomorphisms. Finally, we apply these conditions to planar harmonic mappings in Section 4.

2. Topological properties

Let D be a nonempty domain in the complex plane and let $P(D)$ be the orthogonal projection of the set D onto the imaginary axis. For a given real number a put

$$D^a := D \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}.$$

If D^a is nonempty and has a finite number of connected components, then by $N_a(D)$ we denote the number of connected components of D^a. If $D^a = \emptyset$, then we set $N_a(D) = 0$. If D^a has an infinite number of connected components, then we set $N_a(D) = \infty$.

We consider domains D which can be represented as a finite union of domains D_j, $j = 1, \ldots, n$, convex in the horizontal direction. To study the properties of $N_a(D)$, it is convenient to extend the sets D_j, $j = 1, \ldots, n$, in the following manner. For each $j = 1, 2, \ldots, n$, let the maximal horizontal extension D'_j of the set D_j be defined by

$$D'_j = \bigcup_{a \in \mathbb{R}} D_j(a), \quad j = 1, \ldots, n, \tag{2.1}$$

where

$$D_j(a) := \begin{cases} \emptyset & \text{if } N_a(D_j) = 0, \\ D^a & \text{if } N_a(D) \cdot N_a(D_j) = 1, \\ \text{the connected component of the set } D^a & \\ \text{with a nonempty intersection with } D_j & \text{if } N_a(D) \cdot N_a(D_j) > 1. \end{cases}$$

The extension D'_j inherits some properties of the set D_j. It is clear from the definition that D'_j is a domain convex in the horizontal direction and the inclusion $D_j \subset D'_j$ holds. Moreover, $D = \bigcup_{j=1}^n D'_j = \bigcup_{j=1}^n D_j$ and $\bigcap_{j=1}^n D_j \subset \bigcap_{j=1}^n D'_j$. The maximal horizontal extensions D'_j, $j = 1, 2, \ldots, n$, have some additional properties (not necessarily true for the sets D_j, $j = 1, \ldots, n$), which we prove in the following lemmas.

Lemma 2.1. *Let D be a finite union of domains D_j, $j = 1, \ldots, n$, convex in the horizontal direction and equal to their maximal horizontal extensions defined by (2.1), that is, $D_j = D'_j$ for all $j = 1, \ldots, n$. If I is a nonempty subset of $\{1, 2, \ldots, n\}$ and $\widetilde{D} = \bigcup_{m \in I} D_m$, then*

$$\forall_{a \in \mathbb{R}} \forall_{j \in I} (D^a_j \neq \emptyset) \Rightarrow (N_a(\widetilde{D} \setminus D_j) = N_a(\widetilde{D}) - 1). \tag{2.2}$$

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PROOF. Formula (2.1) ensures that, for each $j = 1, 2, \dots, n$ and for any real number a, either the set D_j^a is a connected component of D^a, or $D_j^a = \emptyset$. This property is inherited by the subsets $\widetilde{D}$ of D in the following sense. If $\emptyset \neq I \subset \{1, 2, \dots, n\}$ and $\widetilde{D} = \bigcup_{m\in I} D_m$, then, for each $m \in I$ and any real a, either the set D_m^a is a connected component of $\widetilde{D}^a$, or $D_m^a = \emptyset$. Hence, (2.2) follows. □

LEMMA 2.2. *Let D be a finite union of domains D_j, $j = 1, \dots, n$, convex in the horizontal direction and equal to their maximal horizontal extensions. Then, for each real number a,*

$$N_a(D) = \sum_{j=1}^{n} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{n} N_a(D_j \cap D_k)\}}. \tag{2.3}$$

PROOF. Fix a. We use induction on n, the number of domains whose union gives D.

First observe that if $n = 1$, then D is a domain convex in the horizontal direction and in that case $N_a(D) \in \{0, 1\}$ and thus $N_a(D) = N_a(D)/\max\{1, N_a(D)\}$.

Let $n > 1$. Assume the hypothesis holds for every positive integer $m < n$, that is,

$$N_a(\widetilde{D}) = \sum_{j=1}^{m} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m} N_a(D_j \cap D_k)\}}$$

for $\widetilde{D} = \bigcup_{j=1}^{m} D_j$, and let $D = \bigcup_{j=1}^{m+1} D_j$, where each of the domains D_j is convex in the horizontal direction. We prove that

$$N_a(D) = \sum_{j=1}^{m+1} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}}.$$

Observe that $N_a(D) - 1 \leq N_a(\widetilde{D}) \leq N_a(D)$, since $\widetilde{D} \subset D$, $(D \setminus D_{m+1}) \subset \widetilde{D}$ and (2.2) holds for every D_j, $j = 1, 2, \dots, m + 1$. We consider three cases.

Case 1. If $N_a(D_{m+1}) = 0$, then $N_a(D) = N_a(\widetilde{D}) = N_a(\widetilde{D}) + N_a(D_{m+1})$, since $\widetilde{D}^a = D^a$.

Case 2. If $N_a(D_{m+1}) = 1$ and $N_a(D_j \cap D_{m+1}) = 0$ for all $j = 1, 2, \dots, m$, then by (2.2),

$$\begin{aligned} N_a(D) = N_a(\widetilde{D}) + 1 &= \sum_{j=1}^{m} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m} N_a(D_j \cap D_k)\}} + N_a(D_{m+1}) \\ &= \sum_{j=1}^{m} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} + \frac{N_a(D_{m+1})}{\max\{1, \sum_{k=1}^{m+1} N_a(D_{m+1} \cap D_k)\}} \\ &= \sum_{j=1}^{m+1} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}}. \end{aligned}$$

Case 3. If $N_a(D_{m+1}) = 1$ and there exists an index $j \in \{1, 2, \dots, m\}$ such that $N_a(D_j \cap D_{m+1}) = 1$, then $D_{m+1}^a = D_j^a$ by (2.1), and consequently $N_a(D) = N_a(\widetilde{D})$. Let $I \subset \{1, 2, \dots, m\}$ be the set of all such indices j for which $N_a(D_j \cap D_{m+1}) = 1$ and

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denote by $I' = \{1, 2, \dots, m\} \setminus I$ the set of all indices j for which $N_a(D_j \cap D_{m+1}) = 0$. Then, denoting by $|I|$ the number of elements in I,

$$\begin{aligned}
N_a(D) = N_a(\widetilde{D}) &= \sum_{j=1}^{m} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m} N_a(D_j \cap D_k)\}} \\
&= \sum_{j\in I} \frac{N_a(D_j)}{\sum_{k=1}^{m} N_a(D_j \cap D_k)} + \sum_{j\in I'} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} \\
&= \sum_{j\in I} \frac{N_a(D_j)}{|I|} + \sum_{j\in I'} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} \\
&= 1 + \sum_{j\in I'} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} \\
&= \sum_{j\in I\cup\{m+1\}} \frac{N_a(D_j)}{|I \cup \{m+1\}|} + \sum_{j\in I'} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} \\
&= \sum_{j\in I\cup\{m+1\}} \frac{N_a(D_j)}{\sum_{k=1}^{m+1} N_a(D_j \cap D_k)} + \sum_{j\in I'} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}} \\
&= \sum_{j=1}^{m+1} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{m+1} N_a(D_j \cap D_k)\}}.
\end{aligned}$$

This completes the proof of (2.3). □

The proof of Theorem 1.1 of Clunie and Sheil-Small relies on the following lemma.

Lemma 2.3. *Let D be a domain convex in the direction of the real axis and let p be a continuous real-valued function in D. Then the mapping $D \ni w \mapsto w + p(w)$ is one-to-one in D if and only if it is locally one-to-one. In this case the image of D is convex in the direction of the real axis.*

To generalise Lemma 2.3 (see Section 3) we need the following auxiliary results.

Let $D = D_1 \cup D_2$, where D_1, D_2 are domains convex in the horizontal direction, and let $q : D \to \mathbb{C}$ be a continuous, locally one-to-one function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D$. Clearly, $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$ if and only if $N_a(D) = N_a(q(D))$ for all real numbers a. In the following lemma we prove a less obvious result.

Lemma 2.4. *Let $D = D_1 \cup D_2$, where D_1, D_2 are domains convex in the horizontal direction, and let $q : D \to \mathbb{C}$ be a continuous, locally one-to-one function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for each $z \in D$. Then for any real number a the following are equivalent:*

(i) $N_a(D) = N_a(q(D))$;

(ii) $N_a(D_1 \cap D_2) = N_a(q(D_1) \cap q(D_2))$.

Proof. Let a be a fixed real number. By Lemma 2.3 we know that $q(D_1)$ and $q(D_2)$ are convex in the horizontal direction.

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We first show that (i) ⇒ (ii). If $N_a(D_1 \cap D_2) = 1$, then $N_a(q(D_1) \cap q(D_2)) = 1$, by the continuity of q. On the other hand, if $N_a(D_1 \cap D_2) = 0$, then by the equality $N_a(D) = N_a(q(D))$ and the continuity of q, we have $N_a(q(D_1) \cap q(D_2)) = 0$.

We now prove that (ii) ⇒ (i). If $N_a(D) = 0$, then clearly $N_a(q(D)) = 0$. If $N_a(D) = 2$, then $N_a(D_1 \cap D_2) = 0$, and by the equality $N_a(D_1 \cap D_2) = N_a(q(D_1) \cap q(D_2))$ and the continuity of q, we get $N_a(q(D)) = 2$. Finally, if $N_a(D) = 1$, then again by the equality $N_a(D_1 \cap D_2) = N_a(q(D_1) \cap q(D_2))$ and the continuity of q, we get $N_a(q(D)) = 1$. □

We now generalise Lemma 2.4 for the case of open sets in the complex plane which can be represented as a finite union of domains convex in the horizontal direction.

Lemma 2.5. *Let D be a finite union of domains D_j, $j = 1, \dots, n$, convex in the horizontal direction and equal to their maximal horizontal extensions. Let $q : D \to \mathbb{C}$ be a continuous, locally one-to-one function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for each $z \in D$. Then for any real number a the following conditions are equivalent:*

(i) $N_a(D) = N_a(q(D))$;

(ii) $N_a(D_j \cup D_k) = N_a(q(D_j) \cup q(D_k))$ *for all indices* $j, k \in \{1, 2, \dots, n\}$;

(iii) $N_a(D_j \cap D_k) = N_a(q(D_j) \cap q(D_k))$ *for all indices* $j, k \in \{1, 2, \dots, n\}$.

Proof. Let a be a fixed real number. By Lemma 2.3 we know that $q(D_j)$ is a domain convex in the horizontal direction for all $j = 1, 2, \dots, n$.

The condition (ii) is equivalent to (iii) by Lemma 2.4. We prove that the condition (i) is equivalent to (iii).

We first show (i) ⇒ (iii). Assume $N_a(D) = N_a(q(D))$. From the continuity of q,

$$N_a(D_j \cap D_k) \le N_a(q(D_j) \cap q(D_k))$$

for all $j, k \in \{1, 2, \dots, n\}$. If there exist indices $j, k \in \{1, 2, \dots, n\}$ such that

$$0 = N_a(D_j \cap D_k) < N_a(q(D_j) \cap q(D_k)) = 1,$$

then $a \in P(D_j) = P(q(D_j))$ and thus

$$\max\Big\{1, \sum_{k=1}^{n} N_a(D_j \cap D_k)\Big\} = \sum_{k=1}^{n} N_a(D_j \cap D_k)\} \ge 1.$$

Consequently,

$$\max\Big\{1, \sum_{k=1}^{n} N_a(D_j \cap D_k)\Big\} = \sum_{k=1}^{n} N_a(D_j \cap D_k) < \sum_{k=1}^{n} N_a(q(D_j) \cap q(D_k)) = \max\Big\{1, \sum_{k=1}^{n} N_a(q(D_j) \cap q(D_k))\Big\}.$$

Moreover, by the continuity of q, we have $N_a(D_j) = N_a(q(D_j))$ and

$$N_a(D) = \sum_{j=1}^{n} \frac{N_a(D_j)}{\max\{1, \sum_{k=1}^{n} N_a(D_j \cap D_k)\}} > \sum_{j=1}^{n} \frac{N_a(q(D_j))}{\max\{1, \sum_{k=1}^{n} N_a(q(D_j) \cap q(D_k))\}} = N_a(q(D)).$$

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This gives a contradiction with the assumption that $N_a(D) = N_a(q(D))$. Thus we have $N_a(D_j \cap D_k) = N_a(q(D_j) \cap q(D_k))$ for all $j, k \in \{1, 2, \dots, n\}$.

The converse (iii) ⇒ (i) follows from the continuity of q and Lemma 2.2. □

3. Main results

We are now ready to prove our main results.

THEOREM 3.1. *Let D be a finite union of domains D_j, $j = 1, \dots, n$, convex in the horizontal direction and equal to their maximal horizontal extensions. Let $q : D \to \mathbb{C}$ be a continuous, locally one-to-one function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for each $z \in D$. Then the following are equivalent:*

(i) *q is one-to-one in D;*
(ii) *$N_a(D) = N_a(q(D))$ for each real number a;*
(iii) *$N_a(D_j \cap D_k) = N_a(q(D_j) \cap q(D_k))$ for all indices $j, k \in \{1, 2, \dots, n\}$ and for each real number a;*
(iv) *$N_a(D_j \cup D_k) = N_a(q(D_j) \cup q(D_k))$ for all indices $j, k \in \{1, 2, \dots, n\}$ and for each real number a.*

PROOF. By Lemma 2.3 we know that $q(D_j)$ is a domain convex in the horizontal direction for all $j = 1, 2, \dots, n$. Since q is continuous, it is clear that (i) ⇒ (ii). We show that (ii) ⇒ (i).

Assume that for every real number a the equality $N_a(D) = N_a(q(D))$ holds. By Lemma 2.5, this implies that the equality $N_a(D_j \cup D_k) = N_a(q(D_j) \cup q(D_k))$ holds for every real a and all $j, k \in \{1, 2, \dots, n\}$. Thus $P(D_j \cap D_k) = P(q(D_j) \cap q(D_k))$ for all $j, k \in \{1, 2, \dots, n\}$. Now, using [12, Lemma 2.2], we see that q is one-to-one in $D_j \cup D_k$ for all $j, k \in \{1, 2, \dots, n\}$, and therefore q is one-to-one in D.

The equivalence of (ii), (iii) and (iv) follows from Lemma 2.5. □

THEOREM 3.2. *Let D be a simply connected domain in the complex plane which is a finite union of domains D_j, $j = 1, \dots, n$, convex in the horizontal direction with a nonempty intersection. Let $q : D \to \mathbb{C}$ be a continuous, locally one-to-one function such that $q(D)$ is simply connected, and $\operatorname{Im} q(z) = \operatorname{Im} z$ for each $z \in D$. Then q is one-to-one in D.*

PROOF. Without any loss of generality we can assume that the sets D_j, $j = 1, 2, \dots, n$, are equal to their maximal horizontal extensions. By Theorem 3.1, q is one-to-one in D if and only if $N_a(D) = N_a(q(D))$ for each real number a. Therefore, we prove the latter statement.

Clearly, $N_a(D) = N_a(q(D))$ for $a \in P(\bigcap_{j=1}^n D_j)$, since q is one-to-one in the domain

$$\bigcup_{a \in P(\bigcap_{j=1}^n D_j)} D^a,$$

by Lemma 2.3.

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To prove that $N_a(D) = N_a(q(D))$ for an arbitrary a, fix $a_0 \in P(\bigcap_{j=1}^n D_j)$. We show that $N_a(D) = N_a(q(D))$ for all $a > a_0$. The proof for $a < a_0$ is analogous.

Let $D^+ := \bigcup_{a>a_0} D^a$. Clearly, the domain D^+ satisfies all the assumptions on D in the theorem, that is, D^+ is a simply connected domain in the complex plane such that $D^+ = \bigcup_{j=1}^n D_j^+$, where $D_j^+ = \bigcup_{a>a_0} D_j^a$, $j = 1, 2, \ldots, n$, are domains convex in the horizontal direction and $\bigcap_{j=1}^n D_j^+ \neq \emptyset$. In addition, each of the sets D_j^+ is equal to its maximal horizontal extension. Moreover,

$$q(D^+) = q(D)^+ = q(D) \cap \{w \in \mathbb{C} : \operatorname{Im} w > a_0\}$$

is simply connected. Obviously, $N_a(D) = N_a(D^+)$ and $N_a(q(D)) = N_a(q(D^+))$ for $a > a_0$.

For all $j, k = 1, 2, \ldots, n$ define

$$a_{j,k} := \begin{cases} \inf\{a > a_0 : D_j^a \cap D_k^a = \emptyset\} & \text{if } \{a > a_0 : D_j^a \cap D_k^a = \emptyset\} \neq \emptyset, \\ a_0 & \text{if } \{a > a_0 : D_j^a \cap D_k^a = \emptyset\} = \emptyset, \end{cases}$$

and let

$$A := \bigcup_{j,k=1}^n \{a_{j,k}\}.$$

Then A has a finite number of elements and $|A| \le n^2$. Put $\mu := |A \setminus \{a_0\}|$ and define a sequence $\{0, 1 \ldots, \mu + 1\} \ni m \mapsto a_m$ as follows:

$$\begin{aligned} a_0 &:= a_0, \\ a_1 &:= \min(A \setminus \{a_0\}), \\ &\ldots \\ a_\mu &:= \min(A \setminus \{a_0, a_1, a_2, \ldots, a_{\mu-1}\}), \\ a_{\mu+1} &:= \infty. \end{aligned}$$

Observe that $N_a(D^+)$, as a function of a variable a, is a step function (constant on the intervals $[a_m, a_{m+1})$, $m = 0, \ldots, \mu$). Since D^+ is simply connected and the sets D_j^+, $j = 1, 2, \ldots, n$, are convex in the horizontal direction, the points a_m, $m = 1, \ldots, \mu$, are the only ones for which $N_a(D^+)$ may be discontinuous.

Using induction on m, we show that $N_a(D^+) = N_a(q(D^+))$ for all $a \in [a_m, a_{m+1})$. We already know that $N_a(D^+) = N_a(q(D^+))$ for all $a \in [a_0, a_1)$, since $[a_0, a_1) \subset P(\bigcap_{j=1}^n D_j)$. Now, assume that $N_a(D^+) = N_a(q(D^+))$ for $a \in [a_0, a_m)$. We show $N_a(D^+) = N_a(q(D^+))$ for any $a \in [a_m, a_{m+1})$ in two steps. First, we show that $N_{a_m}(D^+) = N_{a_m}(q(D^+))$, and then we prove the equality for all $a \in (a_m, a_{m+1})$.

From the continuity of q, we have $N_a(D^+) \ge N_a(q(D^+))$ for all $a \in [a_m, a_{m+1})$. Assume that $N_{a_m}(D^+) > N_{a_m}(q(D^+))$. Then there exist points $z, w \in (D^+)^{a_m}$ such that $q(z) = q(w)$. Since q is (an open mapping) continuous and locally one-to-one, there exist $a \in [a_{m-1}, a_m)$ and points $u, v \in (D^+)^a$ such that $q(u) = q(v)$. Now we consider two cases. If there exists $j \in 1, 2, \ldots, n$ such that $u, v \in (D_j^+)^a$, then we get a contradiction

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with Lemma 2.3. Otherwise, there exist $j, k \in \{1, 2, \ldots, n\}$ such that $u \in (D_j^+)^a$, $v \in (D_k^+)^a$ with $(D_j^+)^a \cap (D_k^+)^a = \emptyset$ and $q((D_j^+)^a) \cap q((D_k^+)^a) \neq \emptyset$. But, by Lemma 2.5, this is a contradiction with the assumption that $N_a(D^+) = N_a(q(D^+))$ for all $a \in [a_0, a_m)$. Thus we get $N_{a_m}(D^+) = N_{a_m}(q(D^+))$.

We now show that $N_a(q(D^+)) = N_b(q(D^+))$ for every $a, b \in [a_m, a_{m+1})$, that is, $N_a(q(D^+))$ is constant on the interval $[a_m, a_{m+1})$ as a function of a. By Lemma 2.5 it is enough to show that $N_a(q(D_j^+) \cap q(D_k^+))$, as a function of a, is constant on the interval $[a_m, a_{m+1})$ for all $j, k \in \{1, 2, \ldots, n\}$. To this end, we show that for all $j, k \in \{1, 2, \ldots, n\}$ we have $N_a(q(D_j^+) \cap q(D_k^+)) = N_a(D_j^+ \cap D_k^+)$. Obviously, from the continuity of q, for all $j, k \in \{1, 2, \ldots, n\}$, we have

$$N_a(q(D_j^+) \cap q(D_k^+)) \geq N_a(D_j^+ \cap D_k^+)$$

for all $a \in (a_m, a_{m+1})$ and

$$N_{a_m}(q(D_j^+) \cap q(D_k^+)) = N_{a_m}(D_j^+ \cap D_k^+).$$

Moreover, by Lemma 2.5, $N_a(D_j^+ \cap D_k^+)$ is constant on $[a_m, a_{m+1})$.

Assume that there exist $j, k \in \{1, 2, \ldots, n\}$ and $\tilde{a} \in (a_m, a_{m+1})$ such that

$$1 = N_{\tilde{a}}(q(D_j^+) \cap q(D_k^+)) > N_{\tilde{a}}(D_j^+ \cap D_k^+) = 0. \tag{3.1}$$

Then $0 = N_{a_m}(q(D_j^+) \cap q(D_k^+)) = N_{a_m}(D_j^+ \cap D_k^+)$, and consequently there exists a point $\xi = x_\xi + ia_m$ such that $\xi \notin q(D^+)$. Additionally, the inequality

$$\min\{\operatorname{Re} q(z), \operatorname{Re} q(w)\} < x_\xi < \max\{\operatorname{Re} q(z), \operatorname{Re} q(w)\}$$

holds for all $z \in (D_k^+)^{a_m}$ and $w \in (D_j^+)^{a_m}$. By formula (3.1), there exists

$$\eta = x_\eta + i\tilde{a} \in q(D_j^+) \cap q(D_k^+),$$

and by the induction assumption, there exists

$$\theta = x_\theta + ia_0 \in q(D^+)^{a_0} = q(D_k^+)^{a_0} \cap q(D_j^+)^{a_0}.$$

Therefore, since $q(D_k^+)$ and $q(D_j^+)$ are simply connected, there are curves $\gamma_k \subset q(D_k^+)$ and $\gamma_j \subset q(D_j^+)$ joining θ with η. Consequently, there exists a curve γ consisting of γ_k and γ_j which is a closed curve and the point ξ is encircled by γ. Hence the complement of $q(D^+)$ in the extended complex plane is not connected, which gives a contradiction with the assumption that $q(D^+)$ is simply connected. Thus we have $N_a(q(D_j^+) \cap q(D_k^+)) = N_a(D_j^+ \cap D_k^+)$ for all $j, k \in \{1, 2, \ldots, n\}$ and for all $a \in [a_m, a_{m+1})$. This completes the proof by Lemma 2.5. □

4. Applications to harmonic mappings

In this section we apply the results obtained in the previous section to the theory of harmonic mappings.

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THEOREM 4.1. *Let $f = h + \overline{g}$ be a harmonic, locally one-to-one function in $\mathbb{D}$. If*

$$N_a((h-g)(\mathbb{D})) = N_a(f(\mathbb{D})) \tag{4.1}$$

for each real number a, then the following statements are equivalent:

(i) *f is a one-to-one mapping and $f(\mathbb{D})$ is a finite sum of domains convex in the horizontal direction with a nonempty intersection;*
(ii) *$h - g$ is a one-to-one analytic mapping and $(h-g)(\mathbb{D})$ is a finite sum of domains convex in the horizontal direction with a nonempty intersection.*

PROOF. (i) ⇒ (ii). Assume $f(\mathbb{D}) = \bigcup_{j=1}^n D_j$, where D_j, $j = 1, 2, \ldots, n$, are domains convex in the horizontal direction with a nonempty intersection and equal to their maximal horizontal extensions. Since f is one-to-one in the unit disc, there exists $f^{-1} : \bigcup_{j=1}^n D_j \to \mathbb{D}$ and the composition $q := (h-g) \circ f^{-1}$ is a well-defined continuous function in $\bigcup_{j=1}^n D_j$. Moreover, $q(w) = (h-g)(f^{-1}(w)) = w - 2\mathrm{Re}\, g(f^{-1}(w))$ for all $w \in \bigcup_{j=1}^n D_j$. Thus q satisfies the assumptions of Theorem 3.1. Additionally, by (4.1),

$$N_a\Big(\bigcup_{j=1}^n D_j\Big) = N_a\Big(q\Big(\bigcup_{j=1}^n D_j\Big)\Big) \tag{4.2}$$

and, in consequence, q is a one-to-one function by Theorem 3.1. Hence $h - g$ is one-to-one in $\mathbb{D}$, since f is. Obviously, the sets $q(D_j)$, $j = 1, 2, \ldots, n$, are domains convex in the horizontal direction, by Lemma 2.3, and their intersection is not empty by (4.2).

The proof of (ii) ⇒ (i) is essentially the same as that of (i) ⇒ (ii). □

THEOREM 4.2. *Let $f = h + \overline{g}$ be a harmonic, locally one-to-one function in $\mathbb{D}$. If $f(\mathbb{D})$ and $(h-g)(\mathbb{D})$ are nonempty simply connected domains, then the following statements are equivalent:*

(i) *f is a one-to-one mapping and $f(\mathbb{D})$ is a finite sum of domains convex in the horizontal direction with a nonempty intersection;*
(ii) *$h - g$ is a one-to-one analytic mapping and $(h-g)(\mathbb{D})$ is a finite sum of domains convex in the horizontal direction with a nonempty intersection.*

PROOF. (i) ⇒ (ii). Assume that $f(\mathbb{D}) = \bigcup_{j=1}^n D_j$, where D_j, $j = 1, 2, \ldots, n$, are domains convex in the horizontal direction with a nonempty intersection and equal to their maximal horizontal extensions. Then the function

$$\bigcup_{j=1}^n D_j \ni w \mapsto q(w) := (h-g)(f^{-1}(w)) = w - 2\mathrm{Re}\, g(f^{-1}(w))$$

is well defined and continuous in $\bigcup_{j=1}^n D_j$, since f is one-to-one in $\mathbb{D}$. Since $(h-g)(\mathbb{D})$ and $f(\mathbb{D})$ are simply connected domains, the desired result follows from Theorems 3.2, 3.1 and 4.1.

The proof of (ii) ⇒ (i) is essentially the same as that of (i) ⇒ (ii). □

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If in Theorem 4.2 one omits the assumption that both $f(\mathbb{D})$ and $(h - g)(\mathbb{D})$ are simply connected, then the theorem is no longer true (see [12]).

Remark 4.3. Recall that Theorem 1.1 can be reformulated and remains valid for a function convex in any fixed direction. Our results can also be rewritten in this fashion.

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