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BOUNDS FOR THE GROWTH OF A PERTURBATION IN SOME DOUBLE-DIFFUSIVE CONVECTION PROBLEMS

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Abstract

Bounds are presented for the modulus of the complex growth rate p of an arbitrary oscillatory perturbation, neutral or unstable, in some double-diffusive problems of relevance in oceanography, astrophysics and non-Newtonian fluid mechanics.

1. Introduction

Strong motivation exists for the investigation of thermohaline convection owing to its interesting complexities as a double-diffusive phenomenon as well as its direct relevance to the hydrodynamics of oceans [2]. The bounds for the growth rate of a perturbation in thermohaline convection is an important problem especially when the boundaries (one or both) are dynamically rigid so that exact solutions in closed form are not obtainable. Recently, a new scheme of combining the governing equations of conservation of mass, momentum, heat and salt was proposed, [1], that leads to the following bounds on the modulus of the complex growth rate p of an arbitrary oscillatory perturbation, neutral or unstable:

$$|p|^2 < R_s\sigma \text{ for Veronis thermohaline configuration [4],}$$

$$|p|^2 < -R\sigma \text{ for Stern thermohaline configuration [3],}$$

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where R and R_s are the thermal and concentration Rayleigh numbers respectively and σ is the Prandtl number.

In this work we extend the above scheme to rotatory and hydromagnetic configurations and to viscoelastic fluids. If a rotation and/or a magnetic field is applied to the above configurations instability may occur first as an overstable, that is, time dependent, maintained, perturbation. This type of instability arises because a steady type of motion may be too restrictive in the sense that it cannot make use of the sources of potential energy that are available to a time-dependent motion. Besides, it is known that these overstable motions, when they do occur, are generally less efficient in transporting heat and mass and in altering the mean gradients than are the steady convective motions. Hence, one expects that, when the determining parameter (usually a Rayleigh number) exceeds the critical value, the overstable motions will occur at first; as the parameter is further increased so that steady convective instability can occur, the observed motions will be the latter.

In Section 3 of this paper we obtain bounds for $|p|$ in rotatory and/or hydromagnetic configurations in the form of four semicircle theorems; in Section 4 the treatement is extended to viscoelastic fluids. These results are new and valid for all combinations of dynamically free and rigid boundaries. They are summarised below:

a) $|p|^2 <$ greater of $(R_s\sigma, T\sigma^2)$ for rotatory, Veronis thermohaline configuration (VTC),

b) $|p|^2 <$ greater of $(-R\sigma, T\sigma^2)$ for rotatory, Stern thermohaline configuration (STC),

c) $|p|^2 <$ greater of $(R_s\sigma, Q^2\sigma^2)$ for hydromagnetic VTC with perfectly conducting boundaries,

d) $|p|^2 <$ greater of $(-R\sigma, Q^2\sigma^2)$ for hydromagnetic STC with perfectly conducting boundaries,

e) $|p|^2 <$ greater of $[\frac{1}{4}\{(4R_s\sigma + R^2\sigma^2\Gamma^2)^{1/2} + R\sigma\Gamma\}^2, T\sigma^2]$ for rotatory VTC for a viscoelastic fluid,

f) $|p|^2 <$ greater of $[\frac{1}{4}\{(-4R\sigma + R_s^2\sigma^2\Gamma^2)^{1/2} - R_s\sigma\Gamma\}^2, T\sigma^2]$ for rotatory STC for a viscoelastic fluid,

g) $|p|^2 <$ greater of $[\frac{1}{4}\{(4R_s\sigma + R^2\sigma^2\Gamma^2)^{1/2} + R\sigma\Gamma\}^2, Q^2\sigma^2]$ for hydromagnetic VTC with perfectly conducting boundaries for a viscoelastic fluid,

h) $|p|^2 <$ greater of $[\frac{1}{4}\{(-4R\sigma + R_s^2\sigma^2\Gamma^2)^{1/2} - R_s\sigma\Gamma\}^2, Q^2\sigma^2]$ for hydromagnetic STC with perfectly conducting boundaries for a viscoelastic fluid,

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where the complex growth rate is $p = p_r + ip_i$ with $p_r \geqslant 0$, $p_i \neq 0$, T is the Taylor number, Q is the Chandrasekhar number and Γ is an elastic parameter. These results are valid for all combinations of dynamically free and rigid boundaries and hence are applicable to most realistic situations.

2. Mathematical formulation

The relevant governing equations and the boundary conditions in non-dimensional forms for the various configurations are given below:

Rotatory Veronis thermohaline configuration

$$(D^2 - a^2)(D^2 - a^2 - p/\sigma)W = Ra^2\theta - R_s a^2\phi + +TD\zeta, \tag{2.1}$$

$$(D^2 - a^2 - p)\theta = -W, \tag{2.2}$$

$$(D^2 - a^2 - p/\tau)\phi = -W/\tau, \tag{2.3}$$

$$(D^2 - a^2 - p/\sigma)\zeta = -DW, \tag{2.4}$$

and

$$W = 0 = \theta = \phi = D^2W = D\zeta \quad \text{at } z = 0 \text{ and } z = 1, \tag{2.5}$$

(both boundaries dynamically free), or

$$W = 0 = \theta = \phi = DW = \zeta \quad \text{at } z = 0 \text{ and } z = 1, \tag{2.6}$$

(both boundaries rigid), or

$$\left.\begin{array}{l} W = 0 = \theta = \phi = D^2W = D\zeta \quad \text{at } z = 1 \\ \text{(upper boundary dynamically free), and} \\ W = 0 = \theta = \phi = DW = \zeta \quad \text{at } z = 0 \\ \text{(lower boundary rigid).} \end{array}\right\} \tag{2.7}$$

Hydromagnetic Veronis thermohaline configuration

$$(D^2 - a^2)(D^2 - a^2 - p/\sigma)W = Ra^2\theta - R_s a^2\phi - QD(D^2 - a^2)h_z, \tag{2.8}$$

$$(D^2 - a^2 - p)\theta = -W, \tag{2.9}$$

$$(D^2 - a^2 - p/\tau)\phi = -W/\tau, \tag{2.10}$$

$$(D^2 - a^2 - p\sigma_1/\sigma)h_z = -DW, \tag{2.11}$$

and (2.5) or (2.6) or (2.7), with

$$h_z = 0 \quad \text{at } z = 0 \text{ and } z = 1 \tag{2.12}$$

(both boundaries perfectly conducting).

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Rotatory viscoelastic Veronis thermohaline configuration

$$(D^2 - a^2)[D^2 - a^2 - p(1 + \Gamma p)/\sigma]W$$
$$= Ra^2(1 + \Gamma p)\theta - R_s a^2(1 + \Gamma p)\phi + T(1 + \Gamma p)D\zeta, \tag{2.13}$$
$$(D^2 - a^2 - p)\theta = -W, \tag{2.14}$$
$$(D^2 - a^2 - p/\tau)\phi = -W/\tau, \tag{2.15}$$
$$[D^2 - a^2 - p(1 + \Gamma p)/\sigma]\zeta = -(1 + \Gamma p)DW, \tag{2.16}$$

with (2.5) or (2.6) or (2.7).

Hydromagnetic viscoelastic Veronis thermohaline configuration

$$(D^2 - a^2)[D^2 - a^2 - p(1 + \Gamma p)/\sigma]W$$
$$= Ra^2(1 + \Gamma p)\theta - R_s a^2(1 + \Gamma p)\phi - Q(1 + \Gamma p)D(D^2 - a^2)h_z, \tag{2.17}$$
$$(D^2 - a^2 - p)\theta = -W. \tag{2.18}$$
$$(D^2 - a^2 - p/\tau)\phi = -W/\tau, \tag{2.19}$$
$$(D^2 - a^2 - p\sigma_1/\sigma)h_z = -DW, \tag{2.20}$$

with (2.5) or (2.6) or (2.7) and (2.12).

In the above equations, z is the vertical coordinate and $0 \leqslant z \leqslant 1$, $D = d/dz$, a^2 is the square of the wave number, τ is the ratio of mass diffusivity to heat diffusivity, W is the vertical velocity, θ is the temperature, ϕ is the concentration, h_z is the vertical magnetic field, ζ is the vertical vorticity and σ_1 is the magnetic Prandtl number. Further, in deriving (2.13)–(2.20) it has been assumed that the viscoelastic fluid is described by Maxwell's constitutive relation.

The corresponding Stern configurations can be obtained by changing the signs of R and R_s in the above equations.

3. Mathematical analysis for rotatory and/or hydromagnetic thermohaline configurations for a viscous fluid

We prove the following theorems:

THEOREM 1. *For the rotatory Veronis thermohaline configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and* $(\text{radius})^2 =$ *greater of* $(R_s\sigma, T\sigma^2)$, *for all combinations of dynamically free and rigid boundaries.*

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PROOF. For an oscillatory perturbation $p_i \neq 0$. Therefore, it follows from (2.3) and (2.4) that

$$\phi = \frac{\tau p^*}{|p|^2}(D^2 - a^2)\phi + \frac{p^* W}{|p|^2}, \tag{3.1}$$

$$D\zeta = \frac{\sigma p^*}{|p|^2} D(D^2 - a^2)\zeta + \frac{\sigma p^*}{|p|^2} D^2 W. \tag{3.2}$$

Substituting from (3.1) and (3.2) in (2.1), multiplying the resulting equation by W^* (an asterisk indicates complex conjugation) and integrating over z, while substituting for W^* appropriately from (2.2) and (2.3), we obtain

$$\begin{aligned}\int_0^1 W^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) W\,dz &= -Ra^2 \int_0^1 \theta(D^2 - a^2 - p^*)\theta^*\,dz \\ &- R_s a^2\left[-\frac{\tau^2 p^*}{|p|^2}\left\{\int_0^1 (D^2 - a^2)\phi\left(D^2 - a^2 - \frac{p^*}{\tau}\right)\phi^*\,dz\right\} + \frac{p^*}{|p|^2}\int_0^1 |W|^2\,dz\right] \\ &+ T\left[\frac{\sigma p^*}{|p|^2}\int_0^1 W^* D(D^2 - a^2)\zeta\,dz + \frac{\sigma p^*}{|p|^2}\int_0^1 W^* D^2 W\,dz\right].\end{aligned} \tag{3.3}$$

Integrating (3.3) by parts a suitable number of times and making use of the boundary conditions and the equality

$$\int_0^1 \psi^* D^{2n}\psi\,dz = (-1)^n \int_0^1 |D^n \psi|^2\,dz,$$
$$\psi = W \quad (n = 1, 2) \quad \text{or} \quad \psi = \theta, \phi, \zeta \quad (n = 1), \tag{3.4}$$

we may rewrite (3.3) in the form

$$\begin{aligned}\int_0^1 \left(|D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2\right) dz &+ \frac{p}{\sigma}\int_0^1 \left(|DW|^2 + a^2 |W|^2\right) dz \\ = Ra^2 &\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + p^*|\theta|^2\right) dz \\ - R_s a^2 &\left[-\frac{\tau^2 p^*}{|p|^2}\int_0^1 |(D^2 - a^2)\phi|^2\,dz \right. \\ &\left. - \frac{\tau p^{*2}}{|p|^2}\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2\right) dz + \frac{p^*}{|p|^2}\int_0^1 |W|^2\,dz\right] \\ + T &\left[-\frac{\sigma p^*}{|p|^2}\int_0^1 DW^*(D^2 - a^2)\zeta\,dz - \frac{\sigma p^*}{|p|^2}\int_0^1 |DW|^2\,dz\right].\end{aligned} \tag{3.5}$$

Further, calculating the value of $\int_0^1 DW^*(D^2 - a^2)\zeta\,dz$ from (2.4) and substituting in (3.5), equating the imaginary part of the resulting equation and cancelling

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p_i ($\neq 0$) throughout, we obtain after a little rearrangement of terms

$$\frac{(|p|^2 - T\sigma^2)}{\sigma |p|^2}\int_0^1 |DW|^2\,dz + \frac{a^2(|p|^2 - R_s\sigma)}{\sigma|p|^2}\int_0^1 |W|^2\,dz + Ra^2\int_0^1 |\theta|^2\,dz$$
$$+ \frac{R_s a^2\tau^2}{|p|^2}\int_0^1 |(D^2 - a^2)\phi|^2\,dz + \frac{2R_s a^2 \tau p_r}{|p|^2}\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2\right) dz$$
$$+ \frac{T\sigma}{|p|^2}\int_0^1 |(D^2 - a^2)\zeta|^2\,dz + \frac{2Tp_r}{|p|^2}\int_0^1 \left(|D\zeta|^2 + a^2|\zeta|^2\right) dz = 0. \tag{3.6}$$

But, since $p_r \geqslant 0$, $R > 0$, $R_s > 0$ and $T > 0$, we have from (3.6) that

$$|p|^2 < \text{greater of } (R_s\sigma, T\sigma^2), \tag{3.7}$$

which proves the theorem.

THEOREM 2. *For rotatory Stern thermohaline configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and* $(radius)^2 =$ *greater of* $(-R\sigma, T\sigma^2)$, *for all combinations of dynamically free and rigid boundaries.*

PROOF. For the Stern configuration $R < 0$, $R_s < 0$. Let $R = -\hat{R}$, $R_s = -\hat{R}_s$ so that $\hat{R} > 0$ and $\hat{R}_s > 0$. Rewriting (2.2) of the governing equations in the form

$$\theta = \frac{1}{p}(D^2 - a^2)\theta + \frac{W}{p},$$

and proceeding as in Theorem 1 (details omitted here) we obtain

$$|p|^2 < \text{greater of } (-R\sigma, T\sigma^2), \tag{3.8}$$

for all combinations of dynamically free and rigid boundaries and this proves the theorem.

THEOREM 3. *For hydromagnetic Veronis thermohaline configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and* $(radius)^2 =$ *greater of* $(R_s\sigma, Q^2\sigma^2)$, *for all combinations of rigid or free perfectly conducting boundaries.*

PROOF. Multiplying (2.8) by W^*, integrating over z and proceeding as in Theorem 1 we get

$$\int_0^1 \left(|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2\right) dz + \frac{p}{\sigma}\int_0^1 \left(|DW|^2 + a^2|W|^2\right) dz + Q\int_0^1 W^* D(D^2 - a^2)h_z\, dz$$
$$= Ra^2\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + p^*|\theta|^2\right) dz$$
$$- R_s a^2\Bigg[-\frac{\tau^2 p^*}{|p|}\int_0^1 |(D^2 - a^2)\phi|^2\, dz - \frac{\tau p^{*2}}{|p|^2}\int_0^1 \left(|D\phi|^2 + a^2|\rho|^2\right) dz + \frac{p^*}{|p|^2}\int_0^1 |W|^2\, dz\Bigg]. \quad (3.9)$$

Integrating the last term on the left hand side of (3.9) by parts once, substituting for DW^* from (2.11) and using (2.12) we obtain

$$\int_0^1 W^* D(D^2 - a^2)h_z\, dz = \int_0^1 |(D^2 - a^2)h_z|^2\, dz + \frac{p^*\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right) dz. \quad (3.10)$$

Substituting from (3.10) in (3.9) and equating the imaginary parts of the resulting equation we have

$$\frac{1}{\sigma}\int_0^1 |DW|^2\, dz + a^2\left(\frac{1}{\sigma} - \frac{R_s}{|p|^2}\right)\int_0^1 |W|^2\, dz$$
$$+ \frac{R_s a^2}{|p|^2}\left[\tau^2\int_0^1 |(D^2 - a^2)\phi|^2\, dz + 2\tau p_r\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2\right) dz\right]$$
$$+ Ra^2\int_0^1 |\phi|^2\, dz = \frac{Q\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right) dz. \quad (3.11)$$

Now

$$\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right) dz = -\int_0^1 h_z(D^2 - a^2)h_z^*\, dz$$
$$\leqslant \int_0^1 |h_z||(D^2 - a^2)h_z|\, dz$$
$$\leqslant \left\{\int_0^1 |h_z|^2\, dz\right\}^{1/2}\left\{\int_0^1 |(D^2 - a^2)h_z|^2\, dz\right\}^{1/2}, \quad (3.12)$$

(by Schwarz's inequality). Further from (2.11) we have

$$\int_0^1 |DW|^2\,dz = \int_0^1 |(D^2-a^2)h_z|^2\,dz + \frac{2p_r\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz + \frac{|p|^2\sigma_1^2}{\sigma^2}\int_0^1 |h_z|^2\,dz. \tag{3.13}$$

It follows from (3.13) that

$$\int_0^1 |(D^2-a^2)h_z|^2\,dz < \int_0^1 |DW|^2\,dz, \tag{3.14}$$

and

$$\int_0^1 |h_z|^2\,dz < \frac{\sigma^2}{\sigma_1^2|p|^2}\int_0^1 |DW|^2\,dz. \tag{3.15}$$

Combining (3.12), (3.14) and (3.15), we get

$$\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz < \frac{\sigma}{\sigma_1|p|}\int_0^1 |DW|^2\,dz. \tag{3.16}$$

Using (3.11) and (3.16), we have

$$|p|^2 < \text{greater of } \left(R_s\sigma,\ Q^2\sigma^2\right), \tag{3.17}$$

which proves the theorem.

THEOREM 4. *For hydromagnetic Stern thermohaline configuration, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and* $(\textit{radius})^2 =$ *greater of* $(-R\sigma,\ Q^2\sigma^2)$, *for all combinations of rigid or free perfectly conducting boundaries.*

PROOF. For the Stern configuration $R < 0$, $R_s < 0$. Let $R = -\hat{R}$, $R_s = -\hat{R}_s$ so that $\hat{R} > 0$ and $\hat{R}_s > 0$. Rewriting (2.9) of the governing equations in the form

$$\theta = \frac{1}{p}(D^2-a^2)\theta + \frac{W}{p},$$

and proceeding as in Theorem 3 (details omitted here) we obtain

$$|p|^2 < \text{greater of } \left(-R\sigma,\ Q^2\sigma^2\right), \tag{3.18}$$

for all combinations of rigid or free perfectly conducting boundaries, and this proves the theorem.

4. Mathematical analysis for rotatory and/or hydromagnetic thermohaline configuration for a viscoelastic fluid

THEOREM 5. *For rotatory Veronis thermohaline configuration for a viscoelastic fluid, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and*

$$(radius)^2 = \text{greater of}\left[\left\{\left(4R_s\sigma + R^2\sigma^2\Gamma^2\right)^{1/2} + R\sigma\Gamma\right\}^2/4,\ T\sigma^2\right],$$

for all combinations of dynamically free and rigid boundaries.

PROOF. It follows by proceeding as in Theorem 1 and using

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2\right) dz < \frac{1}{|p|}\int_0^1 |W|^2\, dz, \tag{4.1}$$

which is derived in a manner similar to the derivation of (3.16).

THEOREM 6. *For rotatory Stern thermohaline configuration for a viscoelastic fluid, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and*

$$(radius)^2 = \text{greater of}\left[\left\{\left(-4R\sigma + R_s^2\sigma^2\Gamma^2\right)^{1/2} - R_s\sigma\Gamma\right\}^2/4,\ T\sigma^2\right],$$

for all combinations of dynamically free and rigid boundaries.

PROOF. It follows by proceeding as in Theorem 2 and using

$$\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2\right) dz < \frac{1}{\tau|p|}\int_0^1 |W|^2\, dz, \tag{4.2}$$

which is derived in a manner similar to the derivation of (3.16).

THEOREM 7. *For hydromagnetic Veronis thermohaline configuration for a viscoelastic fluid, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and*

$$(radius)^2 = \text{greater of}\left[\left\{\left(4R_s\sigma + R^2\sigma^2\Gamma^2\right)^{1/2} + R\sigma\Gamma\right\}^2/4,\ Q^2\sigma^2\right],$$

for all combinations of rigid or free perfectly conducting boundaries.

PROOF. It follows by proceeding exactly as in Theorem 3 and using (4.1).

THEOREM 8. *For hydromagnetic Stern thermohaline configuration for a viscoelastic fluid, the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle in the right half of the complex p-plane whose centre is the origin and*

$$(\textit{radius})^2 = \textit{greater of}\left[\left\{\left(-4R\sigma + R_s^2\sigma^2\Gamma^2\right)^{1/2} - R_s\sigma\Gamma\right\}^2/4,\ Q^2\sigma^2\right],$$

for all combinations of rigid or free perfectly conducting boundaries.

PROOF. It follows by proceeding as in Theorem 4 and using (4.2).

Dedication

This work is contributed towards the establishment of an Institute of Mathematical Sciences on Ganges Bank at Kanpur.

References

[1] M. B. Banerjee, D. C. Katoch, G. S. Dube and K. Banerjee, "Bounds for growth rate of perturbation in thermohaline convection", *Proc. Roy. Soc. London Ser. A* 378 (1981), 301–304.

[2] H. E. Huppert and J. S. Turner, "Double-diffusive convection", *J. Fluid Mech.* 106 (1981), 299–329.

[3] M. E. Stern, "The 'salt-fountain' and thermohaline convection", *Tellus* 12 (1960), 172–175.

[4] G. Veronis, "On finite amplitude instability in thermohaline convection", *J. Mar. Res.* 23 (1965), 1–17.

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