



RESEARCH ARTICLE

Maximal subgroups of a family of iterated monodromy groups

Karthika Rajeev¹ and Anitha Thillaisundaram²

¹Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany, Email: krajeev@math.uni-bielefeld.de

²Centre for Mathematical Sciences, Lund University, Lund, Sweden

Corresponding author: Anitha Thillaisundaram; Email: anitha.thillaisundaram@math.lu.se

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Abstract

The Basilica group is a well-known 2-generated weakly branch, but not branch, group acting on the binary rooted tree. Recently, a more general form of the Basilica group has been investigated by Petschick and Rajeev, which is an s -generated weakly branch, but not branch, group that acts on the m -adic tree, for $s, m \geq 2$. A larger family of groups, which contains these generalised Basilica groups, is the family of iterated monodromy groups. With the new developments by Francoeur, the study of the existence of maximal subgroups of infinite index has been extended from branch groups to weakly branch groups. Here we show that a subfamily of iterated monodromy groups, which more closely resemble the generalised Basilica groups, have maximal subgroups only of finite index.

1. Introduction

Groups acting on rooted trees have drawn a great deal of attention over the last couple of decades because they exhibit prominent features and solve several long-standing problems in group theory. The initial examples studied were Grigorchuk’s groups of intermediate word growth ([13]; answering Milnor’s question) and Gupta and Sidki’s examples of finitely generated infinite p -groups ([17]; providing an explicit family of 2-generated counterexamples to the general Burnside problem). Ever since, attempts have been made to characterise and generalise the groups of automorphisms of rooted trees. Today, the Grigorchuk groups and the Gupta–Sidki groups are known as the first examples of groups in the family of *branch groups*. Branch groups are groups acting level-transitively on a spherically homogeneous rooted tree T and having subnormal subgroups similar to that of the full automorphism group $\text{Aut } T$ of the tree T ; see Section 2 for definitions. The groups studied in this paper belong to a more general class of groups, the *weakly branch groups*, obtained by weakening some of the algebraic properties of branch groups; cf. [2].

The Basilica group is a 2-generated weakly branch, but not branch, group acting on the binary rooted tree, which was introduced by Grigorchuk and Žuk in refs. [14] and [15]. It is the first known example of an amenable [4] but not sub-exponentially amenable group [15]. In contrast to the Grigorchuk and the Gupta–Sidki groups, the Basilica group is torsion-free and has exponential word growth [15]. Moreover, it is the iterated monodromy group of the complex polynomial $z^2 - 1$; see [21, Section 6.12.1]. The generators of the Basilica group are recursively defined as follows:

$$a = (1, b) \quad \text{and} \quad b = (1, a)\sigma,$$

where σ is the cyclic permutation which swaps the subtrees rooted at the first level of the binary rooted tree, and (x, y) represents the independent action on the two maximal subtrees, where $x, y \in \text{Aut } T$. Recently, Petschick and Rajeev [23] introduced a construction which relates the Basilica group and the one-generated dyadic odometer \mathcal{O}_2 (also known as the adding machine). Let $m, s \geq 2$ be integers

and let G be a subgroup of the automorphism group $\text{Aut } T$ of the m -adic tree T . The s th Basilica group of G is given by $\text{Bas}_s(G) = \langle \beta_i^s(g) \mid g \in G, i \in \{0, 1, \dots, s - 1\} \rangle$, where $\beta_i^s : \text{Aut } T \rightarrow \text{Aut } T$ are monomorphisms given by

$$\begin{aligned} \beta_i^s(g) &= (1, \dots, 1, \beta_{i-1}^s(g)) && \text{for } i \in \{1, \dots, s - 1\}, \\ \beta_0^s(g) &= (\beta_{s-1}^s(g_0), \dots, \beta_{s-1}^s(g_{m-1}))g^\epsilon, \end{aligned}$$

where g_x is the restriction of g to the subtree rooted at a first-level vertex $x \in \{0, \dots, m - 1\}$, and g^ϵ is the local action of the element g at the root of T . (In ref. [23], the generators $\beta_i^s(g)$, for $i \in \{1, \dots, s - 1\}$, are defined along the leftmost spine and the element g^ϵ is acting from the left; however, the corresponding s th Basilica group is isomorphic to the one defined here). We obtain the classical Basilica group by applying the operator Bas_2 to the dyadic odometer as follows: let $c = (1, c)\sigma$ be the automorphism of the binary rooted tree generating the dyadic odometer. Then, the generators of the Basilica group are given by

$$a = \beta_1^2(c) \quad \text{and} \quad b = \beta_0^2(c).$$

This gives a natural generalisation of the Basilica group given by $\text{Bas}_s(\mathcal{O}_m)$ for every pair of integers $m, s \geq 2$. Here, \mathcal{O}_m is the m -adic odometer, which is an embedding of the infinite cyclic group into the automorphism group of the m -adic tree T , and is generated by

$$c = (1, \overset{m-1}{\cdot}, 1, c)\sigma$$

where $\sigma = (0 \ 1 \ \dots \ m - 1)$ is the m -cycle that cyclically permutes the m subtrees rooted at the first level of T . The generalised Basilica groups $\text{Bas}_s(\mathcal{O}_m)$ resemble the classical Basilica group, as they are weakly branch, but not branch, torsion-free groups of exponential word growth [23, Theorem 1.6]. They are also weakly regular branch over their derived subgroup.

Now, as we will see below, the generalised Basilica groups lie in the set of iterated monodromy groups of post-critically finite complex polynomials, where a polynomial f is *post-critically finite* if the orbit of the critical point 0 under iterations of f is finite. Iterated monodromy groups are more naturally defined in terms of self-coverings of topological spaces, as done in ref. [21, Chapter 5], and these groups are of interest since they encode information about the dynamics of such self-coverings. However, below we will only abstractly define a subfamily of iterated monodromy groups that are of interest to us, and we only study their algebraic properties. Therefore, we refer the reader to [21] for more information about iterated monodromy groups in general.

We note that the iterated monodromy groups of the quadratic polynomials $f(z) = z^2 + c$ have been studied in more depth. In ref. [3], Bartholdi and Nekrashevych identified the iterated monodromy groups of post-critically finite quadratic polynomials $f(z) = z^2 + c$. Among those groups, we are interested in the groups $\mathfrak{R}(\mathbf{v})$, which are defined as below.

Let $s \geq 2$ and $\mathbf{v} = x_0 \dots x_{s-2}$ be a word over the alphabet $X = \{0, 1\}$. The group $\mathfrak{R}(\mathbf{v})$ is a subgroup of automorphisms of the binary rooted tree generated by the elements a_0, \dots, a_{s-1} , which are defined as

$$a_0 = (1, a_{s-1})\sigma, \quad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0, \\ (1, a_i) & \text{if } x_i = 1, \end{cases}$$

for $i \in \{0, 1, \dots, s - 2\}$. The quadratic polynomial associated with $\mathfrak{R}(\mathbf{v})$ can be precisely defined in terms of \mathbf{v} ; see [3, Section 5]. Note that if $s = 1$, then \mathbf{v} is the empty word, then $\mathfrak{R}(\emptyset)$ is just the cyclic group \mathbb{Z} . Notice that $\mathfrak{R}(1)$ is the classical Basilica group. Furthermore, by setting $\mathbf{v} = 1 \cdot \overset{s'}{\cdot} \cdot 1$, we get that $\mathfrak{R}(\mathbf{v}) = \text{Bas}_{s'+1}(\mathcal{O}_2)$, for $s' \in \mathbb{N}$. The group $\mathfrak{R}(\mathbf{v})$ has a corresponding so-called kneading sequence, by which the symbolic dynamics of quadratic complex polynomials are usually studied; see [3] for more information.

The recurrence relations in the definition of $\mathfrak{K}(\mathbf{v})$ above suggests that one can generalise this notion to groups acting on the m -adic tree, for any $m \geq 2$. Let $\mathbf{v} = x_0 \cdots x_{s-2}$ be a word in the alphabet $X = \{0, 1, \dots, m - 1\}$. Define

$$a_0 = (1, \dots, 1, a_{s-1})\sigma, \quad \text{and} \quad a_{i+1} = (1, \overset{x_i}{\cdot} \overset{-1}{\cdot}, 1, a_i, 1, \dots, 1),$$

for $i \in \{0, 1, \dots, s - 2\}$, and $\mathfrak{K}(\mathbf{v})$ as the group defined by the elements a_0, \dots, a_{s-1} . Then, for $\mathbf{v} = 1 \cdot \overset{s}{\cdot} \cdot 1$, we have $\mathfrak{K}(\mathbf{v}) = \text{Bas}_{s+1}(\mathcal{O}_m)$.

We establish the following basic properties of the groups $\mathfrak{K}(\mathbf{v})$.

Theorem 1.1. *Let $m, s \geq 2$ be positive integers, and let $\mathbf{v} = x_0 \cdots x_{s-2}$ be a word in the alphabet $X = \{0, 1, \dots, m - 1\}$. Then for the group $\mathfrak{K}(\mathbf{v})$ defined by \mathbf{v} , the following assertions hold:*

- (i) $\mathfrak{K}(\mathbf{v})$ is an iterated monodromy group of a post-critically finite polynomial;
- (ii) $\mathfrak{K}(\mathbf{v})$ is super strongly fractal and level-transitive;
- (iii) $\mathfrak{K}(\mathbf{v})$ is not branch, but is weakly regular branch over its commutator subgroup $\mathfrak{K}(\mathbf{v})'$;
- (iv) $\mathfrak{K}(\mathbf{v})/\mathfrak{K}(\mathbf{v})' \cong \mathbb{Z}^s$;
- (v) $\mathfrak{K}(\mathbf{v})$ is torsion-free;
- (vi) $\mathfrak{K}(\mathbf{v})$ is contracting;
- (vii) $\mathfrak{K}(\mathbf{v})$ has exponential word growth.

The above results, apart from part (ii), were known for the case $m = 2$; see [3, Section 3]. We note that the fact that the groups $\mathfrak{K}(\mathbf{v})$ are contracting already follows from [7], but we give a self-contained proof which yields more information about the contracting property of the groups. The property of exponential word growth will not be utilised in this paper, so we instead refer the reader to [2, Chapter 10] for the definition.

In this paper, we study the maximal subgroups of $\mathfrak{K}(\mathbf{v})$ for constant words \mathbf{v} ; the case of non-constant words prove to be much more difficult and would require a different approach. The study of maximal subgroups of branch groups was initiated by Pervova [22], where she proved that the torsion Grigorchuk groups do not contain maximal subgroups of infinite index. Thenceforth, attempts have been made to generalise the results and techniques from [22], for instance see [1], [20], and [10]. Among which our interest lies in the work of Francoeur [10] (or see [9, Section 8.4]), who provided a strategy to study the maximal subgroups of weakly branch groups. In particular, he proved that the classical Basilica group does not contain maximal subgroups of infinite index. Following this technique, we prove that $\mathfrak{K}(\mathbf{v})$, for constant words \mathbf{v} , do not admit maximal subgroups of infinite index.

Theorem 1.2. *Let $m, s \geq 2$ be positive integers, and let $\mathbf{v} = t \overset{s-1}{\cdot} \cdot t$ be the constant word for some $t \in \{0, 1, \dots, m - 1\}$. Then, the group $\mathfrak{K}(\mathbf{v})$ does not admit a maximal subgroup of infinite index.*

Due to the different properties of the family $\mathfrak{K}(\mathbf{v})$, the final stages of our proof differ from previously seen results; compare Theorem 5.6. This is also the first time that maximal subgroups of a weakly branch, but not branch, group G have been considered for a group G with more than 2 generators.

It is interesting to note that there are currently no examples of finitely generated weakly branch, but not branch, groups with maximal subgroups of infinite index. There are only examples of finitely generated branch groups with maximal subgroups of infinite index; see [6, 11, 19]. It remains to be seen whether being a finitely generated weakly branch group with maximal subgroups of infinite index implies the group is branch.

Furthermore, in all known examples of finitely generated weakly branch, but not branch, groups with maximal subgroups only of finite index, these groups have maximal subgroups that are not normal; compare Remark 5.8 and [8, 12]. Therefore, it is also natural to ask if there exists a finitely generated weakly branch, but not branch, group with all maximal subgroups of finite index and normal.

Organisation. Section 2 contains preliminary material on groups acting on the m -adic tree, and Section 3 establishes basic properties of the groups $\mathfrak{R}(\mathfrak{v})$, that is, we prove Theorem 1.1. In Section 4, we record some length-reducing properties of a large subfamily of the groups $\mathfrak{R}(\mathfrak{v})$, and in Section 5, we prove Theorem 1.2.

2. Preliminaries

By \mathbb{N} we denote the set of positive integers, and by \mathbb{N}_0 the set of non-negative integers.

Let $m \in \mathbb{N}_{\geq 2}$ and let $T = T_m$ be the m -adic tree, that is, a rooted tree where all vertices have m children. Using the alphabet $X = \{0, 1, \dots, m - 1\}$, the vertices u_ω of T are labelled bijectively by the elements ω of the free monoid X^* in the following natural way: the root of T is labelled by the empty word and is denoted by ϵ , and for each word $\omega \in X^*$ and letter $x \in X$, there is an edge connecting u_ω to $u_{\omega x}$. More generally, we say that u_ω precedes u_λ whenever ω is a prefix of λ .

There is a natural length function on X^* , which is defined as follows: the words ω of length $|\omega| = n$, representing vertices u_ω that are at distance n from the root, are the n th level vertices and constitute the n th layer of the tree.

We denote by T_u the full rooted subtree of T that has its root at a vertex u and includes all vertices succeeding u . For any two vertices $u = u_\omega$ and $v = u_\lambda$, the map $u_{\omega\tau} \mapsto u_{\lambda\tau}$, induced by replacing the prefix ω by λ , yields an isomorphism between the subtrees T_u and T_v .

Now each $f \in \text{Aut } T$ fixes the root, and the orbits of $\text{Aut } T$ on the vertices of the tree T are the layers of the tree T . The image of a vertex u under f will be denoted by uf . The automorphism f induces a faithful action on X^* given by $(u_\omega)f = u_{\omega f}$. For $\omega \in X^*$ and $x \in X$, we have $(\omega x)f = (\omega f)x'$, for $x' \in X$ uniquely determined by ω and f . This induces a permutation f^ω of X which satisfies

$$(\omega x)f = (\omega f)xf^\omega, \quad \text{and consequently} \quad (u_{\omega x})f = u_{(\omega f)xf^\omega}.$$

More generally, for an automorphism f of T , since the layers are invariant under f , for $u \in X^*$, the equation

$$(uv)f = (uf)vf_u \quad \text{for every } v \in X^*,$$

defines a unique automorphism f_u of T called the *section of f at u* . This automorphism can be viewed as the automorphism of T induced by f upon identifying the rooted subtrees of T at the vertices u and uf with the tree T . As seen here, we often do not differentiate between X^* and vertices of T .

2.1. Subgroups of $\text{Aut } T$

Let G be a subgroup of $\text{Aut } T$ acting *level-transitively*, that is, transitively on every layer of T . The *vertex stabiliser* $\text{st}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u . For $n \in \mathbb{N}$, the *n th level stabiliser* $\text{St}_G(n) = \bigcap_{|\omega|=n} \text{st}_G(u_\omega)$ is the subgroup consisting of automorphisms that fix all vertices at level n .

Each $g \in \text{St}_{\text{Aut } T}(n)$ can be completely determined in terms of its restrictions to the subtrees rooted at vertices at level n . There is a natural isomorphism

$$\psi_n : \text{St}_{\text{Aut } T}(n) \longrightarrow \prod_{|\omega|=n} \text{Aut } T_{u_\omega} \cong \text{Aut } T \times \dots \times \text{Aut } T$$

defined by sending $g \in \text{St}_{\text{Aut } T}(n)$ to its tuple of m^n sections. For conciseness, we will omit the use of ψ_1 , and simply write $g = (g_1, \dots, g_m)$ for $g \in \text{St}_{\text{Aut } T}(1)$.

Let $\omega \in X^n$ be of length n . We further define

$$\varphi_\omega : \text{st}_{\text{Aut } T}(u_\omega) \longrightarrow \text{Aut } T_{u_\omega} \cong \text{Aut } T$$

to be the map sending $f \in \text{st}_{\text{Aut } T}(u_\omega)$ to the section f_{u_ω} .

A group $G \leq \text{Aut } T$ is said to be *self-similar* if for all $f \in G$ and all $\omega \in X^*$ the section f_{u_ω} belongs to G . We will denote G_ω to be the subgroup $\varphi_\omega(\text{st}_G(u_\omega))$.

Recall that for a group G generated by a finite symmetric subset S (i.e. a set for which $S = S^{-1}$), for every $g \in G$, the *length* of g with respect to S is

$$|g| = \min\{n \geq 0 \mid g = s_1 \cdots s_n, \text{ for } s_1, \dots, s_n \in S\}.$$

Now assume that G is a self-similar subgroup of $\text{Aut } T$. Then for every $g \in G$ and every $n \in \mathbb{N}_0 \cup \{0\}$, let

$$\ell_n(g) = \max\{|g_u| \mid |u| = n\}.$$

If there exist $\lambda < 1$ and $C, L \in \mathbb{N}$ such that

$$\ell_n(g) \leq \lambda|g| + C, \quad \text{for every } n > L \text{ and every } g \in G,$$

then we say that the group G is *contracting* with respect to S .

Let G be a subgroup of $\text{Aut } T$ acting level-transitively. Here, the vertex stabilisers at every level are conjugate under G . We say that the group G is *fractal* if $G_\omega := \varphi_\omega(\text{st}_G(u_\omega)) = G$ for every $\omega \in X^*$, after the natural identification of subtrees. Furthermore, we say that the group G is *super strongly fractal* if, for each $n \in \mathbb{N}$, we have $\varphi_\omega(\text{St}_G(n)) = G$ for every word $\omega \in X^n$ of length n .

The *rigid vertex stabiliser* of u in G is the subgroup $\text{rist}_G(u)$ consisting of all automorphisms in G that fix all vertices of T not succeeding u . The *rigid n th level stabiliser* is the direct product of the rigid vertex stabilisers of the vertices at level n :

$$\text{Rist}_G(n) = \prod_{|u|=n} \text{rist}_G(u_\omega) \leq G.$$

We recall that a level-transitive group G is a *branch group* if $\text{Rist}_G(n)$ has finite index in G for every $n \in \mathbb{N}$; and G is *weakly branch* if $\text{Rist}_G(n)$ is non-trivial for every $n \in \mathbb{N}$. If, in addition, the group G is self-similar and there exists a subgroup $1 \neq K \leq G$ with $K \times \cdots \times K \subseteq \psi_1(\text{St}_K(1))$ and $|G : K| < \infty$, then G is said to be *regular branch over K* . If in the previous definition the condition $|G : K| < \infty$ is omitted, then G is said to be *weakly regular branch over K* .

2.2. A basic result

Here we record a general result that will be useful in the sequel. For $g \in \text{Aut } T$, recall that g^ϵ denotes the action induced by g at the root of T .

Lemma 2.1. *For a self-similar group $G \leq \text{Aut } T$, let $z = (z_0, \dots, z_{m-1})z^\epsilon \in G'$. Then $z_0 \cdots z_{m-1} \in G'$.*

Proof. It suffices to prove the result for a basic commutator $[g, h]$, where $g, h \in G$. Write $g = (g_0, \dots, g_{m-1})g^\epsilon$ and $h = (h_0, \dots, h_{m-1})h^\epsilon$. For notational convenience, let us write $\tau = (g^\epsilon)^{-1}$ and $\kappa = (h^\epsilon)^{-1}$, and for $\alpha \in \text{Sym}(X)$ and $x \in X$ we write x^α for the image $\alpha(x)$ of x under α . As

$$\begin{aligned} [g, h] &= \tau(g_0^{-1}, \dots, g_{m-1}^{-1})\kappa(h_0^{-1}, \dots, h_{m-1}^{-1})(g_0, \dots, g_{m-1})g^\epsilon(h_0, \dots, h_{m-1})h^\epsilon \\ &= (g_{0\tau}^{-1}, \dots, g_{(m-1)\tau}^{-1})(h_{0\tau\kappa}^{-1}, \dots, h_{(m-1)\tau\kappa}^{-1})(g_{0\tau\kappa}, \dots, g_{(m-1)\tau\kappa})(h_{0\tau\kappa g^\epsilon}, \dots, h_{(m-1)\tau\kappa g^\epsilon})\tau\kappa g^\epsilon h^\epsilon, \end{aligned}$$

the result follows. □

3. Properties of $\mathfrak{R}(\mathbf{v})$

For any two integers i, j , let $[i, j]$ denote the set $\{i, i + 1, \dots, j - 1, j\}$. In the following sections, we fix $m, s \in \mathbb{N}_{\geq 2}$. Let $X = \{0, 1, \dots, m - 1\}$ and let $\mathbf{v} = x_0 \cdots x_{s-2}$ be a word in X^* . Recall that the group $\mathfrak{R}(\mathbf{v})$ is generated by the elements a_0, \dots, a_{s-1} , where

$$a_0 = (1, \dots, 1, a_{s-1})\sigma, \quad \text{and} \quad a_{i+1} = (1, x_i^{-1}, 1, a_i, 1, \dots, 1),$$

for $i \in [0, s - 2]$.

To prove Theorem 1.1(i), we use the following characterisation of iterated monodromy groups by Nekrashevych [21, Theorem 6.10.8] according to the notation given in [18, Theorem 5.8]:

Theorem 3.1. *A subgroup $G \leq \text{Aut } T$ is isomorphic to a standard action of the iterated monodromy group of a post-critically finite polynomial if and only if G is conjugate in $\text{Aut } T$ to a group generated by a finite invertible automaton A with the following properties:*

- (1) *For each non-trivial $a \in A$, there is a unique $b \in A$ and $x \in X$ with $b_x = a$.*
- (2) *For each $a \in A$ and each cycle $(x_1 x_2 \cdots x_n)$ of the action of a on X , the section a_{x_i} is non-trivial for at most one x_i .*
- (3) *The multi-set of permutations defined by the set of states of A acting on X is tree-like.*
- (4) *Let $a_1 \neq a_2$ be non-trivial states of A with $e \neq v_1, v_2 \in T$ satisfying $(a_i)_{v_i} = a_i$ and $a_i(v_i) = v_i$ for $i \in \{1, 2\}$. Then, there is no $h \in G$ with $h(v_1) = v_2$ and $h_{v_1} = h$.*

We refer the reader to [21] or [18] for any unexplained terminology.

Proof of Theorem 1.1(i). We will show that the above four conditions hold for $\mathfrak{K}(\mathfrak{v})$. Conditions (1) and (2) of Theorem 3.1 are clear from the definition of the generators a_i . For condition (3), notice that a_0 is the only generator of $\mathfrak{K}(\mathfrak{v})$ that acts non-trivially on X . Therefore, according to [18, Section 5.4], the multi-set of permutations of X is given by $\{\sigma\}$. Furthermore, the cycle diagram associated with the multi-set $\{\sigma\}$ is an oriented 2-dimensional CW complex, whose set of 0-cells is X , and with one 2-cell obtained by connecting the elements of X by the action of σ . Hence, the cycle diagram is contractible. Thus, by [18, Definition 5.7] the multi-set of permutations defined by the set of states of $\mathfrak{K}(\mathfrak{v})$ acting on X is tree-like.

For condition (4), the restriction $(a_i)_{v_i} = a_i$ implies that the v_i have to be at level ns for some positive integer n . However, for any a_i , after i levels, the section is a_0 , which does not then fix any vertex below it. So we can never find vertices v_i such that $a_i(v_i) = v_i$. So (4) vacuously holds. □

Lemma 3.2. *The group $\mathfrak{K}(\mathfrak{v})$ is super strongly fractal and level-transitive.*

Proof. Observe that $a_0^m = (a_{s-1}, \dots, a_{s-1})$ and, for any $i \in [1, s - 1]$ and $j \in [0, m - 1]$,

$$\varphi_j \left(a_i^{a_0^{j-x_{i-1}}} \right) = a_{i-1}.$$

It follows that $\varphi_x(\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)) = \mathfrak{K}(\mathfrak{v})$ for all $x \in X$. For $n \in \mathbb{N}$, we write $n = \ell s + r$, where $\ell \in \mathbb{N}_0$ and $r \in [0, s - 1]$. Using similar arguments with the elements

$$a_0^{m^{\ell+1}}, a_1^{m^{\ell+1}}, \dots, a_{r-1}^{m^{\ell+1}}, a_r^{m^\ell}, \dots, a_{s-1}^{m^\ell} \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(n),$$

and their conjugates, we deduce that $\varphi_w(\text{St}_{\mathfrak{K}(\mathfrak{v})}(n)) = \mathfrak{K}(\mathfrak{v})$ for all $w \in X^n$.

The second statement is immediate from the fact that $\varphi_x(\text{st}_{\mathfrak{K}(\mathfrak{v})}(x)) = \mathfrak{K}(\mathfrak{v})$ for some $x \in X$, and since a_0 , and hence, $\mathfrak{K}(\mathfrak{v})$, acts transitively on the first layer. □

Next, we record an elementary but useful result. Recall that $\mathfrak{v} = x_0 \cdots x_{s-2}$ is a word in X^* .

Lemma 3.3. *For distinct $i, j \in [1, s - 1]$ with $x_{i-1} \neq x_{j-1}$, we have $[a_i, a_j] = 1$ in $\mathfrak{K}(\mathfrak{v})$. For $i > \ell \in [1, s - 1]$ with $x_{i-1} = x_{\ell-1}$, we have $[a_i, a_\ell] = 1$ if $x_{i-d} \neq x_{\ell-d}$ for some $d \in [2, \ell]$ and $[a_i, a_\ell] \neq 1$ otherwise.*

Proof. The first statement is a straightforward computation. For the next statement, note that for $i \in [1, s - 1]$,

$$[a_i, a_0] = \begin{cases} (a_{s-1}^{-1} a_{i-1} a_{s-1}, 1, \dots, 1, a_{i-1}^{-1}) & \text{if } x_{i-1} = m - 1, \\ (1, a_{i-1}^{-1}, 1, a_{i-1}^{-1}, a_{i-1}, 1, \dots, 1) & \text{if } x_{i-1} \neq m - 1, \end{cases}$$

so a_0 does not commute with any other a_i . The result then follows, using the fact that

$$[a_i, a_\ell] = (1, \dots, 1, [a_{i-1}, a_{\ell-1}], 1, \dots, 1).$$

□

Lemma 3.4. *The group $\mathfrak{K}(\mathfrak{v})$ is weakly regular branch over $\mathfrak{K}(\mathfrak{v})'$.*

Proof. We use that $\mathfrak{K}(\mathfrak{v})' = \langle [a_i, a_j] \mid i, j \in [0, s - 1] \rangle^{\mathfrak{K}(\mathfrak{v})}$. We have

$$\begin{aligned} [a_i^{a_0^{m-1-x_{i-1}}}, a_j^{a_0^{m-1-x_{j-1}}}] &= (1, \dots, 1, [a_{i-1}, a_{j-1}]), \\ [a_i, a_0^{a_0^{m-1-x_{i-1}}}] &= (1, \dots, 1, [a_{i-1}, a_{s-1}]), \end{aligned}$$

for $i, j \in [1, s - 1]$. Therefore, by transitivity and fractalness, we have $\mathfrak{K}(\mathfrak{v})' \times \dots \times \mathfrak{K}(\mathfrak{v})' \leq \psi(\mathfrak{K}(\mathfrak{v})')$. □

For $i \in [0, s - 1]$, let $A_i = \langle a_j \mid j \in [0, s - 1] \setminus \{i\} \rangle^{\mathfrak{K}(\mathfrak{v})}$.

Lemma 3.5. *For $i \in [0, s - 1]$, we have $\mathfrak{K}(\mathfrak{v})/A_i \cong \mathbb{Z}$. In particular, the elements a_i have infinite order in $\mathfrak{K}(\mathfrak{v})$.*

Proof. We prove simultaneously for all i that $\mathfrak{K}(\mathfrak{v})/A_i \cong \mathbb{Z}$. Assume for a contradiction that, for some $n \in \mathbb{N}$ that $a_i^n \in A_i$, for some $i \in [0, s - 1]$. Choose $n \in \mathbb{N}$ minimal with respect to the property that $a_i^n \in A_i$ for some $i \in [0, s - 1]$. If $i = 0$, then $a_0^n \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$ and, in particular $n \equiv 0 \pmod{m}$. We have

$$a_0^n = (a_{s-1}^{n/m}, \dots, a_{s-1}^{n/m}),$$

and hence, $a_{s-1}^{n/m} \in A_{s-1}$. This is a contradiction to the minimality of n . Now suppose that $i \neq 0$. Then by considering appropriate sections of a_i^n , we see that $a_0^n \in A_0$, which cannot happen as shown above. Therefore, we conclude that $\mathfrak{K}(\mathfrak{v})/A_i \cong \mathbb{Z}$ for all $i \in [0, s - 1]$. □

Lemma 3.6. *We have $\text{Rist}_{\mathfrak{K}(\mathfrak{v})}(1) = A_0$. Hence, the group $\mathfrak{K}(\mathfrak{v})$ is not branch.*

Proof. Analogous to the proof of [8, Theorem 3.5(i)], we clearly have $A_0 \leq \text{Rist}_{\mathfrak{K}(\mathfrak{v})}(1) \leq \text{St}_{\mathfrak{K}(\mathfrak{v})}(1) = A_0 \langle a_0^m \rangle$, where the last equality follows from the fact that $a_i \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$ for all $i \neq 0$, and that $a_0^n \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$ if and only if $n \equiv 0 \pmod{m}$. So $\text{Rist}_{\mathfrak{K}(\mathfrak{v})}(1) = A_0 \langle a_0^{mn} \rangle$ for some n . Since

$$\psi(A_0 \langle a_0^{mn} \rangle) = (A_{s-1} \times \dots \times A_{s-1}) \langle (a_{s-1}^n, \dots, a_{s-1}^n) \rangle$$

and a_{s-1} has infinite order modulo A_{s-1} by Lemma 3.5, it follows from the definition of the rigid stabiliser that $n = 0$. Hence, $\text{Rist}_{\mathfrak{K}(\mathfrak{v})}(1) = A_0$, which has infinite index in $\mathfrak{K}(\mathfrak{v})$, so $\mathfrak{K}(\mathfrak{v})$ is not branch. □

We shall adopt the convention that the subscripts of the a_i 's are taken modulo s . Set $S = \{a_i^{\pm 1} \mid i \in [0, s - 1]\}$ and then $\mathfrak{K}(\mathfrak{v}) = \langle S \rangle$. For each word $w \in S^*$, the length $|w|$ is the usual word length of w over the alphabet S .

Lemma 3.7. *Let w be a word in S representing the identity in $\mathfrak{K}(\mathfrak{v})$. Then the exponent sum of a_i in w must be zero for all $i \in [0, s - 1]$.*

Proof. We proceed by induction on the length of w . Let w be a non-trivial word in S representing the identity in $\mathfrak{K}(\mathfrak{v})$. It is immediate from Lemma 3.5 that $|w| \geq 2$ and w must contain non-trivial powers of at least two distinct $a_i, a_j \in S$. For the base case $|w| = 2$, we have that $w = a_i^{\epsilon_i} a_j^{\epsilon_j}$ for distinct $i, j \in [0, s - 1]$ and $\epsilon_i, \epsilon_j \in \{\pm 1\}$. From considering the image of w in $\mathfrak{K}(\mathfrak{v})/A_i$, we obtain a contradiction to ϵ_i being non-zero.

Assume that the result holds for all words of length $n \geq 2$, and let $|w| = n + 1$. By realising w in $\mathfrak{K}(\mathfrak{v})$, we can see that the exponent sum of a_0 in w must be zero modulo m . By abuse of notation, we write

$$\psi(w) = (w_0, \dots, w_{m-1}),$$

where w_i are reduced words determined by the appropriate sections of the letters of w . Since w represents the identity in $\mathfrak{K}(\mathfrak{v})$, each w_k also represents the identity in $\mathfrak{K}(\mathfrak{v})$. It follows from the definition of the generators a_i that $\sum_{k=0}^{m-1} |w_k| \leq |w|$. In particular, if $|w_k| = |w|$, for some $k \in [0, m - 1]$, then the w_j are trivial for all $j \in [0, m - 1] \setminus \{k\}$, otherwise $|w_k| < |w|$ for all $k \in [0, m - 1]$.

Assume that $|w_k| < |w|$ for all $k \in [0, m - 1]$. By the induction hypothesis, for all $i \in [0, s - 1]$ the exponent sum of a_i is zero in w_k for all $k \in [0, m - 1]$. Since the a_i in w_k are obtained from a_{i+1} in w , we conclude that the exponent sum of a_{i+1} in w is zero for all $i \in [0, s - 1]$.

Now, assume that there exists $k \in [0, m - 1]$ such that $|w_k| = |w|$. By replacing w with w_k , repeatedly if necessary, we may assume that w does involve the letters $a_0^{\pm 1}$. Then, if w has $a_0^{\pm 1}$ as a letter, then the total exponent of a_0 is congruent to 0 modulo m . Suppose first that the total exponent of a_0 is non-zero. Since the total exponent of a_0 is congruent to 0 modulo m , we can find a subword $a_0^{\pm 1} w' a_0^{\pm 1}$ such that the exponent sum of a_0 in w' is zero. By realising this subword in $\mathfrak{K}(\mathfrak{v})$, we immediately see a length reduction among the sections, and it follows by induction that every section w_k has exponent sum of a_i equal to zero for all $i \in [0, s - 1]$. So in particular, the case that total exponent of a_0 in w is non-zero cannot occur. So assume that the total exponent of a_0 is zero. Upon considering each w_k , we may assume that the total exponent of a_0 in w_k is zero, else we are done by the above argument. Hence, the total exponent of a_1 in w is zero. Recursively, we deduce that the total exponent in w of any a_i is zero. The result then follows. □

We proceed to prove parts (iv) and (v) of Theorem 1.1.

Lemma 3.8. *The quotient group $\mathfrak{K}(\mathfrak{v})/\mathfrak{K}(\mathfrak{v})'$ is isomorphic to the free abelian group of rank s .*

Proof. For every $g \in \mathfrak{K}(\mathfrak{v})$, there exist integers n_0, \dots, n_{s-1} such that $g \equiv a_0^{n_0} \cdots a_{s-1}^{n_{s-1}}$ modulo $\mathfrak{K}(\mathfrak{v})'$. It is enough to show that, if $a_0^{n_0} \cdots a_{s-1}^{n_{s-1}} \in \mathfrak{K}(\mathfrak{v})'$, for some $n_0, \dots, n_{s-1} \in \mathbb{Z}$, then $n_0 = \dots = n_{s-1} = 0$.

Assume that $a_0^{n_0} \cdots a_{s-1}^{n_{s-1}} \in \mathfrak{K}(\mathfrak{v})'$ for some $n_0, \dots, n_{s-1} \in \mathbb{Z}$. It is clear that every element in $\mathfrak{K}(\mathfrak{v})'$ can be written as a word in S and the exponent sum of each a_i is zero. It follows from Lemma 3.7 that the exponent sums of the a_i 's in all words representing the same element in $\mathfrak{K}(\mathfrak{v})$ are the same. This, in particular, applies to $a_0^{n_0} \cdots a_{s-1}^{n_{s-1}}$, and hence, $n_0 = \dots = n_{s-1} = 0$. □

Lemma 3.9. *The group $\mathfrak{K}(\mathfrak{v})$ is torsion-free.*

Proof. It follows immediately from [8, Theorem 3.6] that $\mathfrak{K}(\mathfrak{v})$ is torsion-free as the quotient group $\mathfrak{K}(\mathfrak{v})/\mathfrak{K}(\mathfrak{v})'$ is torsion-free and the subgroup $\mathfrak{K}(\mathfrak{v})'$ is contained in $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$. □

Next, we finish the proof of Theorem 1.1. Recall that for $g \in \mathfrak{K}(\mathfrak{v})$, the n th level sections g_0, \dots, g_{m^n-1} of g are determined from expressing g as the product

$$g = \psi_n^{-1}((g_0, \dots, g_{m^n-1}))\tau_g,$$

where $\psi_n^{-1}((g_0, \dots, g_{m^n-1})) \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(n)$ and $\tau_g \in \text{Sym}(X^n)$.

Theorem 3.10. *The group $\mathfrak{R}(\mathfrak{v})$ is contracting with respect to the set of generators S , with $\lambda = \frac{2}{3}$, $L = s - 1$ and $C = 1$.*

Proof. By [8, Lemma 3.7], it suffices to prove that

$$\ell_s(g) \leq \frac{2}{3}|g| + 1, \quad \text{for every } g \in \mathfrak{R}(\mathfrak{v}). \tag{3.1}$$

We can further reduce the problem to showing that $\ell_s(h) \leq 2$ for every $h \in \mathfrak{R}(\mathfrak{v})$ of length 3. Indeed, writing $|g| = 3\alpha + \beta$ with $\beta \in \{0, 1, 2\}$ and $g = h_1 \cdots h_\alpha f$ with $|h_1| = \cdots = |h_\alpha| = 3$ and $|f| = \beta$, we see that (3.1) immediately follows from the subadditivity of ℓ_s ; cf. [8, Equation (3.3)].

Let us then consider an arbitrary element $h \in \mathfrak{R}(\mathfrak{v})$ of length 3 and prove that $\ell_s(h) \leq 2$. Observe that h is of the form

- (i) $a_i^{\epsilon_i} a_j^{\epsilon_j} a_k^{\epsilon_k}$ for $i \neq j$ and $j \neq k$; or
- (ii) $a_i^{2\epsilon_i} a_j^{\epsilon_j}$ or $a_i^{\epsilon_i} a_j^{2\epsilon_j}$ for $i \neq j$; or
- (iii) $a_i^{3\epsilon_i}$,

where $\epsilon_i, \epsilon_j, \epsilon_k \in \{\pm 1\}$. For case (iii), it is clear that $\ell_{i+1}(h) \leq 2$, since the only non-trivial i th level section is $a_0^{3\epsilon_i}$. Similarly for case (ii).

For case (i), suppose for a contradiction that $\ell_n(h) = 3$ for all $n \in \mathbb{N}$. Then note that for every n , the element h has exactly one non-trivial n th level section, which is $a_{i-n}^{\epsilon_i} a_{j-n}^{\epsilon_j} a_{k-n}^{\epsilon_k}$. As usual, the indices of the generators are viewed modulo s , so we effectively only consider the first $(s - 1)$ st level sections.

Consider now the non-trivial i th level section, which is $a_0^{\epsilon_i} a_{j-i}^{\epsilon_j} a_{k-i}^{\epsilon_k}$. Suppose $k = i$. If $\epsilon_i = \epsilon_k$, then clearly the $(i + 1)$ st sections decrease in length. So suppose $\epsilon_i \neq \epsilon_k$. Upon considering the only non-trivial j th level section $a_{i-j}^{\epsilon_i} a_0^{\epsilon_j} a_{i-j}^{\epsilon_k}$, we see that $\ell_{j+1}(h) \leq 2$. So we suppose that $k \neq i$. Consider first the case $\epsilon_i = 1$. Recall that the non-trivial i th level section of h is $a_0 a_{j-i}^{\epsilon_j} a_{k-i}^{\epsilon_k}$. In order for $\ell_{i+1}(h)$ to still be 3, we need

$$a_{j-i} = (a_{j-i-1}, 1, \dots, 1) \quad \text{and} \quad a_{k-i} = (a_{k-i-1}, 1, \dots, 1).$$

If the defining word \mathfrak{v} of the group $\mathfrak{R}(\mathfrak{v})$ does not correspond to the above generators, then this implies that our assumption for this case (i) does not hold, and hence, the section lengths eventually decrease, giving us $\ell_s(h) \leq 2$. So we proceed by assuming that \mathfrak{v} tallies with the above generators and with the other generators singled out below.

Next, we similarly see that in order for $\ell_{s+2i-j}(h)$ to still be 3, we need $a_{j-i} a_{2j-2i}^{\epsilon_j} a_{j+k-2i}^{\epsilon_k}$ to have exactly one non-trivial section. Thus,

$$a_{2j-2i} = (a_{2j-2i-1}, 1, \dots, 1) \quad \text{and} \quad a_{j+k-2i} = (a_{j+k-2i-1}, 1, \dots, 1).$$

In particular, proceeding in this manner, we may assume that

$$a_{\theta(j-i)} = (a_{\theta(j-i)-1}, 1, \dots, 1)$$

for all $\theta \in \mathbb{N}$. Now let $\theta = s - 1$, and so $a_{i-j} = (a_{i-j-1}, 1, \dots, 1)$. Observe that the only non-trivial j th level section of h is $a_{i-j}^{\epsilon_i} a_0^{\epsilon_j} a_{k-j}^{\epsilon_k}$. To avoid a decrease in length at the next level, we must have $\epsilon_j = -1$ and $a_{k-j} = (1, \dots, 1, a_{k-j-1})$. Similarly, for $\ell_{s+2j-k}(h)$ to still be 3, we need $a_{k-2j+i} a_{k-j}^{-1} a_{2(k-j)}^{\epsilon_k}$ to have exactly one non-trivial section. Thus,

$$a_{k-2j+i} = (1, \dots, 1, a_{k-2j+i-1}) \quad \text{and} \quad a_{2(k-j)} = (1, \dots, 1, a_{2(k-j)-1}).$$

As deduced above, since $a_{k-i} = (a_{k-i-1}, 1, \dots, 1)$ we also then have that

$$a_{\theta(k-i)} = (a_{\theta(k-i)-1}, 1, \dots, 1) \quad \text{and} \quad a_{\theta(k-j)} = (1, \dots, 1, a_{\theta(k-j)-1})$$

for all $\theta \in \mathbb{N}$. As before, we let $\theta = s - 1$, and consider the only non-trivial k th level section of h , which is $a_{i-k} a_{j-k}^{-1} a_0^{\epsilon_k}$. Since

$$a_{(s-1)(k-i)} = a_{i-k} = (a_{i-k-1}, 1, \dots, 1) \quad \text{and} \quad a_{(s-1)(k-j)} = a_{j-k} = (1, \dots, 1, a_{j-k-1})$$

Then, we see that $\ell_{k+1}(h) \leq 2$, and we are done in this case $\epsilon_i = 1$.

The remaining case $\epsilon_i = -1$ follows similarly. Hence, the proof is complete. □

Lemma 3.11. *The semigroup generated by a_0, \dots, a_{s-1} is free. In particular, the group $\mathfrak{R}(\mathfrak{v})$ has exponential word growth.*

Proof. We proceed as in the proof of [8, Theorem 6.1], which is based on [15, Lemma 4]. Let u and w be two different words representing the same element in the semigroup generated by a_0, \dots, a_{s-1} , and with $\rho = \max(|u|, |w|)$ minimal. Clearly, $\rho \geq 2$. Now for any word z in the semigroup, let $|z|_{a_0}$ denote the a_0 -length in z , that is, the number of occurrences of a_0 in z . Note that we have $|u|_{a_0} \equiv |w|_{a_0} \pmod{m}$. In the rest of the proof, we may consider the sections of u and w as defined in the proof of Lemma 3.7.

Suppose first that u contains no a_0 's. We may assume that w contains at least one a_0 . Indeed, if both u and w have no a_0 's, by replacing both words with their corresponding non-trivial sections respectively, the condition of both words having no a_0 's cannot hold indefinitely, as then both words u and w would be the trivial word. Thus w contains at least one a_0 . Since $|w|_{a_0}$ must then be a non-zero multiple of m , one deduces that w_j , for $j \in [0, m - 1]$, is a non-empty word, where $\psi(w) = (w_0, \dots, w_{m-1})$. Indeed, every component of $\psi(w)$ contains an a_{s-1} . Certainly $|w_j| < \rho$. Therefore, u must have only one section with a non-empty word, as otherwise, it will contradict the minimality of ρ . Suppose this section is in the i th component, for some $i \in [0, m - 1]$. From considering the j th section of u and w , for $j \neq i$, we obtain a contradiction to the minimality of ρ . So the number of occurrences of a_0 in u is at least one.

If the number of occurrences of a_0 in u , respectively w , is at least 2, then there are at least two sections of u , respectively w , that contain a_{s-1} . Hence, all sections of u , respectively w , have length strictly less than $|u|$, respectively $|w|$, which contradicts the minimality of ρ . So suppose that both u and w have exactly one occurrence of a_0 . Repeating this argument for the sections of u and w , we may assume that every generator appears at most once in u and w and that the same generators appear in both u and w . In other words, the word w is a reordering of the letters in u . Furthermore, as seen above, both u and w have only one non-empty section, which is therefore the rightmost section. Indeed, if this is not the case, then either u or w have more than one non-trivial section, then we contradict the minimality of ρ . Without loss of generality, we may suppose that a_0 is the i th letter of the word u , but the j th letter of the word w where $j < i$. The restrictions on the sections of u imply that the first i letters of u have non-empty section in the rightmost component, and all other letters have non-empty section in the leftmost component. At least one of these first $i - 1$ letters of u appears to the right of a_0 in the word w . Thus, there is a non-empty section in the second last component, which yields a contradiction to the minimality of ρ . Hence, we are done. □

Lastly, we end this section with a result that will be useful for the final section.

Lemma 3.12. *The subgroup $\psi^{-1}(\gamma_3(\mathfrak{R}(\mathfrak{v})) \times \dots \times \gamma_3(\mathfrak{R}(\mathfrak{v})))$ is contained in $\mathfrak{R}(\mathfrak{v})''$.*

Proof. Notice first that, for $i \in [1, s - 1]$ with $x_{i-1} \neq m - 1$, we have

$$[a_i, a_0]^{a_0^{m-2-x_{i-1}}} = (1, \dots, 1, a_{i-1}^{-1}, a_{i-1}).$$

When $x_{i-1} = m - 1$, we get

$$[a_0^{-1}, a_i] = (1, \dots, 1, a_{i-1}^{-1}, a_{i-1}).$$

Together with Lemma 3.4, we see that the elements of the form

$$(1, \dots, 1, [a_i, a_j, a_k])$$

are contained in $\psi(\mathfrak{R}(\mathfrak{v})'')$, for $i, j \in [0, s - 1]$ and $k \in [0, s - 2]$.

We have to find the elements of the form $(1, \dots, 1, [a_i, a_j, a_{s-1}])$. Observe from Lemma 3.4 that

$$\mathfrak{K}(\mathbf{v})'' \times \dots \times \mathfrak{K}(\mathbf{v})'' \leq \psi(\mathfrak{K}(\mathbf{v})'').$$

Then from the Hall-Witt identity, we deduce that

$$\begin{aligned} 1 &= [a_j, a_i^{-1}, a_{s-1}]^{a_i} [a_i, a_{s-1}^{-1}, a_j]^{a_{s-1}} [a_{s-1}, a_j^{-1}, a_i]^{a_j} \\ &= [a_i, a_j, a_{s-1}^{a_i}] [a_{s-1}, a_i, a_j^{a_{s-1}}] [a_j, a_{s-1}, a_i^{a_j}] \\ &\equiv [a_i, a_j, a_{s-1}] [a_{s-1}, a_i, a_j] [a_j, a_{s-1}, a_i] \pmod{\mathfrak{K}(\mathbf{v})''}. \end{aligned}$$

Hence, if $i, j \neq s - 1$, from considering the product of

$$(1, \dots, 1, [a_{s-1}, a_i, a_j]) \quad \text{and} \quad (1, \dots, 1, [a_j, a_{s-1}, a_i])$$

together with an appropriate element from $\mathfrak{K}(\mathbf{v})'' \times \dots \times \mathfrak{K}(\mathbf{v})''$, we obtain

$$(1, \dots, 1, [a_i, a_j, a_{s-1}]) \in \psi(\mathfrak{K}(\mathbf{v})'').$$

If $i = s - 1$ but $j \neq s - 1$, we will show that $[a_{s-1}, a_j, a_{s-1}] = 1$. To this end, in light of Lemma 3.3 it suffices to consider $[a_{s-1-j}, a_0, a_{s-1-j}]$. Recall from the proof of Lemma 3.3 that

$$[a_{s-1-j}, a_0] = \begin{cases} (a_{s-1}^{-1} a_{s-j-2} a_{s-1}, 1, \dots, 1, a_{s-j-2}^{-1}) & \text{if } x_{s-j-2} = m - 1, \\ (1, \overset{x_{s-j-2}-1}{\dots}, 1, a_{s-j-2}^{-1}, a_{s-j-2}, 1, \dots, 1) & \text{if } x_{s-j-2} \neq m - 1. \end{cases}$$

Hence, it is now clear that $[a_{s-1-j}, a_0, a_{s-1-j}] = 1$.

If $j = s - 1$ but $i \neq s - 1$, from the Hall-Witt identity, we have the equivalence

$$1 \equiv [a_i, a_{s-1}, a_{s-1}] [a_{s-1}, a_i, a_{s-1}] \pmod{\mathfrak{K}(\mathbf{v})''}.$$

Hence, we are done from the previous case.

The result now follows from the level-transitivity and the fractalness of $\mathfrak{K}(\mathbf{v})$. □

4. Length-reducing properties

As before, we have $\mathbf{v} = x_0 \dots x_{s-2}$ is a word in the alphabet $X = \{0, 1, \dots, m - 1\}$ and $\mathfrak{K}(\mathbf{v})$ is the group associated with \mathbf{v} . In this section, we establish some length-reducing properties for some of the groups $\mathfrak{K}(\mathbf{v})$. Notice that for every $g \in \mathfrak{K}(\mathbf{v})$, the local action g^ϵ of g at the root is an element of $\langle \sigma \rangle$. Hence, for conciseness, we denote g^ϵ by σ_g . If $g \in \mathfrak{K}(\mathbf{v})$ then $|g|$ denotes the minimal length of all words in the alphabet S representing g . A word $w \in S^*$ is called a *geodesic word* if $|w| = |g|$, where g is the image of the word w in $\mathfrak{K}(\mathbf{v})$.

Lemma 4.1. *Let $g = (g_0, \dots, g_{m-1})\sigma_g \in \mathfrak{K}(\mathbf{v})$. Then $\sum_{k=0}^{m-1} |g_k| \leq |g|$.*

Proof. The proof proceeds by induction on the length of g . Clearly, the result is true if $|g| = 0$ and $|g| = 1$. Assume that $|g| > 1$. Let $w \in S^*$ be a geodesic word representing g . The word w can be written as $w = bw'$ for some $b \in S$ and $w' \in S^*$ such that w' is reduced. Then $|w'| < |w|$ and w' does not represent g in $\mathfrak{K}(\mathbf{v})$. Denote by g' the corresponding element in $\mathfrak{K}(\mathbf{v})$. Then $|g'| \leq |w'| < |w| = |g|$. We obtain

$$\begin{aligned} (g_0, \dots, g_{m-1})\sigma_g &= g = bg' = (b_0, \dots, b_{m-1})\sigma_b(g'_0, \dots, g'_{m-1})\sigma_{g'} \\ &= (b_0g'_0, \dots, b_{m-1}g'_{(m-1)\sigma_b})\sigma_b\sigma_{g'}, \end{aligned}$$

which implies $g_k = b_k g'_{k\sigma_b}$ for all $k \in [0, m - 1]$. It follows by induction that

$$\sum_{k=0}^{m-1} |g_k| = \sum_{k=0}^{m-1} |b_k g'_{k\sigma_b}| \leq \sum_{k=0}^{m-1} |b_k| + \sum_{k=0}^{m-1} |g'_{k\sigma_b}| \leq |b| + |g'| \leq |b| + |w'| = |w| = |g|.$$

□

Lemma 4.2. Let $v = x_0 \cdots x_{s-2}$ be a word in the alphabet X , and let $g = (g_0, \dots, g_{m-1})\sigma_g \in \mathfrak{K}(v)$ with $\sigma_g = \sigma^i$ for some $i \in [1, m - 1]$ such that $\gcd(i, m) = 1$. Let $\alpha_0, \dots, \alpha_{m-1} \in \mathfrak{K}(v)$ be such that $g^m = (\alpha_0, \dots, \alpha_{m-1})$. Then $|\alpha_k| \leq \sum_{\ell=0}^{m-1} |g_\ell| \leq |g|$ for all $k \in [0, m - 1]$.

Proof. Observe that

$$g^m = (g_0 g_{0\sigma^i} g_{0\sigma^{2i}} \cdots g_{0\sigma^{(m-1)i}}, \dots, g_{m-1} g_{(m-1)\sigma^i} g_{(m-1)\sigma^{2i}} \cdots g_{(m-1)\sigma^{(m-1)i}}).$$

By setting $\alpha_k = g_k g_{k\sigma^i} \cdots g_{k\sigma^{(m-1)i}}$, for each $k \in [0, m - 1]$, we obtain

$$|\alpha_k| = |g_k g_{k\sigma^i} \cdots g_{k\sigma^{(m-1)i}}| \leq \sum_{\ell=0}^{m-1} |g_\ell| \leq |g|,$$

where the last inequality follows from Lemma 4.1. □

Lemma 4.3. Let $v = x_0 \cdots x_{s-2}$ be a word in the alphabet $X \setminus \{0\}$. Let $g = (g_0, \dots, g_{m-1})\sigma_g \in \mathfrak{K}(v)$ and let $b_1 \cdots b_\ell \in S^*$ be a geodesic word representing g . If there exist $1 \leq r < r' \leq \ell$ such that $b_r = a_0, b_{r'} = a_0^{-1}$, then $\sum_{k=0}^{m-1} |g_k| < |g|$.

Proof. By assumption, the word $b_1 \cdots b_\ell$ contains a subword of the form $a_0 w a_0^{-1}$, where w is a non-trivial reduced word in the alphabet S . We assume, without loss of generality, that w is a reduced word in the alphabet $S \setminus \{a_0^{\pm 1}\}$. Let w represent an element h in $\mathfrak{K}(v)$. Since $b_1 \cdots b_\ell \in S^*$ is a geodesic word, the word w is also geodesic and so $|h| = |w|$. Notice that $|a_0 w a_0^{-1}| = |w| + 2$. Realising the word $a_0 w a_0^{-1}$ in $\mathfrak{K}(v)$ gives

$$a_0 w a_0^{-1} = (\varphi_1(h), \dots, \varphi_{m-2}(h), \varphi_{m-1}(h), 1);$$

indeed, recall that by assumption we have

$$h = (1, \varphi_1(h), \dots, \varphi_{m-2}(h), \varphi_{m-1}(h)).$$

By Lemma 4.1, we get $\sum_{i=1}^{m-1} |\varphi_i(h)| \leq |h| = |w|$. Therefore, we conclude that

$$\sum_{k=0}^{m-1} |g_k| \leq |g| - 2 < |g|.$$

□

Lemma 4.4. Let $v = 0 \overset{s-1}{\cdot} 0$ and let $g = (g_0, \dots, g_{m-1})\sigma^\epsilon \in \mathfrak{K}(v)$ with $\epsilon \in \{\pm 1\}$. Suppose $b_1 \cdots b_\ell \in S^*$ is a geodesic word representing g . If there exist $1 \leq r < r' \leq \ell$ such that either $b_r = a_0, b_{r'} = a_0^{-1}$ or $b_r = a_0^{-1}, b_{r'} = a_0$, then $|g_0 g_1 \cdots g_{m-1}| < |g|$ if $\epsilon = 1$, and $|g_0 g_{m-1} \cdots g_1| < |g|$ if $\epsilon = -1$.

Proof. Note that we may assume that there are no $1 \leq i < j \leq \ell$ such that $b_i = a_0^{-1}$ and $b_j = a_0$, as then the desired length reduction is clear. It then follows that

$$w := b_1 \cdots b_\ell = *a_0 (*a_0)^\lambda \cdot (*a_0) (*a_0^{-1})^\lambda \cdot (*a_0^{-1}) * \quad \text{if } \epsilon = 1$$

and

$$w := b_1 \cdots b_\ell = (*a_0)^\lambda \cdot (*a_0) (*a_0^{-1})^\lambda \cdot (*a_0^{-1}) * a_0^{-1} * \quad \text{if } \epsilon = -1,$$

for some $\lambda \in \mathbb{N}_0$ and the $*$ in $\langle a_1, \dots, a_{s-1} \rangle$ are arbitrary.

Let

$$w' = (*a_0)^\lambda \cdot (*a_0) (*a_0^{-1})^\lambda \cdot (*a_0^{-1}) *.$$

Write $w' = (w'_0, \dots, w'_{m-1})$ and $\lambda = \mu m + \chi$ for some $\mu \in \mathbb{N}_0$ and $\chi \in [0, m - 1]$. If $\mu \neq 0$, the first-level sections of w' have the following form

$$\begin{aligned} w'_0 &= \varphi_0(*) (a_{s-1}\varphi_0(*)) \cdots (a_{s-1}\varphi_0(*)) a_{s-1}\varphi_0(*) a_{s-1}^{-1}(\varphi_0(*) a_{s-1}^{-1}) \cdots (\varphi_0(*) a_{s-1}^{-1}) \varphi_0(*) \\ w'_1 &= (a_{s-1}\varphi_0(*)) \cdots (a_{s-1}\varphi_0(*)) a_{s-1}\varphi_0(*) a_{s-1}^{-1}(\varphi_0(*) a_{s-1}^{-1}) \cdots (\varphi_0(*) a_{s-1}^{-1}) \\ &\vdots \\ w'_{m-1-\chi} &= (a_{s-1}\varphi_0(*)) \cdots (a_{s-1}\varphi_0(*)) a_{s-1}\varphi_0(*) a_{s-1}^{-1}(\varphi_0(*) a_{s-1}^{-1}) \cdots (\varphi_0(*) a_{s-1}^{-1}) \\ w'_{m-\chi} &= (a_{s-1}\varphi_0(*)) \cdots (a_{s-1}\varphi_0(*)) a_{s-1}\varphi_0(*) a_{s-1}^{-1}(\varphi_0(*) a_{s-1}^{-1}) \cdots (\varphi_0(*) a_{s-1}^{-1}) \\ &\vdots \\ w'_{m-1} &= (a_{s-1}\varphi_0(*)) \cdots (a_{s-1}\varphi_0(*)) a_{s-1}\varphi_0(*) a_{s-1}^{-1}(\varphi_0(*) a_{s-1}^{-1}) \cdots (\varphi_0(*) a_{s-1}^{-1}). \end{aligned}$$

When $\mu = 0$ and hence $\chi \neq 0$, we get

$$\begin{aligned} w'_0 &= \varphi_0(*) \\ w'_1 &= 1 \\ &\vdots \\ w'_{m-1-\chi} &= 1 \\ w'_{m-\chi} &= a_{s-1}\varphi_0(*) a_{s-1}^{-1} \\ &\vdots \\ w'_{m-1} &= a_{s-1}\varphi_0(*) a_{s-1}^{-1}. \end{aligned}$$

Hence, a straightforward computation shows that $|g_0 g_1 \cdots g_{m-1}| < |g|$ if $\epsilon = 1$, and that $|g_0 g_{m-1} \cdots g_1| < |g|$ if $\epsilon = -1$, as required. □

5. Maximal subgroups

It follows from Proposition 5.1 below together with [10, Proposition 2.21] that the group $\mathfrak{K}(\mathfrak{v})$ admits maximal subgroups of infinite index if and only if it admits a proper subgroup $H < \mathfrak{K}(\mathfrak{v})$ such that $HN = \mathfrak{K}(\mathfrak{v})$ for every non-trivial normal subgroup $N \trianglelefteq \mathfrak{K}(\mathfrak{v})$. A subgroup $H \leq \mathfrak{K}(\mathfrak{v})$ satisfying the above condition is called a *prodense subgroup*. As seen below, we prove that $\mathfrak{K}(\mathfrak{v})$, for a constant word \mathfrak{v} , does not admit any proper prodense subgroup, which proves Theorem 1.2.

Proposition 5.1. *The group $\mathfrak{K}(\mathfrak{v})$ is just non-(virtually nilpotent). Hence, maximal subgroups of proper quotients of $\mathfrak{K}(\mathfrak{v})$ are of finite index.*

Proof. As $\mathfrak{K}(\mathfrak{v})$ has exponential word growth from Lemma 3.11, it follows from Bass [5] and Guivarc’h [16] that $\mathfrak{K}(\mathfrak{v})$ is not virtually nilpotent. To see that every proper quotient of $\mathfrak{K}(\mathfrak{v})$ is virtually nilpotent, by [10, Theorem 4.10], it suffices to prove that $\mathfrak{K}(\mathfrak{v})/\mathfrak{K}(\mathfrak{v})''$ is virtually nilpotent. Set $N = \psi^{-1}(\gamma_3(\mathfrak{K}(\mathfrak{v})) \times \cdots \times \gamma_3(\mathfrak{K}(\mathfrak{v})))$. From Lemma 3.12, we have $N \leq \mathfrak{K}(\mathfrak{v})'' < \text{St}_{\mathfrak{K}(\mathfrak{v})}(1) < \mathfrak{K}(\mathfrak{v})$. Therefore, ψ induces a homomorphism

$$\tilde{\psi} : \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/N \longrightarrow \mathfrak{K}(\mathfrak{v})/\gamma_3(\mathfrak{K}(\mathfrak{v})) \times \cdots \times \mathfrak{K}(\mathfrak{v})/\gamma_3(\mathfrak{K}(\mathfrak{v})).$$

Since $\tilde{\psi}$ is injective and $\tilde{\psi}(\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/N)$ is nilpotent (being a subgroup of a nilpotent group), we obtain that $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/N$ is nilpotent. This implies that $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/\mathfrak{K}(\mathfrak{v})''$ is nilpotent as it is a quotient of $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/N$. As the subgroup $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$ has finite index in $\mathfrak{K}(\mathfrak{v})$, the group $\text{St}_{\mathfrak{K}(\mathfrak{v})}(1)/\mathfrak{K}(\mathfrak{v})''$ has finite

index in $\mathfrak{K}(\mathfrak{v})/\mathfrak{K}(\mathfrak{v})''$, and hence, $\mathfrak{K}(\mathfrak{v})/\mathfrak{K}(\mathfrak{v})''$ is virtually nilpotent. The last part of the result follows from [9, Corollary 5.1.3]. \square

Hereafter, for $g, h \in \mathfrak{K}(\mathfrak{v})$, the equivalence $g \equiv h \pmod{\mathfrak{K}(\mathfrak{v})'}$ will simply be denoted by $g \equiv h$. Notice that for every $z \in \mathfrak{K}(\mathfrak{v})'$, we have $\sigma_z = 1$ and $\mathfrak{K}(\mathfrak{v})' \leq \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$. Recall also from Subsection 2.1 the map φ_u for $u \in X^*$. For convenience, we introduce the following notation. For $i \in [0, s - 2]$ and $j \in [0, m - 1]$, we define

$$\delta_j(a_i) = \begin{cases} a_i & \text{if } x_i = j, \\ 1 & \text{otherwise,} \end{cases}$$

and extend δ_j to a homomorphism on S^* .

Lemma 5.2. *Let $\mathfrak{v} = x_0 \cdots x_{s-2}$ be a word in the alphabet X , and let $g \in \mathfrak{K}(\mathfrak{v})$ be such that $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, where $\epsilon_i \in \{\pm 1\}$. For $j \in \mathbb{N}_0$, write $j = \ell s + r$, where $\ell \in \mathbb{N}_0$ and $r \in [0, s - 1]$. If $\psi_j(g^{m^j}) = (g_0, \dots, g_{m^j-1})$ then $g_k \equiv a_{s-1-r}^{\epsilon_{s-1}} \cdots a_{0-r}^{\epsilon_0}$ for all $k \in [0, m^j - 1]$.*

Proof. Since $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, there exists an element $(z_0, \dots, z_{m-1}) = z \in \mathfrak{K}(\mathfrak{v})'$ such that $g = a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0} z$. Suppose that $\epsilon_0 = 1$. We have that the section of g at the vertex j is

$$(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0} z)_j = \begin{cases} \delta_j(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) z_{j+1} & \text{if } j \in [0, m - 2], \\ \delta_{m-1}(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) a_{s-1} z_0 & \text{if } j = m - 1. \end{cases}$$

Therefore, for $j \in [0, m - 1]$, the element $\varphi_j(g^m)$ is equal to the product

$$\begin{aligned} & (\delta_j(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) z_{j+1}) \cdots (\delta_{m-2}(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) z_{m-1}) \cdot \\ & (\delta_{m-1}(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) a_{s-1} z_0) \cdot \\ & (\delta_0(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) z_1) \cdots (\delta_{j-1}(a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1}) z_j). \end{aligned}$$

Since $\delta_k(a_i^{\epsilon_i+1})$ is non-trivial for only one $k \in [0, m - 1]$, we get that $\varphi_j(g^m) \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0} a_{s-1}^{\epsilon_0}$ for all $j \in [0, m - 1]$, since $z_0 \cdots z_{m-1} \in \mathfrak{K}(\mathfrak{v})'$ by Lemma 2.1. The case $\epsilon_0 = -1$ follows in a similar way. The result then follows upon repeating the above process. \square

In the following, we denote by $\text{Sym}(s)$ the symmetric group on $[0, s - 1]$.

Lemma 5.3. *Let $\mathfrak{v} = t \overset{s-1}{\cdot} t$ be a constant word, where $t \in [0, m - 1]$. Let $g = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \in \mathfrak{K}(\mathfrak{v})$ where $\pi \in \text{Sym}(s)$ and $\epsilon_i \in \{\pm 1\}$. For $j \in \mathbb{N}_0$, write $j = \ell s + r$, where $\ell \in \mathbb{N}_0$ and $r \in [0, s - 1]$. Then $\varphi_j(g^{m^j}) = a_{(s-1)^\pi-r}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi-r}^{\epsilon_{0^\pi}}$.*

Proof. Let $i \in [0, s - 1]$ be such that $0 = i^\pi$. Then

$$\begin{aligned} g &= a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi}^{\epsilon_{(i+1)^\pi}} a_0^{\epsilon_0} a_{(i-1)^\pi}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \\ &= \begin{cases} (1, \overset{t-1}{\cdot}, 1, a_{(i-1)^\pi}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}, a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi}^{\epsilon_{(i+1)^\pi}}, 1, \dots, 1, a_{s-1}) \sigma & \text{if } \epsilon_0 = 1 \text{ and } t > 0, \\ (a_{(s-1)^\pi-r}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi-r}^{\epsilon_{(i+1)^\pi}}, 1, \dots, 1, a_{s-1} a_{(i-1)^\pi-r}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi-r}^{\epsilon_{0^\pi}}) \sigma & \text{if } \epsilon_0 = 1 \text{ and } t = 0, \\ (a_{s-1}^{-1}, 1, \overset{t-1}{\cdot}, 1, a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi}^{\epsilon_{(i+1)^\pi}}, a_{(i-1)^\pi}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}, 1, \dots, 1) \sigma^{-1} & \text{if } \epsilon_0 = -1 \text{ and } t > 0, \\ (a_{(s-1)^\pi-r}^{\epsilon_{(s-1)^\pi}} \cdots a_{(i+1)^\pi-r}^{\epsilon_{(i+1)^\pi}} a_{s-1}^{-1} a_{(i-1)^\pi-r}^{\epsilon_{(i-1)^\pi}} \cdots a_{0^\pi-r}^{\epsilon_{0^\pi}}, 1, \dots, 1) \sigma^{-1} & \text{if } \epsilon_0 = -1 \text{ and } t = 0. \end{cases} \end{aligned}$$

By taking the m th power of the element g we get

$$g^m = \begin{cases} (*, \dots, *, a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}, \#, \dots, \#) & \text{if } \epsilon_0 = 1 \text{ and } t > 0, \\ (a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}, \#, \dots, \#) & \text{if } \epsilon_0 = 1 \text{ and } t = 0, \\ (*, \dots, *, a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1}^{-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}, \#, \dots, \#) & \text{if } \epsilon_0 = -1 \text{ and } t > 0, \\ (a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1}^{-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}, \#, \dots, \#) \sigma^{-1} & \text{if } \epsilon_0 = -1 \text{ and } t = 0. \end{cases}$$

where

$$* = \begin{cases} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1}} & \text{if } \epsilon_0 = 1, \\ a_{s-1}^{-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi}} & \text{if } \epsilon_0 = -1, \end{cases}$$

and

$$\# = \begin{cases} a_{s-1} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi}} & \text{if } \epsilon_0 = 1, \\ a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi}} a_{s-1}^{-1} & \text{if } \epsilon_0 = -1. \end{cases}$$

In particular, we have $\varphi_i(g^m) = a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}$, and the result follows recursively. □

We recall that H_u denotes the subgroup $\varphi_u(\text{st}_H(u))$ for a vertex $u \in X^*$. By Lemma 3.2, the group $\mathfrak{K}(\mathfrak{v})$ is fractal, so $\mathfrak{K}(\mathfrak{v})_u = \mathfrak{K}(\mathfrak{v})$ for all $u \in X^*$.

Lemma 5.4. *Let $\mathfrak{v} = t \text{ } \overset{s-1}{\cdot} \text{ } t$ be a constant word, where $t \in [0, m - 1]$, and let H be a subgroup of $\mathfrak{K}(\mathfrak{v})$. Assume that $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi}^{\epsilon_0} \in H$ for some $\pi \in \text{Sym}(s)$, where $\epsilon_i \in \{\pm 1\}$. Then the following assertions hold.*

- (i) *For each $n \in \mathbb{N}$ and vertex u of level ns , the subgroup H_u contains a cyclic permutation of the word $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi}^{\epsilon_0}$.*
- (ii) *Furthermore, if $\epsilon_i = 1$ for some $i \in [0, s]$, then for each $n \in \mathbb{N}$, there is a vertex u of level ns such that the cyclic permutation w of $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi}^{\epsilon_0}$ that is contained in H_u satisfies the following property:*
 - (a) *if $t > 0$, the cyclic permutation w ends with a_i on the right;*
 - (b) *if $t = 0$, the cyclic permutation w starts with a_i on the left.*

Proof. (i) Let $g = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi}^{\epsilon_0} \in H$. Then, $0 = i^\pi$ for some $i \in [0, s - 1]$. Observe from the proof of Lemma 5.3 that

$$\varphi_i(g^m) = a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1}^{\epsilon_0} a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0}}$$

and that $\varphi_j(g^m)$ is a cyclic permutation of $\varphi_i(g^m)$ for every $j \in [0, m - 1]$. By repeating the process of taking powers we get that $\psi_\tau(g^{m^\tau}) = (g_0, \dots, g_{m^\tau-1})$ with $g_\tau = g$, for the component τ corresponding to the vertex t^τ , and g_k is a cyclic permutation of the word $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi} \dots a_{0^\pi}^{\epsilon_0}$ for $k \in [0, m^\tau - 1]$.

(ii) We will only consider the case $t > 0$, as the case $t = 0$ is analogous. In particular, if $\epsilon_0 = 1$, we note from the proof of Lemma 5.3 that

$$\varphi_{t-1}(g^m) = a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi} a_{s-1}},$$

and by Lemma 5.3 we see that

$$\varphi_{(t-1)^\tau-1}(g^m) = a_{(i-1)^\pi-1}^{\epsilon_{(i-1)^\pi} \dots a_{0^\pi-1}^{\epsilon_0} a_{(s-1)^\pi-1}^{\epsilon_{(s-1)^\pi} \dots a_{(i+1)^\pi-1}^{\epsilon_{(i+1)^\pi}} a_0.$$

More generally, suppose that $\epsilon_j = 1$ for $j \in [0, s]$ and let $\ell \in [0, s]$ be such that $\ell^\pi = j$. Writing $v_j = t \cdot \overset{j}{\cdot} \cdot t$ and $w_j = (t - 1)t \cdot \overset{s-j-1}{\cdot} \cdot t$, we recall from Lemma 5.3 that

$$\varphi_{v_j}(g^{mj}) = a_{(s-1)^\pi - j}^{\epsilon_{(s-1)^\pi}} \cdots a_{(\ell+1)^\pi - j}^{\epsilon_{(\ell+1)^\pi}} a_0 a_{(\ell-1)^\pi - j}^{\epsilon_{(\ell-1)^\pi}} \cdots a_{0^\pi - j}^{\epsilon_{0^\pi}}.$$

Then similar to the above, we see that

$$\varphi_{v_j w_j}(g^m) = a_{(\ell-1)^\pi}^{\epsilon_{(\ell-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{(\ell+1)^\pi}^{\epsilon_{(\ell+1)^\pi}} a_j,$$

and as $u_j := v_j w_j$ is a vertex of level s , we have that H_{u_j} contains a cyclic permutation of $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ that ends with a_j on the right.

Now, by using Lemma 5.3 repeatedly, one can see that the result holds for level ns of T , for $n > 1$. \square

Proposition 5.5. *Let $v = t \overset{s-1}{\cdot} t$ be a constant word, where $t \in [0, m - 1]$, and let $g \in \mathfrak{R}(v)$ be such that $g \equiv a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$, where $\epsilon_i \in \{\pm 1\}$. Then, there exists a vertex u of level ns in T , for some $n \in \mathbb{N}_0$, and an element $g' \in \text{st}_{(g)}(u)$ such that $\varphi_u(g') = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}}$ for some $\pi \in \text{Sym}(s)$.*

Proof. The proof proceeds by induction on the length of g . Recall from Lemma 3.8 that $\mathfrak{R}(v)/\mathfrak{R}(v)' = \langle a_0 \mathfrak{R}(v)', \dots, a_{s-1} \mathfrak{R}(v)' \rangle \cong \mathbb{Z}^s$. Hence, if g is equivalent to $a_{s-1}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_0}$ then $|g| > s - 1$, since any word containing each of the distinct generators of $\mathfrak{R}(v)$ has length at least s . Assume that $|g| = s$. Then

$$g \in \{ a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}} \mid \pi \in \text{Sym}(s) \}$$

and the result follows trivially by choosing u as the root vertex. Now, assume that $|g| > s$. Since the exponent sum of a_0 in any word representing g is ϵ_0 , we can write $g = (g_0, \dots, g_{m-1}) \sigma^{\epsilon_0}$ with $g_0, \dots, g_{m-1} \in \mathfrak{R}(v)$. We get

$$g^m = \begin{cases} (g_0 g_1 \cdots g_{m-1}, g_1 \cdots g_{m-1} g_0, \dots, g_{m-1} g_0 g_1 \cdots g_{m-2}) & \text{if } \epsilon_0 = 1, \\ (g_0 g_{m-1} g_{m-2} \cdots g_1, g_1 g_0 g_{m-1} \cdots g_2, \dots, g_{m-1} g_{m-2} \cdots g_0) & \text{if } \epsilon_0 = -1. \end{cases}$$

For every $k \in [0, m - 1]$, we set $\alpha_k = \varphi_k(g^m)$. It follows from Lemma 5.2 that

$$\alpha_k \equiv a_{s-2}^{\epsilon_{s-1}} \cdots a_0^{\epsilon_1} a_{s-1}^{\epsilon_0}$$

for all $k \in [0, m - 1]$. Furthermore, $|\alpha_k| \leq |g|$ for all $k \in [0, m - 1]$ by Lemma 4.2. If there exists $k \in [0, m - 1]$ such that $|\alpha_k| < |g|$, then it follows by induction that there exists a vertex u of level ns in T , for some $n \in \mathbb{N}_0$, and $g' \in \text{st}_{(\alpha_k)}(u)$ such that

$$\varphi_u(g') = a_{(s-1)^\pi - 1}^{\epsilon_{(s-1)^\pi}} a_{(s-2)^\pi - 1}^{\epsilon_{(s-2)^\pi}} \cdots a_{0^\pi - 1}^{\epsilon_{0^\pi}}$$

for some $\pi \in \text{Sym}(s)$. Using Lemma 5.3, we get that

$$\varphi_{g^{s-1}}((g')^{m^{s-1}}) = a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} a_{(s-2)^\pi}^{\epsilon_{(s-2)^\pi}} \cdots a_{0^\pi}^{\epsilon_{0^\pi}},$$

and hence, the result follows.

Assume that $|\alpha_k| = |g|$ for all $k \in [0, m - 1]$. Since $|\alpha_k| \leq \sum_{\ell=0}^{m-1} |g_\ell|$, in particular, we get

$$\sum_{\ell=0}^{m-1} |g_\ell| = |g|.$$

Let $w_g \in S^*$ be a geodesic word representing g . Since for each $i \in [0, s - 1]$ the element $a_i^{\epsilon_i}$ contributes $a_{i-1}^{\epsilon_i}$ in exactly one component, we can obtain words w_{g_k} representing g_k by substituting $a_i^{\epsilon_i}$ in w_g with $a_{i-1}^{\epsilon_i}$ in the appropriate component. Notice that $|g_k| \leq |w_{g_k}|$ for every $k \in [0, m - 1]$. Moreover, the words w_{g_k} are geodesic. Indeed,

$$\sum_{k=0}^{m-1} |g_k| \leq \sum_{k=0}^{m-1} |w_{g_k}| \leq |w_g| = |g| = \sum_{\ell=0}^{m-1} |g_\ell|,$$

which forces that $|w_{g_k}| = |g_k|$. Now, set

$$w_{\alpha_k} = \begin{cases} w_{g_k} w_{g_{k+1}} \cdots w_{g_{k+m-1}} & \text{if } \epsilon_0 = 1, \\ w_{g_k} w_{g_{k-1}} \cdots w_{g_{k-(m-1)}} & \text{if } \epsilon_0 = -1. \end{cases}$$

Clearly, w_{α_k} represents α_k . Therefore, $|\alpha_k| \leq |w_{\alpha_k}|$. Furthermore,

$$|w_{\alpha_k}| \leq \sum_{\ell=0}^{m-1} |w_{g_\ell}| = \sum_{\ell=0}^{m-1} |g_\ell| = |g| = |\alpha_k|.$$

Thus, $|\alpha_k| = |w_{\alpha_k}|$ and w_{α_k} is a geodesic word.

Now, we claim that in order to prove the result, it suffices to consider the situation in which for every $i \in [0, s - 1]$ there exists a unique $k \in [0, m - 1]$ such that w_{g_k} contains a non-trivial power of a_i . First, we consider the case when $i = 0$. Assume to the contrary that there exist distinct $k_1, k_2 \in [0, m - 1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain non-trivial powers of a_0 . We can reduce to the following two cases.

Case 1: Suppose that there exist distinct $k_1, k_2 \in [0, m - 1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0 and a_0^{-1} , respectively. Then for some $k \in [0, m - 1]$, the word w_{α_k} contains a subword of the form $a_0 w a_0^{-1}$ with $w \in S^*$. If $\alpha_k^m = (\beta_0, \dots, \beta_{m-1})$ then, by Lemma 4.2, Lemma 4.3 and Lemma 4.4, we obtain that $|\beta_\ell| < |\alpha|$ for $\ell \in [0, m - 1]$. Again, the result follows by induction.

Case 2: Suppose there exist distinct $k_1, k_2 \in [0, m - 1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0 . Recall that $\mathfrak{K}(\mathbf{v})/\mathfrak{K}(\mathbf{v})' = \langle a_0 \mathfrak{K}(\mathbf{v})', \dots, a_{s-1} \mathfrak{K}(\mathbf{v})' \rangle \cong \mathbb{Z}^s$. Hence, as the exponent sum of a_1 in any word representing g is ϵ_1 , the exponent sum of a_0 in w_{α_k} is equal to ϵ_1 for all $k \in [0, m - 1]$. This implies that there exists $k_3 \in [0, m - 1]$ such that $w_{g_{k_3}}$ contains a_0^{-1} , and we are in the previous case. Analogously, the same argument works if both $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain a_0^{-1} .

We reduce to the case where there exists a unique $k \in [0, m - 1]$ such that w_{g_k} contains a non-trivial power of a_0 . By inducting on $i \in [0, s - 1]$, assume that there exists a unique $k \in [0, m - 1]$ such that w_{g_k} contains a non-trivial power of a_{i-1} . Then suppose that there exist distinct $k_1, k_2 \in [0, m - 1]$ such that $w_{g_{k_1}}$ and $w_{g_{k_2}}$ contain non-trivial powers of a_i . We can find $k_3 \in [0, m - 1]$ such that $w_{\alpha_{k_3}}$ contains a subword of the form $a_i^{\ell_1} w a_i^{\ell_2}$, where $\ell_1, \ell_2 \in \mathbb{Z} \setminus \{0\}$ and $w \in S^*$ with exponent sum of a_0 in w is not equal to $0 \pmod m$. Thanks to Lemma 5.2, we may replace g with α_{k_3} . Then we find more than one w_{g_k} containing non-trivial powers of a_{i-1} , contradicting the induction hypothesis and hence proving the claim.

Thus, we reduce to the situation in which for every $i \in [0, s - 1]$ there exists a unique $k \in [0, m - 1]$ such that w_{g_k} contains a non-trivial power of a_i . An easy computation yields that w_g does not contain a subword of the form $a_i^{\ell_1} w a_i^{\ell_2}$, for some $i \in [0, s - 1]$ where $\ell_1, \ell_2 \in \mathbb{Z} \setminus \{0\}$ and $w \in S^*$ with the exponent sum of a_0 in w being not equal to $0 \pmod m$. Hence, we conclude that w_g must be of the form

$$w_1(a_{i_1}, \dots, a_{i_r}) a_0^{\epsilon_0} w_2(a_{i_{r+1}}, \dots, a_{i_{s-1}}),$$

where w_1 and w_2 are words in the given elements, and $\{i_1, \dots, i_r, i_{r+1}, \dots, i_{s-1}\} = [1, s - 1]$ such that the intersection $\{i_1, \dots, i_r\} \cap \{i_{r+1}, \dots, i_{s-1}\}$ is empty. Consider the element α_t obtained from the element g above. Then, the corresponding w_{α_t} has the form

$$w_1(a_{i_1-1}, \dots, a_{i_r-1}) a_{s-1}^{\epsilon_0} w_2(a_{i_{r+1}-1}, \dots, a_{i_{s-1}-1}),$$

and continuing the above procedure with this word yields the element $a_{(s-1)^\pi}^{\epsilon_{(s-1)^\pi}} \cdots a_0^{\epsilon_0 \pi} \in H_u$ for some u of level ns in T , for some $n \in \mathbb{N}_0$. □

Theorem 5.6. *Let $\mathbf{v} = t \text{ : }^{-1} t$ be a constant word, where $t \in [0, m - 1]$. If H is a prodense subgroup of $\mathfrak{K}(\mathbf{v})$, then $H = \mathfrak{K}(\mathbf{v})$.*

Proof. Note that $H\mathfrak{K}(\mathbf{v})' = \mathfrak{K}(\mathbf{v})$ as H is a prodense subgroup. Therefore, there exists an element $z \in \mathfrak{K}(\mathbf{v})'$ such that $a_{s-1} \cdots a_0 z \in H$. By an application of Proposition 5.5, we can find $u \in T$ such that H_u contains $a_{(s-1)^\pi} \cdots a_0^\pi$ for some $\pi \in \text{Sym}(s)$. We set $g = a_{(s-1)^\pi} \cdots a_0^\pi$. Thanks to [10, Lemma 3.1], the subgroup H_u is again a prodense subgroup of $\mathfrak{K}(\mathbf{v})$. Without loss of generality, we replace H with H_u .

Again, as H is prodense, for some $\tilde{z} \in \mathfrak{K}(\mathfrak{v})'$ we similarly have $a_{s-1} \cdots a_1 a_0^{-1} \tilde{z} \in H$. By Proposition 5.5, there exists a vertex u at level ns , for some $n \in \mathbb{N}$, such that H_u contains an element h_0 of the form

$$h_0 = a_{(s-1)\tau_0}^{\epsilon_{i^{\tau_0}}} \cdots a_{0\tau_0}^{\epsilon_{i^{\tau_0}}},$$

where $\tau_0 \in \text{Sym}(s)$, with $\epsilon_{i^{\tau_0}} = -1$ if $i^{\tau_0} = 0$ and $\epsilon_{i^{\tau_0}} = 1$ otherwise. Now, by Lemma 5.4(i), the subgroup H_u also contains some cyclic permutation of the element g . By abuse of notation, we replace g with this cyclic permutation of g . We again replace H with H_u . Now H contains the elements g and h_0 . Repeating this argument $s - 1$ times, we may assume that H contains the elements g, h_0, \dots, h_{s-1} , where

$$h_j = a_{(s-1)\tau_j}^{\epsilon_{i^{\tau_j}}} \cdots a_{0\tau_j}^{\epsilon_{i^{\tau_j}}},$$

where $\tau_j \in \text{Sym}(s)$ with $\epsilon_{i^{\tau_j}} = -1$ if $i^{\tau_j} = j$ and $\epsilon_{i^{\tau_j}} = 1$ otherwise.

For the remainder of the proof, we will assume that $t > 0$; the case $t = 0$ is analogous, as accounted for in Remark 5.7. Appealing to Lemma 5.4(ii), we now choose a vertex v , with v of level $\tilde{n}s$ for some $\tilde{n} \in \mathbb{N}$, such that the cyclic permutation of g that is contained in H_v ends with a_0 on the right. We rename this element g . So we have $g \in H_v$ and by Lemma 5.4(i) we have a cyclic permutation of each of the elements h_0, \dots, h_{s-1} in H_v . By abuse of notation, we rename these cyclic permutations h_0, \dots, h_{s-1} respectively. As before we replace H with H_v . Now H contains the elements g, h_0, \dots, h_{s-1} , where g ends with a_0 on the right.

For each $n \in \mathbb{N}_0$, let $v_n = t \cdot n \cdot t$. It follows from Lemma 5.3 that for $d \in \mathbb{N}$, $\varphi_{v_{ds}}(g^{m^{ds}}) = g$ and $\varphi_{v_{ds}}(h_i^{m^{ds}}) = h_i$ where $i \in [0, s - 1]$. Furthermore, for any element $f \in \mathfrak{K}(\mathfrak{v})$ of the form

$$f = a_{\iota_1}^{\epsilon_1} \cdots a_{\iota_b}^{\epsilon_b},$$

for pairwise distinct $\iota_1, \dots, \iota_b \in [0, s - 1]$ with $b \in [1, s]$ and $\epsilon_i \in \{\pm 1\}$, we can consider its contribution to H_{v_n} . Specifically, if $f \in \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$, we simply consider its image under φ_t . If $f \notin \text{St}_{\mathfrak{K}(\mathfrak{v})}(1)$, then we consider $\varphi_t(f^m)$. We refer to this general process as projecting along the path $t_\infty := tt \cdots$. By projecting along this path, we observe that if $f \in H_{v_j}$, for $j \in \mathbb{N}$, then $f \in H_{v_{j+s}}$; compare the proof of Lemma 5.3. This observation will be used repeatedly throughout the proof without special mention.

The strategy of the proof is now to consider the contributions from

$$\langle g \rangle, \langle h_0 \rangle, \dots, \langle h_{s-1} \rangle$$

to H_{v_n} and to multiply them appropriately to separate the generators a_0, \dots, a_{s-1} . More specifically, if for some $n \in \mathbb{N}$, suppose we have non-trivial distinct elements $\alpha, \beta \in H_{v_n}$ of the form

$$\alpha = a_{i_1}^{\epsilon_1} \cdots a_{i_q}^{\epsilon_q}, \quad \beta = a_{j_1}^{\delta_1} \cdots a_{j_r}^{\delta_r}$$

where $\epsilon_i, \delta_j \in \{\pm 1\}$ and $2 \leq q, r \leq s$, with $i_1, \dots, i_q \in [0, s - 1]$ pairwise distinct, and also $j_1, \dots, j_r \in [0, s - 1]$ pairwise distinct. We consider two situations below, where we assume always that $\tilde{\alpha}, \tilde{\beta}$ are non-trivial.

- (i) If $\alpha = \tilde{\alpha} a_0$ and $\beta = \tilde{\beta} a_0^{-1} \hat{\beta}$, then

$$\beta \alpha = \tilde{\beta} a_0^{-1} \hat{\beta} \tilde{\alpha} a_0$$

yields

$$\varphi_t(\tilde{\beta}) \in H_{v_{n+1}},$$

and hence

$$\tilde{\beta} \in H_{v_{n+s}} \quad \text{and} \quad a_0^{-1} \hat{\beta} \in H_{v_{n+s}}.$$

- (ii) If $\alpha = a_0 \tilde{\alpha}$ and $\beta = \hat{\beta} a_0^{-1} \tilde{\beta}$, from

$$\alpha \beta = a_0 \tilde{\alpha} \hat{\beta} a_0^{-1} \tilde{\beta},$$

we obtain

$$\varphi_t(\tilde{\beta}) \in H_{v_{n+1}},$$

and similarly,

$$\tilde{\beta} \in H_{v_{n+s}} \quad \text{and} \quad \widehat{\beta} a_0^{-1} \in H_{v_{n+s}}.$$

In other words, upon replacing H_{v_n} with $H_{v_{n+s}}$ we have split $\beta \in H_{v_{n+s}}$ into two non-trivial parts. The plan is to repeatedly perform such operations as in (i) and (ii) above to keep splitting the products of generators. Eventually we will end up with $a_0, \dots, a_{s-1} \in H_u$ for some u , which gives $H_u = \mathfrak{K}(v)$ and equivalently that $H = \mathfrak{K}(v)$, as required.

We begin by first considering the contributions from $\langle g \rangle$ and $\langle h_0 \rangle$ along the path t_∞ of the tree. For convenience, write

$$g = a_{i_1} \cdots a_{i_{s-1}} a_0 \quad \text{and} \quad h_0 = a_{j_1} \cdots a_{j_{d-1}} a_0^{-1} a_{j_{d+1}} \cdots a_{j_s},$$

for some $d \in [1, s]$, where $\{i_1, \dots, i_{s-1}\} = \{j_1, \dots, j_{d-1}, j_{d+1}, \dots, j_s\} = [1, s - 1]$.

Case 1: Suppose $1 < d < s$. Then we are in situation (i) from above, and it follows that

$$a_{j_1-1} \cdots a_{j_{d-1}-1} \in H_{v_1} \quad \text{and} \quad a_{s-1}^{-1} a_{j_{d+1}-1} \cdots a_{j_s-1} \in H_{v_1}.$$

We will now use (the projections of) these two parts of h_0 to split g into two non-trivial parts.

Let $j := j_{d-1}$. We consider the contribution of $a_{j_1} \cdots a_{j_{d-1}}$ to H_{v_j} . In other words, we project along the path t_∞ down to level j , which gives

$$a_{j_1-j} \cdots a_{j_{d-2}-j} a_0 \in H_{v_j}.$$

Recalling that $g = a_{i_1} \cdots a_{i_{s-1}} a_0$, we have that $i_r = j$ for some $r \in [1, s - 1]$. Then setting

$$\beta^{-1} := \varphi_{v_j}(g^{m^j}) = a_{i_1-j} \cdots a_{i_{r-1}-j} a_0 a_{i_{r+1}-j} \cdots a_{i_{s-1}-j} a_{s-j} \in H_{v_j}$$

and

$$\alpha := a_{j_1-j} \cdots a_{j_{d-2}-j} a_0 \in H_{v_j},$$

it follows from situation (i) that

$$a_{i_{r+1}-j-1} \cdots a_{i_{s-1}-j-1} a_{s-j-1} \in H_{v_{j+1}} \quad \text{and} \quad a_{i_1-j-1} \cdots a_{i_{r-1}-j-1} a_{s-1} \in H_{v_{j+1}},$$

so we have split g into two non-trivial parts.

We now use the two parts of g to split the parts of h_0 further. For clarity, let us first project to v_s . Here in H_{v_s} we have the elements

$$a_{j_1} \cdots a_{j_{d-2}} a_j, \quad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \quad a_{i_1} \cdots a_{i_{r-1}} a_j, \quad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0.$$

The left two elements are the two parts of h_0 , and the right two are those of g . Without loss of generality, we replace H with H_{v_s} .

Subcase (a): Suppose $1 < r < s - 1$. Let $k := i_1$. Then either $k = j_q$ for $q \in [1, d - 2]$ or $k = j_q$ for $q \in [d + 1, s]$. Suppose the former; a similar argument works for the latter. If $q > 1$, we let β^{-1} be the k th level projection of $a_{j_1} \cdots a_{j_{d-2}} a_j$ (as usual along the path t_∞) and α be that of $a_k a_{i_2} \cdots a_{i_{r-1}} a_j$, which by (ii) gives, upon replacing H with H_{v_s} , the following elements in H :

$$a_{j_1} \cdots a_{j_{q-1}}, \quad a_k a_{j_{q+1}} \cdots a_{j_{d-2}} a_j, \quad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \quad a_k a_{i_2} \cdots a_{i_{r-1}} a_j, \quad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0.$$

If $q = 1$, before splitting, we have instead the following elements in H :

$$a_k a_{j_2} \cdots a_{j_{d-2}} a_j, \quad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \quad a_k a_{i_2} \cdots a_{i_{r-1}} a_j, \quad a_{i_{r+1}} \cdots a_{i_{s-1}} a_0.$$

Hence, we let $\ell := i_{r+1}$ and let $c \in [2, d - 2] \cup [d + 1, s]$ be such that $j_c = \ell$. We consider the ℓ th projection of $a_\ell a_{i_{r+2}} \cdots a_{i_{s-1}} a_0$ multiplied accordingly with that of $a_j^{-1} a_{j_{d-2}}^{-1} \cdots a_{j_2}^{-1} a_k^{-1}$ or $a_{j_s}^{-1} \cdots a_{j_{d+1}}^{-1} a_0$. This is situation (ii).

Subcase (b): Suppose $r = 1$. Then we have the following elements in H :

$$a_{j_1} \cdots a_{j_{d-2}}, \quad a_0^{-1} a_{j_{d+1}} \cdots a_{j_s}, \quad a_j, \quad a_{i_2} \cdots a_{i_{s-1}} a_0.$$

If $i_2 \neq j_1$, let $k := j_1$, and we proceed according to (ii), with α being the k th projection of $a_k a_{j_2} \cdots a_{j_{d-2}}$ and β that of $(a_{i_2} \cdots a_{i_{s-1}} a_0)^{-1}$. If $i_2 = j_1$, we let instead $k := j_s$ and consider the k th projection of $(a_{i_2} \cdots a_{i_{s-1}} a_0)^{-1}$ multiplied with that of $a_0^{-1} a_{j_{d+1}} \cdots a_{j_{s-1}} a_k$; that is, situation (i).

Subcase (c): Suppose $r = s - 1$. Here we have the following elements in H :

$$a_{j_1} \cdots a_{j_{d-2}} a_j, \quad a_{j_{d+1}} \cdots a_{j_s}, \quad a_{i_1} \cdots a_{i_{s-2}} a_j, \quad a_0.$$

If $i_1 \neq j_1$, we let $k := j_1$, and proceed as in (ii), taking α to be the k th projection of $a_k a_{j_2} \cdots a_{j_{d-2}} a_j$ and β that of $(a_{i_1} \cdots a_{i_{s-2}} a_j)^{-1}$. If $i_1 = j_1$, we instead let $k := j_{d+1}$ and likewise following (ii) we consider the k th level projection of $a_k a_{j_{d+2}} \cdots a_{j_s}$ multiplied with that of $(a_{i_1} \cdots a_{i_{s-2}} a_j)^{-1}$.

We aim to continue in this manner, using newly formed parts of g to split the existing parts of h_0 , and then using the newly formed parts of h_0 to split the existing parts of g . Observe also that if a_i , for some $i \in [0, s - 1]$, is an isolated part of g (i.e. a part of g of length one), then using (i) or (ii), one can further split the parts of h_0 to isolate a_i from the parts of h_0 . Indeed, if a_i or a_i^{-1} occurs as an endpoint of a part of h_0 , then it is clear. If a_i is an interior point of a part $a_{r_1} \cdots a_{r_\xi} a_i a_{r_{\xi+1}} \cdots a_{r_{\xi+z}}$ of h_0 , then projecting to the i th level, we have

$$(a_{r_1-i} \cdots a_{r_\xi-i} a_0 a_{r_{\xi+1}-i} \cdots a_{r_{\xi+z}-i}) a_0^{-1} \in H_{v_i},$$

and thus

$$a_{s-1}, \quad a_{r_1-i-1} \cdots a_{r_\xi-i-1}, \quad a_{r_{\xi+1}-i-1} \cdots a_{r_{\xi+z}-i-1}$$

are elements of $H_{v_{i+1}}$, giving

$$a_i, \quad a_{r_1} \cdots a_{r_\xi}, \quad a_{r_{\xi+1}} \cdots a_{r_{\xi+z}}$$

in $H_{v_{i+s}}$. As usual, we then replace H with $H_{v_{i+s}}$. We proceed similarly in the case when a_i^{-1} is an interior point in a part of h_0 .

Hence, we may assume that the set of length one part of g is equal to the set of length one part of h_0 . Equivalently, the set of parts of g of length at least two involve the same generators that appear in the parts of h_0 of length at least two.

If there are no parts of length at least two, then all generators have been isolated, and we are done, so assume otherwise. Suppose for now that the parts of g of length at least two are labelled as follows:

$$a_{e_1} * \cdots * a_{f_1}, \quad a_{e_2} * \cdots * a_{f_2}, \quad \dots, \quad a_{e_\mu} * \cdots * a_{f_\mu},$$

for some $1 \leq \mu < s$, and similarly for h_0 :

$$a_{p_1}^{\gamma_1} * \cdots * a_{q_1}^{\lambda_1}, \quad a_{p_2}^{\gamma_2} * \cdots * a_{q_2}^{\lambda_2}, \quad \dots, \quad a_{p_\nu}^{\gamma_\nu} * \cdots * a_{q_\nu}^{\lambda_\nu},$$

for some $1 \leq \nu < s$, with $\gamma_j = 1$ if $j \in [1, s - 1]$ and $\gamma_j = -1$ if $j = 0$ and similarly for λ_j . Here $*$ stands for unspecified elements in the alphabet S . Write

$$\mathcal{E}_g = \{(a_{e_1}, a_{f_1}), \dots, (a_{e_\mu}, a_{f_\mu})\}$$

for the set of ordered pairs of the so-called endpoint generators. If a_0 has not been isolated, it follows that the corresponding set \mathcal{E}_{h_0} of endpoint generator pairs for h_0 is of the form

$$\mathcal{E}_{h_0} = \{(a_0^{-1}, a_{q_1}), (a_{p_2}, a_{q_2}), \dots, (a_{p_\nu}, a_{q_\nu})\},$$

subject to reordering the parts of h_0 . Indeed, else we may separate the parts further using (i). Without loss of generality, write

$$\mathcal{E}_g = \{(a_{e_1}, a_0), (a_{e_2}, a_{f_2}), \dots, (a_{e_\mu}, a_{f_\mu})\}.$$

Note that if

$$\{p_2, \dots, p_\nu\} \cup \{q_1, \dots, q_\nu\} \neq \{e_1, \dots, e_\mu\} \cup \{f_2, \dots, f_\mu\},$$

we may proceed as in (i) or (ii), since then an endpoint from a part of g is an interior point in a part of h_0 , or vice versa. Hence, $\mu = \nu$ and

$$\{p_2, \dots, p_\mu\} \cup \{q_1, \dots, q_\mu\} = \{e_1, \dots, e_\mu\} \cup \{f_2, \dots, f_\mu\}.$$

Since $\{p_2, \dots, p_\mu\}$ has less elements than $\{e_1, \dots, e_\mu\}$, it follows that $e_i \in \{q_1, \dots, q_\mu\}$ for some $i \in [1, \mu]$. Then, we proceed as in (ii). Hence, if a_0 is not an isolated part of g (equivalently of h_0), then we can continue splitting the parts of g and h_0 .

So suppose now that a_0 has been isolated. As reasoned above, we have

$$\mathcal{E}_g = \{(a_{e_1}, a_{f_1}), \dots, (a_{e_\mu}, a_{f_\mu})\}$$

and

$$\mathcal{E}_{h_0} = \{(a_{p_1}, a_{q_1}), \dots, (a_{p_\mu}, a_{q_\mu})\}$$

with

$$\{e_1, \dots, e_\mu\} \cup \{f_1, \dots, f_\mu\} = \{p_1, \dots, p_\mu\} \cup \{q_1, \dots, q_\mu\}.$$

Similarly, if

$$\{e_1, \dots, e_\mu\} \cap \{q_1, \dots, q_\mu\} \neq \emptyset,$$

we proceed as in (ii). So we assume that

$$\{e_1, \dots, e_\mu\} = \{p_1, \dots, p_\mu\} \quad \text{and} \quad \{f_1, \dots, f_\mu\} = \{q_1, \dots, q_\mu\}.$$

To proceed, we now consider the element h_{e_1} defined at the beginning of the proof. Proceeding as in (i) and (ii), we use the parts of g and h_0 to split h_{e_1} into parts, and if possible, we likewise use the parts of h_{e_1} to further split the parts of g and h_0 . We claim that a_{e_1} has been isolated through this process. Indeed, analogously to the considerations above for when a_0 was assumed to be an endpoint in \mathcal{E}_g , if we have

$$\mathcal{E}_{h_{e_1}} = \{(a_{k_1}, a_{\ell_1}^{-1}), (a_{k_2}, a_{\ell_2}), \dots, (a_{k_\eta}, a_{\ell_\eta})\}$$

and

$$\mathcal{E}_g = \{(a_{e_1}, a_{f_1}), (a_{e_2}, a_{f_2}), \dots, (a_{e_\eta}, a_{f_\eta})\},$$

where here $\eta \geq \mu$, and by abuse of notation, we still write e_i for the left endpoints and f_i for the right endpoints for the parts of g . Then, as seen before, there is some $f_i \in \{k_1, \dots, k_\eta\}$ for $i \in [1, \eta]$, and we can proceed as in (i) or (ii). If instead $(a_{e_1}^{-1}, a_{\ell_1}) \in \mathcal{E}_{h_{e_1}}$ then we multiply the e_1 th projection of $a_{e_1} * \dots * a_{f_1}$ with that of $a_{e_1}^{-1} * \dots * a_{\ell_1}$ as in (ii). Lastly, if $a_{e_1}^{-1}$ is an interior point in $\mathcal{E}_{h_{e_1}}$, then we proceed as in (ii). In other words, if a_{e_1} is not an isolated part of g (equivalently of h_0 and of h_{e_1}), then we can always continue splitting.

By abuse of notation, we redefine \mathcal{E}_g to be the new set of endpoint pairs, after this further splitting of the parts of g . If $\mathcal{E}_g \neq \emptyset$, pick a left endpoint a_e for some $(a_e, a_*) \in \mathcal{E}_g$. From working in a similar manner with the element h_e , we can isolate a_e .

Proceeding in this manner, we will end up with all individual generators.

Case 2: Suppose $d = 1$. Thus we have

$$g = a_{i_1} \cdots a_{i_{s-1}} a_0 \quad \text{and} \quad h_0 = a_0^{-1} a_{j_2} \cdots a_{j_s}.$$

Write $i := i_1$ and let $r \in [2, s]$ be such that $j_r = i$. As in situation (ii), we consider instead the i th projection of g multiplied with that of h_0^{-1} . We now proceed as in Case 1 with the argument using the pairs of endpoints \mathcal{E}_g .

Case 3: Suppose $d = s$. Here we proceed first using (i) and then following the argument laid out in Case 1. □

Remark 5.7. For the case $t = 0$, as mentioned above it is entirely analogous as compared to the case $t > 0$. Whenever we had applied situation (i) in the case of $t > 0$, we now use situation (ii) in the case $t = 0$. After the first splitting, whenever we consider new endpoints for the next splittings, where we had considered left endpoints in the case $t > 0$, we consider right endpoints in the case $t = 0$, and vice versa.

Remark 5.8. Akin to [8, Proposition 6.6], one can show that the group $\mathfrak{R}(\mathfrak{v})$ has a non-normal maximal subgroup of index q , for infinitely many primes q . Indeed, the group $\mathfrak{R}(\mathfrak{v})$ has a proper quotient isomorphic to $W_m(\mathbb{Z})$, where for \mathcal{G} a group and $m \in \mathbb{N}_{\geq 2}$, we write $W_m(\mathcal{G})$ for the wreath product of \mathcal{G} with a cyclic group of order m . Writing $L = \psi^{-1}(\mathfrak{R}(\mathfrak{v})' \times \cdots \times \mathfrak{R}(\mathfrak{v})')$ and $N = L\langle \alpha_0^m \rangle \trianglelefteq \mathfrak{R}(\mathfrak{v})$, analogous to [8, Lemma 6.4] we have that $\mathfrak{R}(\mathfrak{v})/N \cong W_m(\mathbb{Z}^{s-1})$, which has $W_m(\mathbb{Z})$ as a quotient group; compare also Theorem 1.1(iii).

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