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UNIFORMLY LIPSCHITZIAN SEMIGROUPS IN HILBERT SPACE

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ABSTRACT. Let K be a closed, bounded, convex, nonempty subset of a Hilbert Space $\mathcal{H}$. It is shown that if $\mathcal{T}$ is a left reversible, uniformly k-lipschitzian semigroup of mappings of K into itself, with $k<\sqrt{2}$, then $\mathcal{T}$ has a common fixed point in K.

1. **Introduction.** Let $\mathcal{T}=\{T_\alpha\}_{\alpha\in A}$ be a semigroup of mappings of a metric space (M, d) into itself. Such a semigroup is said to have a *common fixed point* if there exists $x_0\in M$ with $T_\alpha(x_0)=x_0$ for all $\alpha\in A$; $\mathcal{T}$ is said to be *uniformly k-lipschitzian* semigroup if, for each $x, y\in M$ and $\alpha\in A$,

$$d(T_\alpha(x), T_\alpha(y))\le k\, d(x, y).$$

Uniformly k-lipschitzian semigroups were introduced (in a slightly more general form) by K. Goebel, W. A. Kirk, and R. L. Thele in [2], and they also assumed that the semigroup $\mathcal{T}$ was *left reversible* (i.e., every two right ideals in $\mathcal{T}$ have non-empty intersection). This latter is automatically fulfilled if, for example $\mathcal{T}$ is commutative, and in particular if $\mathcal{T}=\{T_s\}_{s\in[0,\infty)}$. The basic result of [2] asserts that if E is a uniformly convex Banach space then there is a $k_0>1$ such that, whenever $K\subseteq E$ is a closed, bounded, convex set and $\mathcal{T}$ is a left reversible uniformly k-lipschitzian semigroup of mappings from K into K with $k<k_0$, then $\mathcal{T}$ has a common fixed point in K. Precisely how large k_0 may be taken to be remains, even in Hilbert space, an open question; the estimate provided for Hilbert space in [2] was $\sqrt{5}/2$, with an upper bound of 2. In the special case where $\mathcal{T}$ consists of iterates of a single mapping $T: K\to K$, T is said to be uniformly k-lipschitzian mapping. These mappings were first studied by K. Goebel and W. A. Kirk in [1]. In [4], E. Lifschitz proved, using a technique different from the one we employ below, that in Hilbert space a uniformly k-lipschitzian mapping with $k<\sqrt{2}$ has a fixed point. Our main purpose in this note, accomplished in Section 2, is to show that the estimate of $\sqrt{2}$ is valid under the more general semigroup assumptions.

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2. **Uniformly k-Lipschitzian semigroups.** The main result of this note may be stated as follows:

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space and let K be a nonempty, closed, convex, bounded subset of $\mathcal{H}$. Let $\mathcal{T}=\{T_\alpha\}_{\alpha\in A}$ be a left reversible semigroup of mappings: $T_\alpha: K\to K$ for each $\alpha\in A$. If $\mathcal{T}$ is uniformly k-lipschitzian with $k<\sqrt{2}$, then there exists $x_0\in K$ with $T_\alpha(x_0)=x_0$ for all $\alpha\in A$.*

The basic idea of our proof is the same as the proof of Theorem 2.1 in [2], and we include many of the details only for the sake of completeness; our result requires, however, somewhat more refined bounds on the quantity $d(x)$ defined below. These bounds, in turn, are motivated by a result of N. Routledge [5] (c.f., [3], page 192) which asserts that, in Hilbert space, the diameter of a set is equal to $\sqrt{2}$ times the optimal Chebyshev radius of the set.

Proof of Theorem 1. We may assume $k>1$. For each $\alpha\in A$, let $\mathcal{T}_\alpha=\{T_\alpha\circ T: T\in\mathcal{T}\}$ and for each $x\in K$, let $\mathcal{T}_\alpha(x)=\{T(x): T\in\mathcal{T}_\alpha\}$. In addition set $d(x)=\inf_{\alpha\in A}\{\sup\|x-Tx\|: T\in\mathcal{T}_\alpha\}$. It will suffice to show $d(x_0)=0$ for some $x_0\in K$. For suppose this is the case; since $\mathcal{T}$ is left reversible, the family $\{T_\alpha\}_{\alpha\in A}$ forms a directed set under the relation:

$$(1) \qquad \mathcal{T}_\alpha\geq\mathcal{T}_\beta \quad \text{if and only if} \quad \mathcal{T}_\alpha\subseteq\mathcal{T}_\beta.$$

Now if $x_\alpha\in\mathcal{T}_\alpha(x_0)$ for each $\alpha\in A$, the fact that $d(x_0)=0$ yields that the net $\{x_\alpha\}_{\alpha\in A}$ converges to x_0; and thus, if $T\in\mathcal{T}$, $\{Tx_\alpha\}_{\alpha\in A}$ converges to Tx_0. But for $T\in\mathcal{T}$, $\{T\mathcal{T}_\alpha(x_0)\}_{\alpha\in A}$ is a subset of $\{\mathcal{T}_\alpha(x_0)\}$ and $Tx_\alpha\in T\mathcal{T}_\alpha(x_0)$ for all $\alpha\in A$. This implies that the net $\{Tx_\alpha\}_{\alpha\in A}$ converges to x_0, whence $Tx_0=x_0$ for all $T\in\mathcal{T}$.

Now to see $d(x_0)=0$ for some $x_0\in K$, fix $x\in K$. Let $R(x)=\{r>0: \mathcal{T}_\alpha(x)\subseteq B(y;r)$ for some $\alpha\in A$ and $y\in K\}$ and let $r_0(x)=r_0=\inf R(x)$. Note that if $r<r_0(x)$ and $z\in K$, then for all $\alpha\in A$, there exists $T\in\mathcal{T}_\alpha$ with

$$(2) \qquad \|z-Tx\|>r.$$

Let $\varepsilon>0$ and set

$$D(r_0,\alpha,\varepsilon)=\bigcap_{T\in\mathcal{T}_\alpha} B(Tx; r_0+\varepsilon)\cap K.$$

Clearly for each $\varepsilon>0$, there exists $\alpha\in A$ with $D(r_0,\alpha,\varepsilon)\neq\emptyset$. Also, for fixed ε, the family $\{D(r_0,\alpha,\varepsilon)\}_{\alpha\in A}$ is an increasing net when directed as in (1). Thus if $C_\varepsilon=\bigcup_{\alpha\in A} D(r_0,\alpha,\varepsilon)$, C_ε is nonempty and convex. It then follows that if $C=\bigcap_{\varepsilon>0}(clC_\varepsilon\cap K)$, C is also nonempty. Let $g(x)\in C$. We may assume $r_0>0$; for if $r_0=0$, then for each $\varepsilon>0$ there exists $\alpha\in A$ with $\|g(x)-Tx\|\leq\varepsilon$ for all $T\in\mathcal{T}_\alpha$. Thus for each $T\in\mathcal{T}_\alpha$,

$$\begin{aligned}\|g(x)-Tg(x)\| &\leq \|g(x)-T^2x\|+\|T^2x-Tg(x)\|\\ &\leq \varepsilon+k\,\|Tx-g(x)\|\\ &\leq \varepsilon(1+k)\end{aligned}$$

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and so $d(g(x))=0$. Fix $\varepsilon>0$, $\varepsilon<\min\{r_0/2, d(g(x))/2\}$, and $\lambda\in[0,1]$. Choose $\alpha\in A$ so that

$$\|g(x)-T_\alpha(g(x))\|\geq d(g(x))+\varepsilon, \tag{3}$$

and choose $\beta\in A$ with

$$\|g(x)-Tx\|\leq r_0+\varepsilon \quad \text{for all} \quad T\in\mathcal{T}_\beta.$$

Now, since $\mathcal{T}$ is a semigroup, $T_\alpha\circ T_\beta=T_\gamma$ for some $\gamma\in A$. Let $\mu\in A$ so that $T_\mu\in\mathcal{T}_\gamma\cap\mathcal{T}_\beta$; then $\mathcal{T}_\mu\subseteq\mathcal{T}_\gamma\cap\mathcal{T}_\beta$. If $T\in\mathcal{T}_\mu$, there exists $\tilde{T}\in\mathcal{T}_\beta$ with $T=T_\alpha\circ\tilde{T}$. This yields for $T\in\mathcal{T}_\mu$

$$\begin{aligned}\|T_\alpha(g(x))-Tx\|&=\|T_\alpha(g(x))-T_\alpha\circ\tilde{T}x\|\\&\leq k\,\|g(x)-\tilde{T}x\|\leq k(r_0+\varepsilon),\end{aligned} \tag{4}$$

and since $\mathcal{T}_\mu\subseteq\mathcal{T}_\beta$,

$$\|g(x)-Tx\|\leq r_0+\varepsilon \tag{5}$$

for all $T\in\mathcal{T}_\mu$. Finally, by (2), we may choose $T_0\in\mathcal{T}_\mu$ with

$$\|(1-\lambda)T_\alpha(g(x))+\lambda g(x)-T_0(x)\|\geq r_0-\varepsilon. \tag{6}$$

Set $u=g(x)-T_0x$, $v=T_\alpha(g(x))-T_0x$, so $u-v=g(x)-T_\alpha(g(x))$. By (6), $\|\lambda u+(1-\lambda)v\|\geq r_0-\varepsilon$ and so by (4) and (5) we have

$$(r_0-\varepsilon)^2\leq\|\lambda u+(1-\lambda)v\|^2\leq\lambda^2(r_0+\varepsilon)^2+2\lambda(1-\lambda)\langle u,v\rangle+k^2(1-\lambda)^2(r_0+\varepsilon)^2$$

thus

$$(r_0-\varepsilon)^2-\lambda^2(r_0+\varepsilon)^2-k^2(1-\lambda)^2(r_0+\varepsilon)^2\leq 2\lambda(1-\lambda)\langle u,v\rangle$$

or

$$-2\langle u,v\rangle\leq\frac{-(r_0-\varepsilon)^2+\lambda^2(r_0+\varepsilon)^2+k^2(1-\lambda)^2(r_0+\varepsilon)^2}{\lambda(1-\lambda)}.$$

Using this, we obtain

$$\begin{aligned}\|u-v\|^2&\leq(r_0+\varepsilon)^2-2\langle u,v\rangle+k^2(r_0+\varepsilon)^2\\&\leq(r_0+\varepsilon)^2+\frac{-(r_0+\varepsilon)^2+\lambda^2(r_0+\varepsilon)^2+k^2(1-\lambda)^2(r_0+\varepsilon)^2}{\lambda(1-\lambda)}+k^2(r_0+\varepsilon)^2\\&=\frac{\lambda(1-\lambda)(r_0+\varepsilon)^2-(r_0-\varepsilon)^2+\lambda^2(r_0+\varepsilon)^2+k^2(1-\lambda)^2(r_0+\varepsilon)^2+\lambda(1-\lambda)k^2(r_0+\varepsilon)^2}{\lambda(1-\lambda)}\end{aligned} \tag{7}$$

By (3),

$$(d(g(x))+\varepsilon)^2\leq\|u-v\|^2$$

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Combining this inequality with (7) above and taking the limit as $\varepsilon \to 0$,

$$d(g(x))^2 \le \frac{\lambda(1-\lambda)r_0^2 - r_0^2 + \lambda^2 r_0^2 + k^2(1-\lambda)^2 r_0^2 + \lambda(1-\lambda)k^2 r_0^2}{\lambda(1-\lambda)}$$
$$= \frac{\lambda(1-\lambda)r_0^2 - (1-\lambda^2)r_0^2 + k^2(1-\lambda)^2 r_0^2 + \lambda(1-\lambda)k^2 r_0^2}{\lambda(1-\lambda)}$$
$$= \frac{\lambda r_0^2 - (1+\lambda)r_0^2 + k^2(1-\lambda)r_0^2 + \lambda k^2 r_0^2}{\lambda}.$$

Letting $\lambda \to 1$,

$$d(g(x))^2 \le (k^2-1)r_0^2 \quad \text{or} \quad d(g(x)) \le \sqrt{(k^2-1)}\, r_0$$

It is clear that $r_0(x) \le d(x)$ and that

$$\|g(x)-x\| \le r_0(x) + d(x) \le 2d(x).$$

Thus, for some $\xi < 1$ ($\xi = \sqrt{(k^2-1)}$) and for each $x \in K$, we have shown that there exists $g(x) \in K$ with

$$d(g(x)) \le \xi d(x), \qquad \|g(x)-x\| \le 2d(x).$$

Define a sequence $\{x_n\}$ in K by fixing $x_0 \in K$ and letting $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$. If $r_0(x_n)$ or $d(x_n) = 0$ for any n, we are done. Otherwise note

$$\|x_{n+1} - x_n\| \le 2d(x_n) \le 2\xi^n\, d(x_0),$$

so that $\{x_n\}$ is a Cauchy sequence. Therefore $x_n \to z \in K$ as $n \to \infty$. Let $\{\varepsilon_m\}$ be a sequence of positive numbers with $\varepsilon_m \to 0$, and for each n, choose $\alpha \in A$ so that

$$\|x_n - Tx_n\| \le d(x_n) + \varepsilon_n \quad \text{for all} \quad T \in \mathcal{T}_{\alpha_n}.$$

Then for $T \in \mathcal{T}_{\alpha_n}$,

$$\|z - Tz\| \le \|z - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tz\|$$
$$\le (1-k)\,\|z - x_n\| + d(x_n) + \varepsilon_n.$$

This quantity can be made arbitrarily small, hence $d(z) = 0$.

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