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A COMPARISON OF EIGENVALUES OF TWO STURM-LIOUVILLE PROBLEMS

BY
YISONG YANG

ABSTRACT. We compare, under some assumptions on mass density, the eigenvalues of the Sturm-Liouville problems satisfying homogeneous Dirichlet and Neumann boundary condition.

1. **Introduction.** We consider in this note the following two eigenvalue problems satisfying the Dirichlet and the Neumann boundary condition respectively

$$\phi''(x) + \lambda p(x)\phi(x) = 0 \text{ in } (-1,1), \quad \phi(-1) = \phi(1) = 0 \tag{1}$$

and

$$\psi''(x) + \mu p(x)\psi(x) = 0 \text{ in } (-1,1), \quad \psi'(-1) = \psi'(1) = 0 \tag{2}$$

where $p(x) > 0$ is continuous over $[-1,1]$. We have two countable sets of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ and $0 = \mu_1 < \mu_2 < \cdots$ with $\lambda_n, \mu_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$$\mu_n < \lambda_n, \quad n = 1, 2, \cdots. \tag{3}$$

The recent work of Bandle and Philippin [1] sharpens the inequality (3) which states that for the mass density $p(x)$ satisfying $p(-x) = p(x)$ and $p(x)$ increasing in $(-1,0)$, we have

$$\mu_n \le \lambda_n - 2\lambda_1, \quad n = 2, 3, \cdots.$$

The aim of the present note is to continue their work and study another aspect of the problem: we establish the comparison inequality $\lambda_n \le \mu_{n+2} - 2\mu_2$, for $n = 1, 2, \cdots$. It is interesting to compare our condition on $p(x)$ below with that in [1] stated above.

2. **Main Result.** In the following three preliminary lemmas we assume $p(x) \in C^1[-1,1]$.

LEMMA 1. *Let $n \ge 2$. If (ψ_n, μ_n) is the n-th eigenpair of the problem (2), then (v_{n-1}, μ_n) is the $(n-1)$-st eigenpair of*

$$\left(\frac{v'}{p}\right)' + \mu v = 0 \textit{ in } (-1,1), \quad v(-1) = v(1) = 0 \tag{4}$$

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where

$$v_{n-1}(x) = \int_{-1}^{x} p(s)\psi_n(s)\,ds. \tag{5}$$

PROOF. Substituting (ψ_n, μ_n) into (2) and integrating, we get

$$\psi_n{}'(x) + \mu_n \int_{-1}^{x} p(s)\psi_n(s)\,ds = 0, \tag{6}$$

that is,

$$\left(\frac{v'_{n-1}}{p}\right)' + \mu_n v_{n-1} = 0.$$

The boundary condition of v_{n-1} follows obviously from Eq. (6).

It is an elementary fact that $\{v_n\}$ forms an orthogonal basis of $L^2(-1,1)$.

LEMMA 2. *Let* (v_1, μ_2) *and* (v_n, μ_{n+1}) *be the first and the n-th eigenpairs of the problem* (4). *Then* $w_n = v_n/v_1$, $\sigma_n = \mu_{n+1} - \mu_2$ *is the n-th eigenpair of the singular boundary value problem*

$$\left(\frac{v_1^2 w'}{p}\right)' + \sigma v_1^2 w = 0 \text{ in } (-1,1), \qquad \lim_{x\to -1^+} w'(x) = \lim_{x \to 1^-} w'(x) = 0. \tag{7}$$

PROOF. It is easily checked that w_n satisfies the equation in (7) over $(-1,1)$ with $\sigma = \sigma_n$.

For $x \longrightarrow 1^-$, using L'Hôpital's rule, (4), and (5), we have

$$\lim_{x\to 1^-} w'_n(x) = \lim_{x\to 1^-} \frac{v'_n v_1 - v_n v'_1}{v_1^2} = \frac{1}{2v'_1(1)^2} \lim_{x\to 1^-} (v'_n v_1 - v_n v'_1)'' = 0.$$

Similarly, $w'_n(x) \longrightarrow 0$ for $x \longrightarrow -1^+$.

Conversely, from the equality

$$\int_{-1}^{1} f w_n v_1^2\,dx = \int_{-1}^{1} (f v_1) v_n\,dx, \quad f \in L^2\big((-1,1), v_1^2\,dx\big), \quad n = 1, 2, \cdots,$$

we can verify that $\{w_n\}$ forms an orthogonal basis of $L^2\big((-1,1), v_1^2\,dx\big)$.

The observations given above lead us to the conclusion that $\{(w_n, \sigma_n)\}$ is a complete set of eigenpairs of the singular boundary value problem (7).

LEMMA 3. $u_n = v_1 w'_{n+1}/p$, $\gamma_n = \mu_{n+2} - 2\mu_2$ $(n \geq 1)$ *is a solution of the singular eigenvalue problem*

$$u'' - \left[2\left(\frac{v'_1}{v_1}\right)^2 - \frac{p' v'_1}{p v_1}\right] u + \gamma p u = 0 \text{ in } (-1,1), \qquad \lim_{x\to -1^+} u(x) = \lim_{x\to 1^-} u(x) = 0. \tag{8}$$

The verification of this lemma is straightforward.

Now we can state our main result of this note:

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THEOREM 1. *If $p(x)$ satisfies (i) $p(-x) = p(x)$ and (ii) $p(x)$ is increasing in* $(0, 1)$, *then*

$$\lambda_n \leq \mu_{n+2} - 2\mu_2, \quad n = 1, 2, \cdots. \tag{9}$$

PROOF. First we assume $p(x) \in C^1[-1, 1]$. Since v_1 satisfies the problem

$$(\frac{v'}{p})' + \mu_2 v = 0 \text{ in } (-1, 1), \quad v(-1) = v(1) = 0 \tag{10}$$

where $p(-x) = p(x)$ and the solution space of (10) is one-dimensional, we can conclude that $v_1(-x) = v_1(x)$. In particular $v_1'(0) = 0$. Consequently, from (10) follows:

$$\frac{v_1'(x)}{p(x)} = -\mu_2 \int_0^x v_1(s)\, ds. \tag{11}$$

As the first eigenfunction of the problem (4), v_1 is of constant sign in the interval $(-1, 1)$; so (11) gives us $v_1'/v_1 < 0$ for $x > 0$. Under the hypothesis, that $p(x)$ is increasing in $(0, 1)$, we have $p'v_1'/v_1 \leq 0$ for $x > 0$. By symmetry, we obtain $p'v_1'/v_1 \leq 0$ for $x < 0$. In particular,

$$2(\frac{v_1'}{v_1})^2 - \frac{p'v_1'}{pv_1} \geq 0 \text{ in } (-1, 1). \tag{12}$$

Because (8) is a singular boundary value problem, we cannot apply the classical monotonicity theorem (cf., e.g., [2, p. 174]) directly to the problems (1) and (8) and using (12) to conclude that

$$\gamma_n \geq \lambda_n,\ n = 1, 2, \cdots, \tag{13}$$

and hence (9). But, still, the inequality (13) can be established by imitating the argument in the proof of the classical monotonicity theorem ([2, p. 174]).

In fact, it follows from the well-known oscillation theorem ([2, p. 174]) that, as the n-th eigenfunction of (4), v_n has exactly $n - 1$ zeros in $(-1, 1)$. Hence so does w_n. Consequently, $w_n'(x)$ has at least $\max(n - 2, 0)$ zeros in $(-1, 1)$. This proves that u_n has at least $n + 1$ zeros on $[-1, 1]$.

Suppose, otherwise, $\gamma_n < \lambda_n$ for some $n \geq 1$. Let ϕ_n be the n-th eigenfunction of the problem (1) and $\alpha < \beta$ two consecutive zeros of u_n. We claim that there exists at least one zero of ϕ_n in (α, β). Otherwise we can find two consecutive zeros $\alpha_1 < \beta_1$ of ϕ_n such that $(\alpha, \beta) \subset (\alpha_1, \beta_1)$. Since ϕ_n is the first eigenfunction of (1) over (α_1, β_1), we have, by virtue of (8) and the standard minimax principle for regular eigenvalue problems, the inequality

$$\begin{aligned}
\gamma_n &= \int_\alpha^\beta \left((u_n')^2 + [2(\frac{v_1'}{v_1})^2 - \frac{p'v_1'}{pv_1}]u_n^2\right) dx \Big/ \int_\alpha^\beta p u_n^2\, dx \\
&\geq \inf_{u \in W_0^{1,2}(\alpha_1, \beta_1)} \left\{ \int_{\alpha_1}^{\beta_1} \left((u')^2 + [2(\frac{v_1'}{v_1})^2 - \frac{p'v_1'}{pv_1}]u^2\right) dx \Big/ \int_{\alpha_1}^{\beta_1} p u^2\, dx \right\} \\
&\geq \inf_{u \in W_0^{1,2}(\alpha_1, \beta_1)} \left\{ \int_{\alpha_1}^{\beta_1} (u')^2\, dx \Big/ \int_{\alpha_1}^{\beta_1} p u^2\, dx \right\} \\
&= \lambda_n.
\end{aligned}$$

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This achieves a contradiction.

Now, since u_n has at least $n+1$ zeros on $[-1,1]$, ϕ_n has at least n zeros in $(-1,1)$. This contradicts the assertion of the oscillation theorem ([2, p. 174]) that ϕ_n has exactly $n-1$ zeros in $(-1,1)$.

Therefore the inequality (13) is proved for $p(x) \in C^1[-1,1]$.

If $p(x) \in C^0[-1,1]$, we can approximate p in $C^0[-1,1]$ by a suitable sequence of functions $\{p_j\}_{j=1}^{\infty}$ taken from $C^1[-1,1]$. The continuous dependence of λ_n and μ_n on p again yields the inequality (13) (cf. [1]).

The proof of Theorem 1 is complete.

3. **A More General Theorem.** We can also apply Theorem 1 to some other problems.

First observe that the theorem holds on any interval $[a,b]$ provided we assume that $p(x)$ is even about the point $x=(a+b)/2$ and increasing over the interval $\big((a+b)/2, b\big)$.

Consider the problems

$$(14) \qquad \big(p(x)\phi'(x)\big)' + \lambda q(x)\phi(x) = 0 \text{ in } (-1,1), \quad \phi(-1)=\phi(1)=0$$

and

$$(15) \qquad \big(p(x)\psi'(x)\big)' + \mu q(x)\psi(x) = 0 \text{ in } (-1,1), \quad \psi'(-1)=\psi'(1)=0$$

THEOREM 2. *If $p(-x)=p(x)$, $q(-x)=q(x)$ and $p(x)q(x)$ is increasing in $(0,1)$, then the inequality (9) still holds. Here we keep the assumption $p,q>0$.*

PROOF. Under the change of variables:

$$t = \int_{-1}^{x} \frac{ds}{p(s)}, \quad L = \int_{-1}^{1} \frac{ds}{p(s)},$$

the problems (14) and (15) become

$$(16) \qquad \frac{d^2\phi}{dt^2} + \lambda p\big(x(t)\big)q\big(x(t)\big)\phi = 0 \text{ in } (0,L), \quad \phi(0)=\phi(L)=0$$

and

$$(17) \qquad \frac{d^2\psi}{dt^2} + \mu p\big(x(t)\big)q\big(x(t)\big)\psi = 0 \text{ in } (0,L), \quad \psi'(0)=\psi'(L)=0.$$

Now since p is even with respect to $x=0$, so $x=0$ corresponds to $t=L/2$. Because $(pq)\big(x(t)\big)$ is even with respect to $t=L/2$ and increasing in $(L/2, L)$, applying Theorem 1 to (16) and (17) we see immediately that λ_n, μ_n satisfy (9).

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COROLLARY 1. *Under the assumption of Theorem 2, we have* $\mu_3 > 2\mu_2$.

COROLLARY 2. *Under the assumption of Theorem 2, we have the following lower bound estimate for the gap of the first two nonzero eigenvalues of the Neumann problem (15):*

$$\mu_3 - \mu_2 \geq \lambda_1 + \mu_2.$$

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Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003

Current address:

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
U. S. A.