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HERMITE CONJUGATE FUNCTIONS AND REARRANGEMENT INVARIANT SPACES

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The Hermite conjugate Poisson integral $\tilde{f}(x, y)$ of a given $f \in L^1(\mu)$, $d\mu(y) = \exp(-y^2)\, dy$, was defined by Muckenhoupt [5, p. 247] as

$$\tilde{f}(x, y) = \int_{-\infty}^{\infty} Q(x, y, z) f(z)\, d\mu(z) \qquad x > 0,\ y \in \Omega = (-\infty, \infty)$$

where

$$Q(x, y, z) = \int_0^1 \frac{2^{1/2}(z-ry)\exp(x^2/2 \log r)}{\pi(-\log r)^{1/2}(1-r^2)^{3/2}} \exp\left(\frac{-r^2y^2+2ryz-r^2z^2}{1-r^2}\right) dr.$$

If the *Hermite conjugate function operator* T is defined by $(Tf)(y) = \lim_{x\to 0+} \tilde{f}(x, y)$ a.e., then one of the main results of [5] is that T is of weak-type (1, 1) and strong-type (p, p) for all $p > 1$. This result together with a theorem of Boyd [3, Theorem 1] shows that if $L^\rho(\Omega)$ is a rearrangement invariant space with upper and lower indices α and β respectively (see [3] for definitions and notation) which satisfy $0 < \beta \le \alpha < 1$, then T maps $L^\rho(\Omega)$ continuously into itself. The purpose of this note is to give an elementary proof of the converse which then results in the following generalization of Muckenhoupt's result:

THEOREM. *Let $L^\rho(\Omega)$ be a rearrangement invariant space with upper index α and lower index β. Then $0 < \beta \le \alpha < 1$ is a necessary and sufficient condition for T to be bounded as a linear operator from $L^\rho(\Omega)$ into itself.*

In general, the indices α and β will depend not only on the particular function norm ρ defining L^ρ but also on the nature of the underlying measure space. However, it is known that the conditions $0 < \beta \le \alpha < 1$ are equivalent to uniform convexity in the case of the Lorentz spaces $\Lambda(\varphi, p)$, $p > 1$, and to reflexivity in the case of the Orlicz spaces. For the infinite non-atomic measure spaces this was proved by Boyd [2], and using similar methods Kerman [4] and the author [1] obtained the same results for the finite non-atomic and the purely atomic cases respectively. Thus, Kerman's result applies in the present situation, and in analogy with known results for the classical Hilbert transform [2], the classical conjugate function

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operator [4], [6] and the discrete Hilbert transforms [1] we have:

COROLLARY 1. *T is bounded from $\Lambda(\varphi, p)$, $p>1$, into itself if and only if $\Lambda(\varphi, p)$ is uniformly convex.*

COROLLARY 2. *T is bounded from an Orlicz space into itself if and only if the space is reflexive.*

For the proof of the Theorem we require the following lemma.

LEMMA. *Denote by Ω^* the interval $[0, a]$ equipped with Lebesgue measure m, where $a=\mu(\Omega)$, and if $f\in L^p(\Omega^*)$ let τf denote the function in $L^p(\Omega)$ given by*

$$(\tau f)(z) = \begin{cases} f(z) & \text{if } z \in [0, a] \\ 0 & \textit{otherwise} \end{cases}.$$

Then f and τf have equivalent norms, that is,

$$(\exp(-a^2))\rho_{\Omega^*}(f) \leq \rho_\Omega(\tau f) \leq \rho_{\Omega^*}(f).$$

Proof. Denote by f^* and $(\tau f)'$ respectively, the nonincreasing equimeasurable rearrangements of f and τf onto Ω^*. Then we clearly have

$$m\{z\in[0,a]:f^*(z)>y\} = m\{z\in[0,a]:|f(z)|>y\} \leq \frac{1}{s}\mu\{z\in(-\infty,\infty):|\tau f(z)|>y\}$$
$$= \frac{1}{s}m\{z\in[0,a]:(\tau f)'(z)>y\}$$

so that

$$f^*(z) \leq (\tau f)'(sz) \tag{1}$$

where we have put $s=\exp(-a^2)$. On the other hand we also have

$$\begin{aligned} m\{z\in[0,a]:(\tau f)'(z)>y\} &= \mu\{z\in(-\infty,\infty):|\tau f(z)|>y\} \\ &\leq m\{z\in[0,a]:|f(z)|>y\} \\ &= m\{z\in[0,a]:f^*(z)>y\} \end{aligned}$$

so that

$$(\tau f)'(z) \leq f^*(z). \tag{2}$$

Now it follows immediately from Lemma 3(a) and (47) of [3] that $\rho_{\Omega^*}((\tau f)'(s\cdot))\leq (1/s)\rho_{\Omega^*}((\tau f)')$ and hence from (1) and (2) it follows that

$$\rho_{\Omega^*}(f) = \rho_{\Omega^*}(f^*) \leq (1/s)\rho_{\Omega^*}((\tau f)') = (1/s)\rho_\Omega(\tau f)$$

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and

$$\rho_\Omega(\tau f) = \rho_{\Omega^*}((\tau f)') \le \rho_{\Omega^*}(f^*) = \rho_{\Omega^*}(f)$$

which proves the lemma.

Proof of the Theorem. (necessity). According to [3, p. 1253, (50) and (51)] it is sufficient to show that if T is bounded, then there is a constant A independent of $f \in L^p(\Omega^*)$ such that

$$\rho_{\Omega^*}(|(P+P')f|) \le A\rho_{\Omega^*}(|f|) \tag{3}$$

where for $y \in \Omega^*$,

$$(Pf)(y) = \int_0^1 f(yz)\, dz \quad \text{and} \quad (P'f)(y) = \int_1^{a/y} f(yz)\frac{dz}{z}.$$

Now if $0<y\le a$, $0\le yz\le a$ and $0<c<1$ then

$$\begin{aligned} -yQ(0, y, -yz) &\ge \frac{2^{1/2}}{\pi} y^2 \int_c^1 \frac{(z+r)}{(-\log r)^{1/2}(1-r^2)^{3/2}} \exp\left(-y^2\frac{r^2+2rz+r^2z^2}{1-r^2}\right) dr \\ &= \frac{1}{2^{1/2}\pi}\int_c^1 \frac{(1+r)^{1/2}}{(1+rz)}\left(\frac{1-r}{-\log r}\right)^{1/2}\left(-\frac{d}{dr}\exp\left(-y^2\frac{r^2+2rz+r^2z^2}{1-r^2}\right)\right) dr \end{aligned}$$

and since $((1-r)/-\log r)^{1/2}$ is bounded below for $c\le r\le 1$ we have

$$-yQ(0, y, -yz) \ge \frac{2K}{(1+z)} \tag{4}$$

where K is a positive constant, depending only on c. Now if $f \in L^p(\Omega^*)$ with $f\ge 0$, let $g(z)=-(\tau f)(-z)$. Then g and τf are equimeasurable which together with the lemma shows that

$$\rho_\Omega(|g|) = \rho_\Omega(\tau f) \le \rho_{\Omega^*}(f) \tag{5}$$

and according to Theorem 2 of [5], for almost all y, $0<y\le a$, we have

$$\begin{aligned} (Tg)(y) &= \lim_{\varepsilon\to 0}\int_{|y-z|>\varepsilon} Q(0, y, z)g(z)\, d\mu(z) \\ &= -\int_0^a Q(0, y, -z)f(z)\, d\mu(z) \end{aligned}$$

(note that the principal value is not required in this last integral since $|z-y|\ge y>0$ for all z in the support of g). Now making the change of variable $z\to yz$ and using (4) we get

$$\begin{aligned} (Tg)(y) &= \left(\int_0^1 + \int_1^{a/y}\right) -yQ(0, y, -yz)f(yz)\exp(-(yz)^2)\, dz \\ &\ge K\exp(-a^2)[(P+P')f](y) \end{aligned}$$

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for $0<y\leq a$, so that for almost all $y \in \Omega$ we have

$$[\tau(P+P')f](y) \leq \frac{\exp a^2}{K} |(Tg)(y)|$$

and hence, by the Lemma and (5)

$$\rho_{\Omega^*}((P+P')f) \leq (\exp a^2)\rho_\Omega(\tau(P+P')f) \leq \frac{\exp 2a^2}{K} \rho_\Omega(|Tg|)$$

$$\leq \frac{\exp 2a^2}{K} \|T\| \rho_{\Omega^*}(f)$$

from which (3) follows, noting that $|(P+P')f(y)| \leq [(P+P')|f|](y)$, $y \in \Omega^*$.

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