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YOSIDA FUNCTIONS AND PICARD VALUES OF INTEGRAL FUNCTIONS AND THEIR DERIVATIVES

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In this paper we improve and generalise a result of J. Clunie by proving that if $f(z)$ is a transcendental integral function with only zeros of order at least $k+1$, then $f^{(k)}(z)$ assumes every finite non-zero complex value infinitely often. Also, the related criterion for normality of a family of holomorphic functions is given, and the value distribution of $f^2 + af^{(k)}$ is discussed.

Hayman's conjecture [6, 7] that if f is a transcendental meromorphic function, then ff' assumes every finite non-zero complex value infinitely often, has been proved. First, Bergweiler and Eremenko proved the conjecture for functions of finite order; and then, Bergweiler and Eremenko [1], the author of this paper and Fang [2], and Zalcman [18] proved the conjecture for the case of infinite order simultaneously and independently. The principal aim of the present paper is to improve and generalise this result further in the case of integral functions. We prove the following theorem.

THEOREM 1. *If $f(z)$ is a transcendental integral function with only zeros of order at least $k+1$, then $f^{(k)}(z)$ assumes every finite non-zero complex value infinitely often.*

Obviously, Hayman's conjecture in the case of integral functions, which was proved by Clunie [4] many years ago, is a direct consequence of the above theorem for $k=1$. In fact, if $f(z)$ is a transcendental integral function, then $\{f(z)\}^2$ has only zeros of order $\geqslant 2$. Thus, applying Theorem 1 to $\{f(z)\}^2$ and $k=1$, we see that $2f(z)f'(z)$ assumes every finite non-zero complex value infinitely often.

We also give a criterion for normality corresponding to Theorem 1, which is stated as follows.

THEOREM 2. *Let $\mathcal{F}$ be a family of holomorphic functions. If, for every function $f \in \mathcal{F}$, f has only zeros of order at least $k+1$ and $f^{(k)}$ does not assume the value 1, then $\mathcal{F}$ is normal.*

Yang and Zhang [15] have proved a criterion for normality of a family of holomorphic functions in which, for every function f in the family, f and $f^{(k)} - 1$ have

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only zeros of order at least l and p respectively, where the positive integers k, l and p satisfies $(k+1)/l + 1/p < 1$. Let $p = \infty$, this means that $f^{(k)} - 1$ has no zero, then k and l should satisfy $k+1 < l$. So, in this special case, Theorem 2 weakens the assumption of their criterion. The example $\{nz^k : n = 1, 2, \cdots\}$ shows that the lower bound $k+1$ of the order of zeros in the assumption of Theorem 2 is best possible.

Besides the above theorems, we discuss the value distribution of $f^2 + af^{(k)}$. We us the standard notation of value distribution theory [**8, 14**].

1. Yosida Functions

The method of proving Theorem 1 is to prove it for a Yosida function first and, then, to reduce the case of an arbitrary integral function to that of a Yosida function. A meromorphic function $f(z)$ defined on the complex plane $\mathbf{C}$ is called a Yosida function, if its spherical derivative $f^{\#}(z) = |f'(z)| / \left(1 + |f(z)|^2\right)$ is uniformly bounded on $\mathbf{C}$. This notion was introduced by Yosida [**17**] who established some basic properties of this class of functions. For example, the order of a Yoshida function is at most two and this bound is sharp. Also, f is a Yosida function if and only if the family $\{f(z' + z) : z' \in \mathbf{C}\}$ is normal on $\mathbf{C}$. Clunie and Hayman [**5**] proved that the order a holomorphic Yosida function is at most 1. Minda [**10**] gave a precise estimate for the growth of a holomorphic Yosida function, and the proof is based on a distortion theorem for holomorphic normal functions on the unit disk, which is due to Pommerenke [**13**]. This fact as well as the following theorem, which reduces an arbitrary meromorphic function to a Yosida function inheriting some properties of the former, plays a key role in the proof of Theorem 1.

Theorem 3. *Let $f(z)$ be a meromorphic function on $\mathbf{C}$ with the property that its zeros all have order at least k (a positive integer). If $f(z)$ is not a Yosida function, then for $0 \leqslant \alpha < k$, there exist a sequence of complex numbers $z_n \to \infty$ and a sequence of positive numbers $\rho_n \to 0$, such that $\rho_n^{-\alpha} f(z_n + \rho_n \zeta)$ converges to a non-constant Yosida function $g(\zeta)$ spherically and locally uniformly on $\mathbf{C}$.*

By Marty's criterion, $f(z)$ is a Yosida function if and only if the family $\{f(a+z) : a \in \mathbf{C}\}$ is normal on $\mathbf{C}$. Thus, Theorem 3 is a direct consequence of the following result.

Theorem 4. *Let $\mathcal{F}$ be a family of meromorphic functions with the property that every function $f \in \mathcal{F}$ has only zeros of order at least k (a positive integer). If $\mathcal{F}$ is not normal at a point z_0, then for $0 \leqslant \alpha < k$, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z'_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that $\rho_n^{-\alpha} f(z'_n + \rho_n \zeta)$ converges to a non-constant Yosida function $g(\zeta)$ spherically and locally uniformly on $\mathbf{C}$.*

The above theorem is a generalisation of Zalcman's lemma [**19**] and is almost the same as [3, Theorem 2]. It differs in that the limit function $g(\zeta)$ is not only a non-constant meromorphic function, but also a Yosida function. The proof in [3] did show that $g(\zeta)$ is a Yosida function. The cases "$k = 1$" and "$k = \infty$" in [3, Theorem 2] were first proved by Pang [11] and Pang and Xue [12] respectively, in a different way.

One can obtain Theorem 3 from Theorem 4 directly. However, It remains to show that $z_n \to \infty$ as $n \to \infty$. Suppose that $z_{n_j} \to z_0$. Since $f(z)$ has only zeros of order at least k, by Hurwitz' theorem, $g(\zeta)$ has no zero of order less than k. Thus $g(\zeta)$ is not a polynomial of degree less than k, and $g^{(k)}(\zeta) \not\equiv 0$. Let $g(\zeta_0) \neq \infty$, $g^{(k)}(\zeta_0) \neq 0$. Since

$$g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_0) = \lim_{n\to\infty} \rho_n^{k-\alpha} f^{(k)}(z_n + \rho_n\zeta_0),$$

we have

$$f^{(k)}(z_0) = \lim_{j\to\infty} f^{(k)}\left(z_{n_j} + \rho_{n_j}\zeta_0\right) = \infty.$$

On the other hand, since

$$g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_0) = \lim_{n\to\infty} \rho_n^{-\alpha} f(z_n + \rho_n\zeta_0),$$

we have

$$f(z_0) = \lim_{j\to\infty} f\left(z_{n_j} + \rho_{n_j}\zeta_0\right) = 0.$$

This contradiction shows that $z_n \to \infty$ as $n \to \infty$.

2. Proof of Theorem 1

Now, we are going to prove Theorem 1. First, we assume that $f(z)$ is a Yosida function. Then $f(z)$ and $f^{(k)}(z)$ have order at most 1. Suppose that $f^{(k)}(z) - a$ has only finitely many zeros for a finite non-zero value a. Then $f^{(k)}(z) - a = p(z)e^{bz}$, where $b \neq 0$ is a constant and $p(z) \not\equiv 0$ is a polynomial. Thus,

$$f(z) = az^k + q(z) + p_1(z)e^{bz},$$
$$f'(z) = akz^{k-1} + q'(z) + p_2(z)e^{bz},$$

where, $q(z)$, $p_1(z)$ and $p_2(z)$ are polynomials, $q(z)$ has degree at most $k-1$ and $p_2(z) \not\equiv 0$ has the same degree as $p_1(z) \not\equiv 0$. Since $p_1(z)e^{bz}$ has the Picard exceptional values 0 and ∞, $f(z) = az^k + q(z) + p_1(z)e^{bz}$ must have infinitely many zeros $z_n \to \infty$, for $az^k + q(z)$ is a small function with respect to $p_1(z)e^{bz}$.

According to the assumption that $f(z)$ has only zeros of order at least $k+1$, we have

$$az_n^k + q(z_n) + p_1(z_n)e^{bz_n} = 0,$$
$$akz_n^{k-1} + q'(z_n) + p_2(z_n)e^{bz_n} = 0$$

for $n = 1, 2, \cdots$. Hence,

$$\frac{p_1(z_n)}{p_2(z_n)} = z_n \cdot \frac{a + z_n^{-k} q(z_n)}{ak + z_n^{-(k-1)} q'(z_n)}.$$

This is impossible, for the left side tends to $1/b$, while the right side tends to ∞ as $n \to \infty$. We have proved the theorem for a transcendental and holomorphic Yosida function $f(z)$.

If $f(z)$ is not a Yosida function, then, according to Theorem 3, we have a sequence $g_n(\zeta) = \rho_n^{-k} f(z_n + \rho_n \zeta)$ with $\rho_n \to 0$ and $z_n \to \infty$, which converges to a non-constant and holomorphic Yosida function $g(\zeta)$ locally uniformly on **C**. By Hurwitz' theorem, $g(\zeta)$ has only zeros of order at least $k+1$ and cannot be a polynomial of degree less than $k+1$.

For a given finite non-zero complex value a, if $g(\zeta)$ is a polynomial of degree at least $k+1$, then $g^{(k)}(\zeta)$ assumes a finitely many times; if $g(\zeta)$ is transcendental, then, as proved for the first case, $g^{(k)}(\zeta)$ assumes a infinitely often. Let ζ' be a point such that $g^{(k)}(\zeta') = a$. Since $g^{(k)}(\zeta) \not\equiv a$, by Hurwitz' theorem, there exists a sequence of points $\zeta_n' \to \zeta'$ such that $g_n^{(k)}(\zeta_n') = f^{(k)}(z_n + \rho_n \zeta_n') = a$ for $n \geqslant n_0$. Note that $z_n + \rho_n \zeta_n' \to \infty$ as $n \to \infty$. This completes the proof of Theorem 1. □

The proof of Theorem 2 is an application of Theorem 4. We omit it as the reasoning is almost the same as the second part of the proof of Theorem 1 for the case that f is not a Yosida function.

3. Zeros of $f^2 + af^{(k)}$

Let g be a transcendental meromorphic function. Then a well-known result of Hayman [6] asserts that either g itself assumes every finite complex value infinitely often, or $g^{(k)}$ assumes every finite non-zero value infinitely often for any positive integer k. As a consequence of this theorem, we have that if f is a transcendental integral function. Then $f^2 + af'$ has infinitely many zeros for any finite non-zero complex value a. In fact, for an integral function f, $g = 1/f$ has no zero and the zeros of $g' - 1/a$ are zeros of $f^2 + af'$. Ye [16], Hua and Chen [9] independently proved that this conclusion can be generalised by substituting $f^{(k)}$ for f'. However, this generalisation is no longer a consequence of Hayman's theorem again. Now, We shall prove the generalisation in the same way as Theorem 1 is proved. In fact, our proof produces a stronger conclusion in the case that f is not a Yosida function.

Theorem 5. *Let f be a transcendental integral function. Then $f^2 + af^{(k)}$ has infinitely many zeros for any finite non-zero complex number a and any positive integer k. Furthermore, if f is not a Yosida function (in particular, if f is a function of order* *greater than 1), then, for any finite non-zero complex number a and any positive integer k, $f^2 + af^{(k)}$ assumes every finite complex value infinitely often.*

PROOF: First we assume that f is a Yosida function. Then f and $f^2+af^{(k)}$ both have order at most 1. If $f^2 + af^{(k)}$ has only finitely many zeros, then we have

$$f^2 + af^{(k)} = pe^{bz}, \tag{1}$$

$$2ff' + af^{(k+1)} = (bp + p')e^{bz}, \tag{2}$$

where $p \not\equiv 0$ is a polynomial and b is a complex number. If $b = 0$, then from (1)

$$f^2 + af^{(k)} = p,$$

and consequently

$$\begin{aligned} 2m(r,f) = m(r,f^2) &\leqslant m\left(r, f^2 + af^{(k)}\right) + m\left(r, af^{(k)}\right) + O(1) \\ &\leqslant m(r,f) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1), \\ m(r,f) &\leqslant m\left(r, f^2 + af^{(k)}\right) + O(\log r). \end{aligned}$$

This contradicts the assumption that f is transcendental, and it is proved that $b \neq 0$.

Eliminating e^{bz} between (1) and (2) gives

$$f - 2qf' = A \tag{3}$$

with $q = p/(bp + p')$ and
$$A = aq\frac{f^{(k+1)}}{f} - a\frac{f^{(k)}}{f}.$$

By differentiating (3) $k-1$ times, we obtain

$$\begin{gathered} q_{2,1}f' - 2qf^{(2)} = A', \\ q_{3,1}f' + q_{3,2}f^{(2)} - 2qf^{(3)} = A^{(2)}, \\ \cdots \quad \cdots \\ q_{k,1}f' + q_{k,2}f^{(2)} + \cdots + q_{(k,k-1)}f^{(k-1)} - 2qf^{(k)} = A^{(k-1)}, \end{gathered}$$

where, $q_{j,l}$ are rational functions, $A^{(j)}$ are polynomials in $f'/f, f^{(2)}/f, \cdots, f^{(2k)}/f$ with coefficients of rational functions. Eliminating $f', f^{(2)}, \cdots, f^{(k-1)}$ among the above k equalities, we arrive at

$$f^{(k)} = hf + A_0, \tag{4}$$

where h is a rational function and A_0 is a polynomial like $A^{(j)}$.

Substituting (4) in (1), we have

$$\left(f+\frac{ah}{2}\right)^2+aA_0-\frac{a^2h^2}{4}=pe^{bz}. \tag{5}$$

Set $g=f+ah/2$, $B=aA_0-a^2h^2/4$, then (5) becomes

$$g^2+B=pe^{bz}. \tag{6}$$

Differentiating (6) gives

$$2gg'+B'=(bp+p')e^{bz}. \tag{7}$$

By eliminating e^{bz} between (6) and (7), we have

$$g(g-2qg')=qB'-B. \tag{8}$$

Since B and B' are polynomials like $A^{(j)}$,

$$m(r,g(g-2qg'))=m(r,qB'-B)=O(\log r). \tag{9}$$

To estimate $m(r,g-2qg')$, we note the inequality

$$|g-2qg'|\leqslant|g|\left(1+2|q|\left|\frac{g'}{g}\right|\right). \tag{10}$$

If $|g(z)|>1$, then, from (8),

$$\left|g(z)-2q(z)g(z)'\right|\leqslant|qB'(z)|+|B(z)|. \tag{11}$$

If $|g(z)|\leqslant 1$, then, from (10),

$$|g(z)-2q(z)g'(z)|\leqslant\left(1+2|q(z)|\left|\frac{g'(z)}{g(z)}\right|\right). \tag{12}$$

It follows from (11) and (12) that

$$m(r,g-2qg')=O(\log r). \tag{13}$$

If $g-2qg'\equiv 0$, then

$$\frac{g'}{g}=\frac{1}{2q}=\frac{bp+p'}{2p}=\frac{b}{2}+\frac{p'}{2p}.$$

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Integrating the differential equation gives $g^2 = \lambda p e^{bz}$. By substituting this equality in (6), we obtain $B = (1-\lambda)pe^{bz}$. This is possible only if $\lambda = 1$, for B is a polynomial like $A^{(j)}$ such that $m(r, B) = O(\log r)$. From $g^2 = pe^{bz}$ we obtain $f = \sqrt{p}e^{bz/2} - ah/2$. Thus, from (1),

$$(-ah\sqrt{p} + h_1)e^{bz/2} + \frac{1}{4}a^2h^2 - \frac{1}{2}a^2h^{(k)} = 0,$$

where $h_1 e^{bz/2} = a\left(\sqrt{p}e^{bz/2}\right)^{(k)}$. Hence, h_1 is a polynomial in z of same degree as $\sqrt{p}$. Since both of $-ah\sqrt{p} + h_1$ and $a^2h^2/4 - a^2h^{(k)}/2$ are rational functions, we have, from the above equality, that these two functions are all equal to zero identically. From $-ah\sqrt{p} + h_1 \equiv 0$, we obtain $h = h_1/\left(a\sqrt{p}\right)$, and $h(\infty) \neq 0, \infty$ for h_1 and $\sqrt{p}$ are polynomials of the same degree. Consequently, $\left(a^2h^2/4 - a^2h^{(k)}/2\right)|_\infty = a^2\{h(\infty)\}^2/4 \neq 0$, a contradiction. Thus, we have $g - 2qg' \not\equiv 0$.

Now, from (13) and (9), using Jensen's formula, we have

$$\begin{aligned} m(r, g) &\leqslant m(r, g(g - 2qg')) + m\left(r, \frac{1}{g - 2qg'}\right) \\ &\leqslant N(r, g - 2qg') + O(\log r) = O(\log r). \end{aligned}$$

Consequently,

$$m(r, f) \leqslant m(r, g) + m\left(r, \frac{ah}{2}\right) + O(1) = O(\log r).$$

We arrive at a contradiction for f is transcendental according to the assumption, and the theorem is proved in case that f is a Yosida function. ▯

Now suppose that f is not a Yosida function. Then, $\phi = 1/f$ is not a Yosida function either, and ϕ has no zero. For a given positive integer k, by applying Theorem 3 to the function ϕ, we have a sequence $h_n(\zeta) = \rho_n^{-k}\phi(z_n + \rho\zeta)$ with $\rho_n \to 0$ and $z_n \to \infty$, which converges to a non-constant Yoshida function $h(\zeta)$, without zero, spherically and locally uniformly on **C**. Consequently, $g_n(\zeta) = 1/h_n(\zeta) = \rho_n^k f(z_n + \rho_n\zeta)$ converges to the non-constant and holomorphic Yosida function $g(\zeta) = 1/h(\zeta)$.

For a given finite non-zero complex number a, if $g^2 + ag^{(k)} \equiv 0$, then

$$g = -a \cdot \frac{g^{(k)}}{g}, \quad T(r, g) = m(r, g) = o(T(r, g))$$

outside of a set of finite linear measure, a contradiction. We have proved that $g^2 + ag^{(k)} \not\equiv 0$. If g is a polynomial, then $g^2 + ag^{(k)}$ has finite many zeros; if g is a transcendental and holomorphic Yosida function, then $g^2 + ag^{(k)}$ has infinitely many zeros by the result just proved. Let $g^2(\zeta_0) + ag^{(k)}(\zeta_0) = 0$.

For a given finite complex value b, the sequence of functions

$$g_n^2(\zeta) + a g_n^{(k)}(\zeta) - \rho_n^{2k} b = \rho_n^{2k}\Big(\{f(z_n + \rho_n\zeta)\}^2 + a f^{(k)}(\zeta) - b\Big)$$

uniformly converges to $g^2(\zeta) + a g^{(k)}(\zeta)$ near the point ζ_0. Using Hurwitz's theorem, we know that there exists a sequence of points $\zeta_n \to \zeta_0$ such that $\{f(z_n + \rho_n\zeta_n)\}^2 + a f^{(k)}(\zeta_n) - b = 0$ for $n \geqslant n_0$. Note that $z_n + \rho_n\zeta_n \to \infty$. This completes the proof of Theorem 5. ∎

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