BULL. AUSTRAL. MATH. SOC. 35J55, 58E99, 35A15
VOL. 50 (1994) [501–519]

EXISTENCE OF ENTIRE SOLUTIONS FOR SOME ELLIPTIC SYSTEMS

DING YANHENG AND LI SHUJIE

We establish the existence of solutions for the elliptic systems on $\mathbb{R}^N$:

$$-\triangle u = \frac{\partial H}{\partial v}(x,u,v)$$
$$-\triangle v = \frac{\partial H}{\partial u}(x,u,v)$$

such that $u,v \in W^{1,2}(\mathbb{R}^N)$, where $H(x,u,v) = -q(x)uv + \overline{H}(x,u,v)$ with $q(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$ and $\overline{H}(x,u,v)$ being superlinear or sublinear as $(u^2+v^2)^{1/2} \longrightarrow \infty$.

1. INTRODUCTION

In this paper we consider the existence of solutions for the following elliptic systems on $\mathbb{R}^N$:

$$(\mathrm{ES})_1 \qquad \begin{cases} -\triangle u = \dfrac{\partial H}{\partial v}(x,\, u,\, v) \\ -\triangle v = \dfrac{\partial H}{\partial u}(x,\, u,\, v) \end{cases}$$

such that $u,v \in W^{1,2}(\mathbb{R}^N)$ where $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ is superlinear or sublinear as $(u^2+v^2)^{1/2} \longrightarrow \infty$.

The existence of solutions (u,v) to the elliptic systems like $(\mathrm{ES})_1$ on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$ has been studied earlier by Benci-Rabinowitz [1], Clement-de Figueiredo-Mitidieri [2], de Figueiredo-Felmer [4] and Szulkin [8] using a variational approach.

First, we deal with the superlinear case. We are interested in the Hamiltonian of the type

$$H(x,u,v) = -q(x)uv + \overline{H}(x,u,v),$$

Received 17th February, 1994
Li Shujie would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. This research was partially supported by the Chinese National Science Foundation.

where $q(x)$ satisfies

(Q) $q \in C(\mathbb{R}^N)$ and $q(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$.

We denote $(u,v) \in \mathbb{R}^2$ by z and $(u^2+v^2)^{1/2}$ by $|z|$, and suppose that H satisfies

(H_1) there is $\mu > 2$ such that

$$0 < \mu \overline{H}(x,z) \leqslant \overline{H}_z(x,z)z$$

for all $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^2 \setminus \{0\}$, where $\overline{H}_z(x,z) = \nabla_z \overline{H}(x,z)$;

(H_2) $0 < \underline{b} \equiv \inf_{x \in \mathbb{R}^N, |z|=1} \overline{H}(x,z)$;

(H_3) $|\overline{H}_z(x,z)| = o(|z|)$ as $|z| \longrightarrow 0$ uniformly in $x \in \mathbb{R}^N$;

(H_4) there are $0 \leqslant a_1(x) \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $a_2 > 0$ such that

$$|\overline{H}_z(x,z)|^q \leqslant a_1(x) + a_2 \overline{H}_z(x,z)z, \quad \forall (x,z) \in \mathbb{R}^N \times \mathrm{R}^2$$

where $q > 1$, $\mu \leqslant q/(q-1) \equiv \gamma < \overline{N} \equiv (2N)/(N-2)$ if $N > 2$ and $\gamma < \infty$ if $N = 1,2$.

We point out that, by (H_4), there are $\beta_1, \beta_2 > 0$ such that

$$|\overline{H}_z(x,z)| \leqslant \beta_1 + \beta_2 |z|^{\gamma-1}, \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{1.1}$$

Our result reads:

Theorem 1.1. *Under the assumptions (Q) and (H_1)–(H_4) on H, $(ES)_1$ has at least one nontrivial $W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$ solution.*

Next, we deal with the sublinear case. We again consider the Hamiltonian with the form

$$H(x,u,v) = -q(x)uv + G(x,u,v).$$

Suppose that $q(x)$ satisfies

(Q_α) $q \in C(\mathbb{R}^N)$ and there exists $\alpha < 2$ such that $q(x)|x|^{\alpha-2} \longrightarrow \infty$ as $|x| \longrightarrow \infty$, and G satisfies

(G_1) there is $1 < \beta \in ((2N)/(2-\alpha+N), 2)$ such that

$$0 < G_z(x,z)z \leqslant \beta G(x,z), \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad z \in \mathbb{R}^2 \setminus \{0\};$$

(G_2) there are $a_1, a_2 > 0$ and $\nu > \max\{0, (\alpha - 2 + N)/(2-\alpha+N)\}$ such that

$$G(x,z) \geqslant a_1 |z|^\beta \quad \text{and} \quad |G_z(x,z)| \leqslant a_2 |z|^\nu$$

for all $x \in \mathbb{R}^N$ and $|z| \leqslant 1$;

(G_3) there are $1 < \overline{\beta} \in ((2N)/(2-\alpha+N), \beta]$, $a_3 > 0$, $\overline{r} > 0$ such that

$$G(x,z) \geqslant a_3 |z|^{\overline{\beta}}, \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad |z| \geqslant \overline{r};$$

(G_4) $|G_z(x,z)| \in L^\infty(\mathbb{R}^N \times B_R)$ for any $R > 0$, where $B_R = \{z \in \mathbb{R}^2; |z| \leqslant R\}$, and

$$|z|^{-1} |G_z(x,z)| \longrightarrow 0 \quad \text{as} \quad |z| \longrightarrow \infty \quad \text{uniformly in} \quad x \in \mathbb{R}^N.$$

Then we have

Theorem 1.2. *Under the assumptions (Q_α) and (G_1)–(G_4) on H, $(ES)_1$ has at least one nontrivial $W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$ solution.*

We remark that the study of $(ES)_1$ is equivalent to that of the following systems on $\mathbb{R}^N$:

$$(ES)_2 \qquad \begin{cases} -\triangle w &= \dfrac{\partial H}{\partial w}(x,w,y) \\ -\triangle y &= -\dfrac{\partial H}{\partial y}(x,w,y). \end{cases}$$

However, it seems convenient for us to handle $(ES)_1$ and $(ES)_2$ separately. For example, one can consider $(ES)_2$ for the Hamiltonian being of the type

$$H(x,w,y) = -\frac{1}{2}q_1(x)w^2 + \frac{1}{2}q_2(x)y^2 + \overline{H}(x,w,y) \tag{1.2}$$

with different q_1 and q_2. In the sequel we shall show some results for $(ES)_2$ which are similar to those for $(ES)_1$.

The paper is organised as follows. In section 2 we give some preliminary results, particularly, a compact embedding lemma which enables us to apply standard critical point theory to handling the problems. In section 3 and section 4 we shall deal with the superlinear case and the sublinear case respectively.

2. Preliminaries

In order to study $(ES)_1$ and $(ES)_2$, we first recall some facts about the Schrödinger operators.

Suppose q satisfies (Q) and let A denote the self-adjoint extension of $-\triangle + q(x)$ acting in $L^2 \equiv L^2(\mathbb{R}^N)$. Let $|A|$ be the absolute value of A, $|A|^{1/2}$ the square root of $|A|$, $\{E(\nu); -\infty < \nu < \infty\}$ the resolution of the identity corresponding to A, and $U = I - E(0) - E(-0)$. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = |A|U$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

is the polar decomposition of A (see [5]). Set $E = \mathcal{D}\left(|A|^{1/2}\right)$. E is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_0 = \left(|A|^{1/2} u, |A|^{1/2} v\right)_{L^2} + (u, v)_{L^2}$$

and norm

$$\|u\|_0^2 = \langle u, u\rangle_0$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. Clearly $C_0^\infty(\mathbb{R}^N)$ is dense in E and E is continuously embedded in $W^{1,2}(\mathbb{R}^N)$. Moreover we have

Lemma 2.1. *If q satisfies (Q) then E is compactly embedded in L^p for $p \in [2, \overline{N})$ where $\overline{N} = (2N)/(N-2)$ if $N \geqslant 3$, $\overline{N} = \infty$ if $N = 2$, and $p \in [2, \infty]$ if $N = 1$.*

Proof: It is known that E is compactly embedded in L^2, see, for example, [7]. Moreover for $N = 1$ we refer, for example, to [3]. For $N \geqslant 2$ and $p > 2$ it follows from the interpolation inequality

$$\|u\|_{L^p} \leqslant c\|u\|_{L^2}^{1-\theta} \|u\|_{W^{1,2}}^{\theta}$$

where $\theta = ((p-2)N)/(2p)$ and c is independent of u. ∎

Lemma 2.2. *Suppose q satisfies (Q_α). Then E is compactly embedded in L^p for all $1 \leqslant p \in ((2N)/(2-\alpha+N), 2)$.*

Remark. Since (Q_α) implies (Q), E is already compactly embedded in L^p for $p \in [2, \overline{N})$ by Lemma 2.1. Moreover, since $\alpha < 2, (2N)/(2-\alpha+N) < 2$, and if $\alpha < 2-N$ then $(2N)/(2-\alpha+N) < 1$.

Proof: First we assume $q(x) \geqslant 1$ for all $x \in \mathbb{R}^N$. Let $k = (2-\alpha)/(2-p)$. Then

$$pk > N. \tag{2.1}$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

For any $R > 0$, one has

$$
\begin{aligned}
\int_{|x|>R} |u|^\rho &= \int_{\substack{|x|>R \\ \{|x|^k|u(x)|>1\}}} |u|^\rho + \int_{\substack{|x|>R \\ \{|x|^k|u(x)|\leqslant 1\}}} |u|^\rho \\
&\leqslant \int_{|x|>R} \frac{1}{|x|^{pk}} + \int_{\substack{|x|>R \\ \{|x|^k|u|>1\}}} \left(|x|^k|u|\right)^p |x|^{-k\rho} \\
&\leqslant \int_{|x|>R} \frac{1}{|x|^{pk}} + \int_{|x|>R} |x|^{(2-p)k} |u|^2 \\
&= \int_{|x|>R} \frac{1}{|x|^{pk}} + \int_{|x|>R} |x|^{2-\alpha} |u|^2 \\
&= \int_{|x|>R} \frac{1}{|x|^{pk}} + \int_{|x|>R} \frac{q(x)\,|u|^2}{q(x)\,|x|^{\alpha-2}} \\
&\leqslant \int_{|x|>R} \frac{1}{|x|^{pk}} + \frac{1}{\beta(R)} \|u\|_0^2
\end{aligned}
\tag{2.2}
$$

where $\beta(R) = \inf_{|x|\geqslant R} q(x)\,|x|^{\alpha-2}$.

Let $K \subset E$ be a bounded set,

$$\|u\|_0 \leqslant M \quad \forall u \in K.$$

We shall show that, for any $\varepsilon > 0$, K has a finite ε-net.

Since by (2.1)

$$\int_{|x|>R} \frac{1}{|x|^{pk}} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty$$

and by (Q_α)

$$\frac{1}{\beta(R)} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,$$

one can take R_0 large such that

$$\int_{|x|\geqslant R_0} \frac{1}{|x|^{pk}} + \frac{4M^2}{\beta(R_0)} < \frac{\varepsilon^2}{2}. \tag{2.3}$$

By the Sobolev compact embedding theorem, there are $u_1, \cdots, u_m \in K$ such that for any $u \in K$, there is u_i satisfying

$$\|u - u_i\|^2_{L^p(B(R_0))} < \frac{\varepsilon^2}{2}, \tag{2.4}$$

where $B(R_0) = \{x \in \mathbb{R}^N;\ |x| < R_0\}$. Now (2.2)–(2.4) shows

$$\|u - u_i\|_{L^p(\mathbb{R}^N)} < \varepsilon$$

that is, K has a finite ε-net in L^p and so is precompact in L^p.

In general, by (Q_α), $q(x)$ is bounded from below, $q(x) \geqslant -a+1$ for some $a > 0$ and all $x \in \mathbb{R}^N$. Since $E = \mathcal{D}\Big((A+a)^{1/2}\Big)$, we can introduce a norm on E by setting

$$\|u\|_a^2 = \left((A+a)^{1/2}u, (A+a)^{1/2}u\right)_{L^2} + (u,u)_{L^2}.$$

By the above argument we know that $(E, \|\cdot\|_a)$ is compactly embedded in L^p for $1 \leqslant p \in ((2N)/(2-\alpha+N), 2)$. Therefore in order to prove the Lemma it suffices to show that the norms $\|\cdot\|_a$ and $\|\cdot\|_0$ are equivalent to each other. In fact, for $u \in \mathcal{D}(A)$,

$$\begin{aligned}
\left\| |A|^{1/2} u \right\|_{L^2}^2 &= \left(|A|^{1/2} u, |A|^{1/2} u\right)_{L^2} = (|A|\, u, u)_{L^2} \\
&= \left(U(A+a)^{1/2}u, (A+a)^{1/2}u\right)_{L^2} - a(Uu,u)_{L^2} \\
&\leqslant \left\|(A+a)^{1/2}u\right\|_{L^2}^2 + a\,\|u\|_{L^2}^2,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\left\|(A+a)^{1/2}u\right\|_{L^2}^2 &= ((A+a)u,u)_{L^2} = (Au,u)_{L^2} + a(u,u)_{L^2} \\
&= \left(U\,|A|^{1/2} u, |A|^{1/2} u\right)_{L^2} + a(u,u)_{L^2} \\
&\leqslant \left\| |A|^{1/2} u \right\|_{L^2}^2 + a\,\|u\|_{L^2}^2 .
\end{aligned}$$

Hence $c_1\|u\|_a \leqslant \|u\|_0 \leqslant c_2\|u\|_a$ for all $u \in \mathcal{D}(A)$, and so for all $u \in E$ since $\mathcal{D}(A)$ is dense in E and by continuity. The proof is complete. ∎

By Lemma 2.1 A has a compact resolution, and so $\sigma(A)$, the spectrum of A, consists of eigenvalues (repeated according to their multiplicities)

$$\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_n \leqslant \cdots \longrightarrow \infty$$

with a corresponding system of eigenfunctions (h_n), $Ah_n = \lambda_n h_n$, which forms an orthonormal basis in L^2. Let n^- (respectively n^0) denote the number of negative (respectively 0) eigenvalues, and $\overline{n} = n^- + n^0$. Set

$$\begin{aligned}
E^- &= \mathrm{span}\{h_1, \cdots, h_{n^-}\} \\
E^0 &= \mathrm{span}\{h_{n^-+1}, \cdots, h_{\overline{n}}\} \\
E^+ &= \left(E^- \oplus E^0\right)^{\perp_E} = Cl_E(\mathrm{span}\{h_{\overline{n}+1}, \cdots\})
\end{aligned}$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

where $Cl_E S$ is the closure of S in E. Then, clearly

$$E = E^- \oplus E^0 \oplus E^+$$

is a natural orthogonal decomposition. Based on this decomposition, we introduce the following inner product in E

$$\langle u, v\rangle = \left(|A|^{1/2} u, |A|^{1/2} v\right)_{L^2} + \left(u^0, v^0\right)_{L^2}$$

and norm

$$\|u\|^2 = \langle u, u\rangle$$

for all $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$. It is easy to see that for any $u \in E$

(2.5) $$\|u\|_{L^2}^2 \leqslant \underline{\lambda} \|u\|^2$$

where $\underline{\lambda} = \max\{1, 1/(|\lambda_{n^-}|), 1/(\lambda_{\overline{n}+1})\}$ and

(2.6) $$\|u\| \leqslant \|u\|_0 \leqslant (1 + \underline{\lambda})^{1/2} \|u\|,$$

that is, $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

Let

$$a(u, v) = \left(|A|^{1/2} Uu, |A|^{1/2} v\right)_{L^2}$$

be the quadratic form associated with A. Then for $u \in \mathcal{D}(A)$ and $v \in E$

(2.7) $$a(u, v) = (Au, v)_{L^2} = \int_{\mathbb{R}^N} (\nabla u \nabla v + q(x)uv)$$

and so for all $u, v \in E$ by continuity. Clearly E^-, E^0 and E^+ are orthogonal to each other with respect to $a(\cdot, \cdot)$ and moreover,

(2.8) $$\begin{aligned} a(u, v) &= \langle (p^+ - p^-)u, v\rangle \\ a(u, u) &= \|u^+\|^2 - \|u^-\|^2 \end{aligned}$$

where $p^\pm : E \longrightarrow E^\pm$ are the orthogonal projectors.

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

3. The Superlinear Case

In this section we give the proof of Theorem 1.1. Suppose that the assumptions are satisfied. Let $(E, \|\cdot\|)$ be as in the previous section. Define the product space $\mathbb{E} = E \times E = (E)^2$ with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle = \langle u,\varphi\rangle + \langle v,\psi\rangle$$

and norm

$$\|(u,v)\|^2 = \|u\|^2 + \|v\|^2 .$$

Consider the quadratic form defined on $\mathbb{E} \times \mathbb{E}$:

$$\begin{aligned} Q((u,v),(\varphi,\psi)) &= a(u,\psi) + a(v,\varphi) \\ &= \int_{\mathbb{R}^N} \nabla u \nabla \psi + q(x)u\psi + \nabla v \nabla \varphi + q(x)v\varphi. \end{aligned} \tag{3.1}$$

By (2.8)

$$\begin{aligned} Q((u,v),(\varphi,\psi)) &= \langle (p^+ - p^-)u,\psi\rangle + \langle (p^+ - p^-)v,\varphi\rangle \\ &= \langle ((p^+ - p^-)v,(p^+ - p^-)u),(\varphi,\psi)\rangle. \end{aligned}$$

Hence the self-adjoint bounded operator L, reduced by Q, is given by

$$L : \mathbb{E} \longrightarrow \mathbb{E}, \quad (u,v) \longrightarrow ((p^+ - p^-)v,(p^+ - p^-)u).$$

Consider the eigenvalue problem

$$Lz = \lambda z$$

where $z = (u,v)$. It is easy to see that

$$(p^+ - p^-)v = \lambda u, \quad (p^+ - p^-)u = \lambda v.$$

Therefore $\lambda = \pm 1, 0$, and we can define

$$\begin{aligned} \mathbb{E}^0 &= E^0 \times E^0, \\ \mathbb{E}^- &= \{(u^- + u^+, u^- - u^+) ; u^- + u^+ \in E^- \oplus E^+\}, \\ \mathbb{E}^+ &= \{(u^- + u^+, -u^- + u^+) ; u^- + u^+ \in E^- \oplus E^+\}. \end{aligned}$$

Then

$$\mathbb{E} = \mathbb{E}^- \oplus \mathbb{E}^0 \oplus \mathbb{E}^+ \tag{3.2}$$

is an orthogonal decomposition of $\mathbb{E}$. For any $z = (u, v) \in \mathbb{E}$, let

$$\begin{aligned} z^- &= \frac{1}{2}\left(u^- + v^- + u^+ - v^+, u^- + v^- - u^+ + v^+\right), \\ z^0 &= \left(u^0, v^0\right), \\ z^+ &= \frac{1}{2}\left(u^- - v^- + u^+ + v^+, -u^- + v^- + u^+ + v^+\right). \end{aligned}$$

Then we have the unique decomposition

$$z = (u, v) = \left(u^- + u^+, v^- + v^+\right) + \left(u^0, v^0\right) = z^- + z^0 + z^+$$

with $z^\pm \in \mathbb{E}^\pm$ and $z^0 \in \mathbb{E}^0$. It is easy to check that

$$Q(z) \equiv Q((u, v), (u, v)) = \left\|z^+\right\|^2 - \left\|z^-\right\|^2 \tag{3.3}$$

for any $z \in \mathbb{E}$.

Let

$$J(z) = \int_{\mathbb{R}^N} \overline{H}(x, z)\, dx \quad \forall z \in \mathbb{E}.$$

By a standard argument it is easy to show that $J \in C^1(\mathbb{E}, \mathbb{R})$,

$$\nabla J(z) y = \int_{\mathbb{R}^N} \overline{H}_z(x, z) y\, dx \quad \forall z, y \in \mathbb{E}. \tag{3.4}$$

Here ∇J represents the gradient of J. Moreover J is weakly continuous and ∇J is compact. For the reader's convenience, we show that $J(z)$ is weakly continuous. By (H_4) (see (1.1)) we have

$$\left|\overline{H}_z(x, z)\right| \leqslant \beta_1 + \beta_2 |z|^{\gamma - 1} \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, \tag{3.5}$$

and by (H_3), (3.5) we have

$$\left|\overline{H}(x, z)\right| \leqslant c_1 |z|^2 + c_2 |z|^\gamma \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, \tag{3.6}$$

(Here and after, the c_i stand for positive constants.) Let $z_n \longrightarrow z$ weakly in $\mathbb{E}$.

By Lemma 2.1, one can assume that $z_n \longrightarrow z$ strongly in $\left(L^2(\mathbb{R}^N)\right)^2$ and $\left(L^\gamma(\mathbb{R}^N)\right)^2$. Note that, by (3.6), for any $R > 0$,

$$\left|\int_{|x| \geqslant R} \left(\overline{H}(x, z_n) - \overline{H}(x, z)\right)\right| \leqslant c_1 \int_{|x| \geqslant R} \left(|z_n|^2 + |z|^2\right) + c_2 \int_{|x| \geqslant R} \left(|z_n|^\gamma + |z|^\gamma\right). \tag{3.7}$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

For any $\varepsilon > 0$, by (3.7), one can take R_0 large such that

$$\left|\int_{|x|\geqslant R_0} \left(\overline{H}(x,z_n) - \overline{H}(x,z)\right)\right| < \frac{\varepsilon}{2} \tag{3.8}$$

for all $n \in \mathbb{N}$. On the other hand, it is well-known that

$$\int_{|x|\leqslant R_0} \overline{H}(x,z_n) \longrightarrow \int_{|x|\leqslant R_0} \overline{H}(x,z)$$

as $n \longrightarrow \infty$. Therefore, there is $n_0 \in \mathbb{N}$ such that

$$\left|\int_{|x|\leqslant R_0} \left(\overline{H}(x,z_n) - \overline{H}(x,z)\right)\right| < \frac{\varepsilon}{2} \quad \forall n \geqslant n_0. \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$|J(z_n) - J(z)| = \left|\int_{\mathbb{R}^N} \left(\overline{H}(x,z_n) - \overline{H}(x,z)\right)\right| < \varepsilon \quad \forall n \geqslant n_0.$$

We have proved that J is weakly continuous. Now an abstract theorem [**6**] implies immediately that ∇J is compact.

Define

$$f(z) = \frac{1}{2}Q(z) - J(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \int_{\mathbb{R}^N} \overline{H}(x,z) \tag{3.10}$$

for $z = (u,v) \in \mathbb{E}$. Then

$$\begin{aligned} \nabla f((u,v),(\varphi,\psi)) = &\int_{\mathbb{R}^N} (\nabla u \nabla \psi + q(x)u\psi + \nabla v \nabla \varphi + q(x)v\varphi) \\ &- \int_{\mathbb{R}^N} \left(\frac{\partial \overline{H}}{\partial v}(x,u,v)\varphi + \frac{\partial \overline{H}}{\partial u}(x,u,v)\psi\right). \end{aligned} \tag{3.11}$$

Clearly, any critical point of f corresponds to a $W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$ solution of $(\mathrm{ES})_1$.

Let $e_1, e_2, \cdots$ be an orthonormal basis for $\mathbb{E}^+$, $g_1, g_2, \cdots$ be an orthonormal basis for $\mathbb{E}^- \oplus \mathbb{E}^0$. Denote $\mathbb{E}_n^+ = \mathrm{Span}\{e_1, \cdots, e_n\}$, $\mathbb{E}_n^{-,0} = \mathrm{Span}\{g_1, g_2, \cdots, g_n\}$ and $\mathbb{E}_n = \mathbb{E}_n^+ \oplus \mathbb{E}_n^{-,0}$. Let $f_n = f|_{\mathbb{E}_n}$. We say that f satisfies the $(\mathrm{PS})^*$ condition if any sequence (z_n) in E, $z_n \in \mathbb{E}_n$, $f_n(z_n) \leqslant c < +\infty$, $\nabla f_n(z_n) \longrightarrow 0$ possesses a convergent subsequence. The following Proposition is a slight variant version of a theorem of Benci-Rabinowitz [**1**, Theorem 0.1].

Proposition 3.1. *Suppose*

(f_1) $f \in C^1(\mathbb{E}, \mathbb{R})$ *and satisfies* $(PS)^*$;

(f_2) *there are constants* $\rho, \delta > 0$ *such that*

$$f(u) \geqslant \delta \quad \forall u \in S_\rho,$$

where $S_\rho = \{z \in \mathbb{E}^+ ; \quad \|z\| = \rho\}$;

(f_3) *there are constants* $r > \rho$, $M > 0$, $e \in \mathbb{E}_1^+$ $\|e\| = 1$ *such that*

$$f|_{\partial Q} \leqslant 0 \quad \text{and} \quad f|_Q \leqslant M$$

where $Q = \left(B(0,r) \cap \mathbb{E}^- \oplus \mathbb{E}^0\right) \oplus \{se\,;\, 0 \leqslant s \leqslant r\}$.

Then f *has a critical point* z *with* $f(z) \geqslant \delta$.

Proof: By applying the Benci-Rabinowitz Theorem to f_n, one gets a sequence $(z_n) \subset \mathbb{E}$ such that $z_n \in \mathbb{E}_n, \nabla f_n(z_n) = 0$, $\delta \leqslant f_n(z_n) \leqslant M$. By $(\mathrm{PS})^*$, z_n possesses a convergent subsequence. The proof is complete. ∎

In the three lemmas below we shall show that f (given by (3.10)) satisfies the hypotheses of Proposition 3.1.

Lemma 3.2. f *satisfies* $(PS)^*$.

Proof: Suppose (z_n) is a sequence in $\mathbb{E}$ such that $|f(z_n)| \leqslant c$, $\varepsilon_n = \|\nabla f(z_n)\| \longrightarrow 0$. From ($\mathrm{H}_1$)

(3.12)
$$\begin{aligned} f(z_n) - \frac{1}{2}\nabla f(z_n) z_n &= \int_{\mathbb{R}^N} \left(\frac{1}{2}\overline{H}_z(x, z_n) z_n - \overline{H}(x, z_n)\right) \geqslant \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^N} \overline{H}(x, z_n) \\ &= \left(\frac{\mu}{2} - 1\right) \int_{|z_n(x)|<1} \overline{H}(x, z_n) + \left(\frac{\mu}{2} - 1\right) \int_{|z_n(x)|\geqslant 1} \overline{H}(x, z_n). \end{aligned}$$

Let

$$z_n^1 = \begin{cases} z_n & \text{if} \quad |z_n(x)| < 1 \\ 0 & \text{if} \quad |z_n(x)| \geqslant 1, \end{cases} \qquad z_n^2 = \begin{cases} 0 & \text{if} \quad |z_n(x)| < 1 \\ z_n & \text{if} \quad |z_n(x)| \geqslant 1. \end{cases}$$

Then

$$f(z_n) - \frac{1}{2}\nabla f(z_n) z_n \geqslant \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^N} \overline{H}(x, z_n^1) + \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^N} \overline{H}(x, z_n^2). \tag{3.13}$$

From (H_1) and (H_4) there exists $\bar{b} > 0$ such that

$$\overline{H}(x, z) \leqslant \bar{b}\,|z|^\mu \quad \text{if} \quad |z| < 1 \tag{3.14}$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

and from (H_1) and (H_2)

$$\overline{H}(x,z) \geqslant \underline{b}|z|^{\mu} \quad \text{if} \quad |z| \geqslant 1. \tag{3.15}$$

Then for n large

$$\left(\frac{\mu}{2}-1\right)\underline{b}\int_{\mathbb{R}^N} |z_n^2|^{\mu} \leqslant \left(\frac{\mu}{2}-1\right)\int_{\mathbb{R}^N} \overline{H}(x,z_n) \leqslant c+\|z_n\|. \tag{3.16}$$

We denote by c various positive constants independent of n. From (H_1) (H_4) for n large

$$\begin{aligned} c+\|z_n\| &\geqslant f(z_n)-\frac{1}{2}\nabla f(z_n)z_n \geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} \overline{H}_z(x,z_n)z_n \\ &\geqslant c\left\|\overline{H}_z(x,z_n)\right\|_{L^q}^q - c. \end{aligned} \tag{3.17}$$

Then

$$\begin{aligned} \left\|z_n^+\right\|^2 &= \int_{\mathbb{R}^N} \overline{H}_z(x,z_n)z_n^+ + \nabla f(z_n)z_n^+ \\ &\leqslant \left\|z_n^+\right\| + \left\|z_n^+\right\|_{L^\gamma}\left\|\overline{H}_z(x,z_n)\right\|_{L^q} \\ &\leqslant \left\|z_n^+\right\| + \left\|z_n^+\right\|\left(c+c\|z_n\|^{1/q}\right) \quad \text{(by Lemma 2.1 and (3.17))}. \end{aligned} \tag{3.18}$$

Namely,

$$\left\|z_n^+\right\|^2 \leqslant c\|z_n\|^{1+(1/q)} + c. \tag{3.19}$$

Similarly,

$$\left\|z_n^-\right\|^2 \leqslant c\|z_n\|^{1+(1/q)} + c. \tag{3.20}$$

Since $\dim \mathbb{E}^0 < \infty$, for any $\overline{N} > \beta > 2$, $1/\beta + 1/\beta' = 1$ and $1/\mu' + 1/\mu = 1$

$$\begin{aligned} \left\|z_n^0\right\|_{L^2}^2 = \left(z_n^0, z_n\right)_{L^2} &= \left(z_n^0, z_n^1\right)_{L^2} + \left(z_n^0, z_n^2\right)_{L^2} \\ &\leqslant \left(\left\|z_n^0\right\|_{L^{\beta'}}\left\|z_n^1\right\|_{L^\beta} + \left\|z_n^0\right\|_{L^{\mu'}}\left\|z_n^2\right\|_{L^\mu}\right) \\ &\leqslant c\left\|z_n^0\right\|_{L^2}\left(\left\|z_n^1\right\|_{L^\beta} + \left\|z_n^2\right\|_{L^\mu}\right). \end{aligned}$$

By Lemma 2.1,

$$\int_{\mathbb{R}^N} |z_n^1|^\beta \leqslant \int_{\mathbb{R}^N} |z_n^1|^2 \leqslant \int_{\mathbb{R}^N} |z_n|^2 \leqslant c\|z_n\|^2$$

and by (3.16),

$$\left\|z_n^2\right\|_{L^\mu} \leqslant c + \|z_n\|^{1/\mu}.$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

We get

$$\left\|z_n^0\right\|_{L^2} \leqslant c\left(1 + \left\|z_n\right\|^{2/\beta} + \left\|z_n\right\|^{1/\mu}\right). \tag{3.21}$$

By combining (3.19), (3.20) and (3.21) we see that (z_n) is bounded in $\mathbb{E}$.

Since ∇J is compact we conclude immediately that (z_n) has a convergent subsequence. ∎

LEMMA 3.3. *f satisfies (f_2).*

PROOF: For $z \in \mathbb{E}^+$

$$f(z) = \frac{1}{2}\left\|z\right\|^2 - \int_{\mathbb{R}^N} \overline{H}(x,z). \tag{3.22}$$

From (H_3) and (H_4), for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\overline{H}(x,z) \leqslant \varepsilon\left|z\right|^2 + c_\varepsilon\left|z\right|^\gamma. \tag{3.23}$$

By Lemma 2.1

$$f(z) \geqslant \frac{1}{2}\left\|z\right\|^2 - \varepsilon\cdot c\left\|z\right\|^2 - c\cdot c_\varepsilon\left\|z\right\|^\gamma.$$

The lemma then follows. ∎

LEMMA 3.4. *f satisfies (f_3).*

PROOF: Let $Q = \left(B(0,r) \cap \mathbb{E}^- \oplus \mathbb{E}^0\right) \oplus \{se_1; 0 \leqslant s \leqslant r\}$. From (H_1) and (H_3), for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\overline{H}(x,z) \geqslant c_\varepsilon\left|z\right|^\mu - \varepsilon\left|z\right|^2. \tag{3.24}$$

Therefore, for $z = z^- + z^0 + se$, we have

$$\begin{aligned} f(z) &= \frac{s^2}{2} - \frac{1}{2}\left\|z^-\right\|^2 - \int_{\mathbb{R}^N} \overline{H}(x,z) \\ &\leqslant \frac{s^2}{2} - \frac{1}{2}\left\|z^-\right\|^2 + \varepsilon\left\|z\right\|_{L^2}^2 - c_\varepsilon\left\|z\right\|_{L^\mu}^\mu. \end{aligned} \tag{3.25}$$

Since by the Hölder inequality and $\dim \mathbb{E}^0 < \infty$,

$$\begin{aligned} \left\|z^0 + se_1\right\|_{L^2}^2 &= \left(z^0 + se_1, z\right)_{L^2} \leqslant \left\|z^0 + se_1\right\|_{L^{\mu'}}\left\|z\right\|_{L^\mu} \\ &\leqslant c\left\|z^0 + se_1\right\|_{L^2}\left\|z\right\|_{L^\mu}, \end{aligned}$$

we see that

$$\left\|z^0 + se_1\right\|_{L^2}^\mu \leqslant c\left\|z\right\|_{L^\mu}^\mu,$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

or

$$\left\|z^0\right\|^{\mu} + s^{\mu} \leqslant c\left\|z\right\|_{L^{\mu}}^{\mu}. \tag{3.26}$$

Combining (3.25) and (3.26) shows

$$f\left(z^- + z^0 + se_1\right) \leqslant \frac{s^2}{2} - \frac{1}{2}\left\|z^-\right\|^2 + \varepsilon\left(\left\|z^-\right\|_{L^2}^2 + \left\|z^0\right\|_{L^2}^2 + \left\|se_1\right\|_{L^2}^2\right) - c'_{\varepsilon}\left(\left\|z^0\right\|^{\mu} + s^{\mu}\right).$$

The lemma then follows by taking ε small enough and noting that $\mu > 2$. ∎

Now we can give the following

PROOF OF THEOREM 1.1: Lemmas 3.2, 3.3 and 3.4 show that the f satisfies all the hypotheses of Proposition 3.1. Hence f has a nontrivial critical point which gives rise to a $W^{1,2}$ solution for $(\mathrm{ES})_1$. ∎

Next we deal with the system $(\mathrm{ES})_2$ in a similar way. Suppose H has the form of (1.2) with the q_1 and q_2 satisfying (Q). Let $A_i = -\triangle + q_i(x)$ $(i = 1, 2)$ be the Schrödinger operators acting in L^2, and let $E_i = \mathcal{D}\left(|A_i|^{1/2}\right)$. Along the lines of Section 2, we introduce on E_i inner products and norms denoted by $\langle\cdot,\cdot\rangle_i$ and $\|\cdot\|_i$ respectively, such that the E_i become Hilbert spaces. In addition, we denote by $a_i(\cdot,\cdot)$ the quadratic forms associated with A_i. Set $\widetilde{\mathbb{E}} = E_1 \times E_2$ equipped with the inner product

$$\langle(u, v), (\varphi, \psi)\rangle = \langle u, \varphi\rangle_1 + \langle v, \psi\rangle_2$$

and norm

$$\|(u, v)\|^2 = \|u\|_1^2 + \|v\|_2^2,$$

and consider the quadratic form on $\widetilde{\mathbb{E}}$

$$\widetilde{Q}((u, v), (\varphi, \psi)) = a_1(u, \varphi) - a_2(v, \psi).$$

Moreover, let

$$\begin{aligned} \widetilde{\mathbb{E}}^0 &= E_1^0 \times E_2^0, \\ \widetilde{\mathbb{E}}^- &= E_1^- \times E_2^+, \\ \widetilde{\mathbb{E}}^+ &= E_1^+ \times E_2^-. \end{aligned}$$

Clearly $\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}^- \oplus \widetilde{\mathbb{E}}^0 \oplus \widetilde{\mathbb{E}}^+$ is an orthogonal decomposition of $\widetilde{\mathbb{E}}$, and for any $z = (u, v) \in \widetilde{\mathbb{E}}$, $z = z^- + z^0 + z^+$ where $z^0 = \left(u^0, v^0\right), z^- = (u^-, v^+)$ and $z^+ = (u^+, v^-)$. It is easy to check that

$$\widetilde{Q}(z) = \left\|z^+\right\|^2 - \left\|z^-\right\|^2$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

for any $z = z^- + z^0 + z^+ \in \widetilde{\mathbb{E}}$. Define

$$f(z) = \frac{1}{2}\widetilde{Q}(z) - \int_{\mathbb{R}^N} \overline{H}(x, z)$$

for $z \in \widetilde{\mathbb{E}}$. Then critical points of f are solutions of $(\mathrm{ES})_2$. Now repeating the procedure of the proof of Theorem 1.1, one can get

Theorem 3.5. *Suppose that H has the form of (1.2) such that q_1 and q_2 satisfy (Q) and $\overline{H}$ satisfies (H_1)–(H_4). Then $(ES)_2$ has at least one nontrivial $W^{1,2}$ solution.*

4. The Sublinear Case

In this section we consider the sublinear case. Let the assumptions of Theorem 1.2 be satisfied. Below, the symbols $\mathbb{E}$, $\mathbb{E}^-$, $\mathbb{E}^0$, $\mathbb{E}^+$, z^-, z^0, z^+, $\mathbb{E}_n$, f_n still have the same meaning as in Section 3. The following propsition is a slightly variant version of Benci-Rabinowtz [**1**, Theorem 1.33].

Proposition 4.1. *Suppose*

(f_1) *$f \in C^1(\mathbb{E}, \mathbb{R})$ and satisfies $(PS)^*$;*

(f_2) *there are constants $\rho > 0$, $\sigma > 0$ and a $v \in \mathbb{E}_1^+$ and $v \in Q \equiv B_\rho \cap \mathbb{E}^+$ such that*

$$f \geqslant \sigma \quad \text{for all} \quad z \in S$$

where $S = \mathbb{E}^- \oplus \mathbb{E}^0 + v$;

(f_3) *there is a $M > 0$ such that*

$$f \leqslant 0 \quad \text{for all} \quad z \in \partial Q$$
$$f \leqslant M \quad \text{for all} \quad z \in Q.$$

Then f has a critical point z with $f(z) \geqslant \sigma$.

The proof is very easy and we omit the details.

We shall apply proposition 4.1 to the functional

$$f(z) = J(z) - \frac{1}{2}\left\|z^+\right\|^2 + \frac{1}{2}\left\|z^-\right\|^2 \quad \text{for} \quad z \in \mathbb{E}$$

where $J(z) = \int_{\mathbb{R}^N} G(x, z)dx$.

Proof of Theorem 1.2: The proof will be accomplished in several steps.

Step 1. Assumptions (G_1)–(G_4) imply that there are positive constants $a_i \leqslant \overline{a}_i \quad (i = 1, 2)$ such that

$$a_1 |z|^\beta \leqslant G(x, z) \leqslant \overline{a}_1 |z|^{1+\nu} \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad |z| \leqslant 1, \tag{4.1}$$

$$a_2 |z|^{\overline{\beta}} \leqslant G(x, z) \leqslant \overline{a}_2 |z|^\beta \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad |z| \geqslant 1. \tag{4.2}$$

Clearly (4.1) implies $1+\nu \leqslant \beta$. Note also that $1+\nu > (2N)/(2-\alpha+N)$ by (G_2).

Hence by Lemma 2.2, J is well-defined and $J \in C^1(\mathbb{E}, \mathbb{R})$,

$$\nabla J(z)\varphi = \int_{\mathbb{R}^N} G_z(x,z)\varphi dx, \quad \forall z, \varphi \in \mathbb{E}.$$

Moreover J is weakly continuous and ∇J is compact. We only show that J is weakly continuous. Let $z_n \in \mathbb{E}$ be such that $z_n \longrightarrow z$ weakly in $\mathbb{E}$. By definition

$$|J(z_n) - J(z)| = \left| \int_{\mathbb{R}^N} (G(x,z_n) - G(x,z)) \right|.$$

Note that, by (G_2) and (G_4)

$$|G(x,z)| \leqslant \bar{a}_2 |z|^{1+\nu} + c|z|^2. \tag{4.3}$$

For any $R > 0$, it follows from (4.3) and the Hölder inequality that

$$\left| \int_{|x| \geqslant R} (G(x,z_n) - G(x,z)) \right| \leqslant c \int_{|x| \geqslant R} \left(|z_n|^{1+\nu} + |z|^{1+\nu} + |z_n|^2 + |z|^2 \right).$$

By Lemma 2.2, for any $\varepsilon > 0$ one can take R_0 large such that

$$\left| \int_{|x| \geqslant R_0} (G(x,z_n) - G(x,z)) \right| < \frac{\varepsilon}{2}. \tag{4.4}$$

It is known that the functional

$$\int_{|x| < R_0} G(x,z) \in C^1(W^{1,2}(B_{R_0}, \mathbb{R}^2), \mathbb{R}),$$

and it is weakly continuous. Therefore there exists n_0 such that

$$\left| \int_{|x| < R_0} (G(x,z_n) - G(x,z)) \right| < \frac{\varepsilon}{2} \quad \forall n \geqslant n_0$$

which, together with (4.4), yields

$$|J(z_n) - J(z)| < \varepsilon \quad \forall n \geqslant n_0.$$

Hence J is weakly continuous and ∇J is compact.

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

STEP 2. By step 1, $f \in C^1(\mathbb{E}, \mathbb{R})$. Similarly to the previous section, one can easily check that any nontrivial critical point z of f on $\mathbb{E}$ is an entire solution of $(\mathrm{ES})_1$ with $z \in W^{1,2}$, since $\mathbb{E} \subset W^{1,2}$. We shall verify that f satisfies the all assumptions of Proposition 4.1.

STEP 3. f satisfies the condition(PS)*. Let $(z_n) \subset \mathbb{E}$ with $z_n \in \mathbb{E}_n$ be such that

$$f(z_n) \leqslant \text{const}, \quad \varepsilon_n \equiv \|\nabla f_n(z_n)\| \longrightarrow 0. \tag{4.5}$$

Then by (G_1)

$$\begin{aligned} f_n(z_n) - \frac{1}{2}\nabla f_n(z_n)z_n &= J(z_n) - \frac{1}{2}\nabla J(z_n)z_n \\ &= \int_{\mathbb{R}^N} G(x, z_n) - \frac{1}{2}G_z(x, z_n)z_n \\ &\geqslant \left(1 - \frac{\beta}{2}\right)\int_{\mathbb{R}^N} G(x, z_n). \end{aligned} \tag{4.6}$$

For any $z \in \mathbb{E}$, we write

$$z^1(x) = \begin{cases} z(x) & \text{if} \quad |z(x)| < 1 \\ 0 & \text{if} \quad |z(x)| \geqslant 1, \end{cases} \qquad z^2(x) = \begin{cases} 0 & \text{if}\, |z(x)| < 1 \\ z(x) & \text{if}\, |z(x)| \geqslant 1. \end{cases}$$

Then (4.1), (4.2), (4.5), (4.6) imply

$$c(1 + \|z_n\|) \geqslant \|z_n^1\|_{L^\beta}^\beta + \|z_n^2\|_{L^{\overline{\beta}}}^{\overline{\beta}}, \quad \forall n. \tag{4.7}$$

Note that since $\dim \mathbb{E}^0 < \infty$, by the Hölder inequality,

$$\|z_n^0\|_{L^2}^2 = (z_n^0, z_n^1)_{L^2} + (z_n^0, z_n^2)_{L^2} \leqslant c\,\|z_n^0\|_{L^2}\left(\|z_n^1\|_{L^\beta} + \|z_n^2\|_{L^{\overline{\beta}}}\right), \tag{4.8}$$

which, together with (4.7), shows

$$\|z_n^0\| \leqslant c\left(1 + \|z_n\|^{1/\beta} + \|z_n\|^{1/\overline{\beta}}\right). \tag{4.9}$$

Let b be the constant such that

$$\|z\|_{L^2}^2 \leqslant b\,\|z\|^2 \quad \forall z \in \mathbb{E}$$

by Lemma 2.2. Let $\delta = 1/(2b)$. By (G_2) and (G_4),

$$|G_z(x, z)| \leqslant \delta\,|z| + c\,|z|^{\overline{\beta}-1} \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad |z| \geqslant 1. \tag{4.10}$$

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

Now by (G_2) and (4.10)

$$\left\|z_n^+\right\|^2 = \int_{\mathbb{R}^N} G_z(x, z_n)z_n^+ - \nabla f(z_n)z_n^+$$
$$\leqslant \int_{\mathbb{R}^N} G_z\left(x, z_n^1\right)z_n^+ + \int_{\mathbb{R}^N} G_z\left(x, z_n^2\right)z_n^+ + \varepsilon_n \left\|z_n^+\right\|$$
$$\leqslant c[\left\|z_n^1\right\|_{L^{1+\nu}}^{\nu} + \left\|z_n^2\right\|_{L^{\overline{\beta}}}^{\overline{\beta}-1}] \left\|z_n^+\right\| + \frac{1}{2}\left\|z_n\right\| \left\|z_n^+\right\| + \varepsilon_n \left\|z_n^+\right\|.$$

Clearly, there exists a similar estimate for (z_n^-). Therefore, by (4.7)

$$\left\|z_n^\pm\right\| \leqslant c\Big(1 + \|z_n\|^{\nu/(1+\nu)} + \|z_n\|^{(\overline{\beta}-1)/(\overline{\beta})}\Big). \tag{4.11}$$

Combining (4.9) and (4.11), we have

$$\|z_n\| \leqslant c\Big(1 + \|z_n\|^{1/\beta} + \|z_n\|^{1/\overline{\beta}} + \|z_n\|^{\nu/(1+\nu)} + \|z_n\|^{(\overline{\beta}-1)/(\overline{\beta})}\Big). \tag{4.12}$$

Therefore $\|z_n\|$ is bounded, and by Lemma 2.2, without loss of generality we can suppose that $z_n \longrightarrow z$ weakly in $\mathbb{E}$. Since ∇J is compact, dim $\mathbb{E}^0 < \infty$, and for any $n, m \in \mathbb{N}$

$$\left\|z_n^\pm - z_m^\pm\right\|^2 \leqslant (\varepsilon_n + \varepsilon_m)\left\|z_n^\pm - z_m^\pm\right\| + \|\nabla J(z_n) - \nabla J(z_m)\| \left\|z_n^\pm - z_m^\pm\right\|,$$

one sees that (z_n) has a Cauchy subsequence. This proves that f satisfies $(\mathrm{PS})^*$.

STEP 4. f satisfies (f_2). Choose $e \in \mathbb{E}_1^+$ with $\|e\| = 1$ and $X = \mathbb{E}^- \oplus \mathbb{E}^0 \oplus Re$. For $z = z^- + z^0 + se \in X$,

$$f(z) = \int_{\mathbb{R}^N} G(x, z) - \frac{1}{2}s^2 + \frac{1}{2}\left\|z^-\right\|^2. \tag{4.13}$$

Similarly to (4.8), we have

$$\left\|z^0 + se\right\| \leqslant c\Big(\left\|z^1\right\|_{L^\beta} + \left\|z^2\right\|_{L^{\overline{\beta}}}\Big).$$

Hence by (4.1) and (4.2), there is $\underline{b} > 0$ such that

(i) $\underline{b}\Big(\left\|z^0\right\|^{\overline{\beta}} + s^{\overline{\beta}}\Big) \leqslant \int_{\mathbb{R}^N} G(x, z)$ if $\left\|z^1\right\|_{L^\beta} \leqslant \left\|z^2\right\|_{L^{\overline{\beta}}}$, or

(ii) $\underline{b}\Big(\left\|z^0\right\|^{\beta} + s^{\beta}\Big) \leqslant \int_{\mathbb{R}^N} G(x, z)$ if $\left\|z^1\right\|_{L^\beta} > \left\|z^2\right\|_{L^{\overline{\beta}}}$.

Therefore

$$f(z) \geqslant \begin{cases} \Big(\underline{b} - \frac{1}{2}s^{2-\overline{\beta}}\Big)s^{\overline{\beta}} + \underline{b}\Big(\left\|z^0\right\|^{\overline{\beta}} + \left\|z^-\right\|^2\Big) & \text{if (i)} \\ \big(\underline{b} - \frac{1}{2}s^{2-\beta}\big)s^{\beta} + \underline{b}\Big(\left\|z^0\right\|^{\beta} + \left\|z^-\right\|^2\Big) & \text{if (ii)} \end{cases}$$

and so one can take $s_0 > 0$ small, such that

$$f(z) \geqslant \sigma > 0 \quad \text{for all} \quad z \in S,$$

where $S = \mathbb{E}^- \oplus \mathbb{E} + s_o e \equiv \mathbb{E}^- \oplus \mathbb{E}^0 + v$.

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press

STEP 5. For any $z \in \mathbb{E}^+$, by (4.1) and (4.2)

$$f(z) = \int_{\mathbb{R}^N} G(x,z) - \frac{1}{2}\|z^2\|$$
$$\leqslant c\int |z|^\beta - \frac{1}{2}\|z\|^2 \leqslant c\Big(\|z\|^\beta - \|z\|^2\Big) \longrightarrow -\infty$$

as $\|z\| \longrightarrow \infty$ since $\beta < 2$. One can take $\rho > s_0$ such that

$$f|_{\partial Q} \leqslant 0 \quad \text{for all} \quad z \in \partial Q,$$
$$f \leqslant M \quad \text{for all} \quad z \in Q$$

where $Q \equiv B_\rho \cap \mathbb{E}^+$.

STEP 6. From Proposition 4.1 we immediately get Theorem 1.2. □

REMARK. Concerning $(\mathrm{ES})_2$, it is easy to see that if q_1 and q_2 satisfy (Q_α) and G satisfies (G_1)–(G_4), then $(\mathrm{ES})_2$ possesses at least one nontrivial $W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)$ solution.

REFERENCES

[1] V. Benci and P.H. Rabinowitz, 'Critical point throrems for indefinite functional', *Invent. Math.* **52** (1979), 241–273.

[2] Ph. Clément, D.G. de Figueiredo and E. Mitidieri, 'Positive solutions of semilinear elliptic systems', (preprint).

[3] Y.H. Ding, 'Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems', *Nonlinear Anal.* (to appear).

[4] D.G. de Figueiredo and P.L. Felmer, 'On superquadratic elliptic systems', (preprint).

[5] T. Kato, *Perturbation theory for linear operators* (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[6] M.A. Krasnoselski, *Topological methods in the theory of nonlinear integral equations* (Macmillan, New York, 1964).

[7] M. Reed and B. Simon, *Methods of modern mathematical physics: IV, Analysis of Operators* (Academic Press, 1978).

[8] A. Szulkin, 'Bifurcation for strongly indefinite functionals and Liapunov type theorem for Hamiltonian systems', *Differential Integral Equations* **7** (1994), 217–234.

Institute of Mathematics
Academia Sinica
100080 Beijing
People's Republic of China

International Centre for
Theoretical Physics
Trieste
Italy

https://doi.org/10.1017/S0004972700013605 Published online by Cambridge University Press