Canad. Math. Bull. Vol. **18** (1), 1975

ON EXTENSIONS OF THE RIEMANN AND LEBESGUE INTEGRALS BY NETS

BY
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1. **Introduction.** In this note our principal interest is in using nets to give spaces of non-absolutely convergent integrals as extensions of the spaces of absolutely convergent Riemann and Lebesgue integrals. For this purpose we develop a general theory of extensions, by nets, of functions defined on the open intervals with closures in the complement of a fixed closed set, the nets being directed by inclusion for finite disjoint collections of such intervals. Two cases are considered leading to open extension (OE-) and conditional open extension (COE-) nets, the latter being subnets of the former. Necessary and sufficient conditions for the convergence of the OE- and COE-nets are given, those for the COE-nets being similar to conditions that arise in the definition of the restricted Denjoy integral. Properties of inner continuity, weak additivity and the existence of a continuous integral are defined and studied. These relate to the more specialized nets that are suitable for the extension of integrals.

When applied to the Riemann and Lebesgue integrals the OE-extensions do not lead beyond the Lebesgue integral. On the other hand the COE-extensions lead to scales of nonabsolutely converging Cauchy-Riemann and Cauchy-Lebesgue integrals.

We use the definitions of directed set, net, subnet, etc. as given in ([2], Chapter 2). In particular we note

(N1) If $(S, \geq)=(S_n, D, \geq)$ is a net in $\mathbb{R}$ (with the usual topology) then $(S, \geq)$ converges to α if and only if the net is eventually in every interval $(\alpha-\delta, \alpha+\delta)$, $\delta>0$.

(N2) If $D'\subset D$ and N is the identity map from D' into D then $(S\circ N, D', \geq)$ is a subnet of $(S, D, \geq)$ if and only if for each n in D there exists n' in D' with $n'\geq n$. This implies that if the net $(S, D, \geq)$ converges to α then every subnet $(S, N, D', \geq)$ also converges to α.

(N3) ([2], p. 77). A net $(S_n, D, \geq)$ in $\mathbb{R}$ is monotone increasing (decreasing) if $m\geq n$ implies that $S_m\geq S_n (S_m\leq S_n)$. A monotone increasing (decreasing) net converges to the supremum (infimum) of its range (in $\overline{\mathbb{R}}$).

[1] Partially supported by the National Research Council of Canada under Grant No. A3007.
Received by the editors November 17, 1972 and, in revised form, December 5, 1973.

https://doi.org/10.4153/CMB-1975-002-1 Published online by Cambridge University Press

2. **Open extension nets.** We consider below nets directed by inclusion for open sets. Let

$I=(a, b), -\infty \le a < b \le \infty$;

E denote a closed set in $\mathbb{R}$;

$\mathscr{I}(\tilde{E} \cap I)$ the set of all finite open intervals J with closures $\bar{J} \subset \tilde{E} \cap I$ (compact closure in $\tilde{E} \cap I$);

V_a a finite union of disjoint intervals in $\mathscr{I}(\tilde{E} \cap I)$;

F a function from $\mathscr{I}(\tilde{E} \cap I)$ into $\mathbb{R}$, extended additively to the unions V_a, and

$V(E)=V(E, I)$ the set of all collections V_a directed by inclusion. Then $(F, V(E, I), \le)$ is a net in $\mathbb{R}$. We call it an OE-(open extension) net or more precisely the OE-net associated with I, E and F. We shall usually abbreviate the notation for an OE-net to (F, E, I) below.

We shall use the notation $\tilde{E} \cap I = \cup_1^\infty I_i$, $I_i=(a_i, b_i)$ for the canonical expression of the open set $\tilde{E} \cap I$ as a union of disjoint open intervals. (If $\tilde{E} \cap I$ is a finite union, $\cup^n I_i$ say, then in our notation we assume that $I_i=\varnothing$, $i>n$.)

If $V^*(E, I)$ denotes the set of $V_a \in V(E, I)$ with at most one interval in each I_i, then $(F, V^*(E, I), \le)$ is a net in $\mathbb{R}$. It will be called a conditional open extension net (COE-net), the COE-net associated with I, E and F. This will usually be abbreviated to $(F, E, I)^*$. It is easy to verify that $(F, E, I)^*$ is a subnet of (F, E, I)

If $(F, V'(E, I), \le)$ is an OE-net or subnet of an OE-net we write

$$\bar{F}(\tilde{E} \cap I) = \lim(F, V'(E, I), \le),$$

when the limit exists in the extended reals. When (F, E, I) converges in $\mathbb{R}$, and in particular when (F, E, I) is monotone,

$$\bar{F}(\tilde{E} \cap I) = \lim(F, E, I) = \lim(F, E, I)^*. \tag{2.1}$$

We shall call a net or subnet convergent if the limit is finite.

For $I' \subset \tilde{E} \cap I$ we define

$$\omega_F(I') = \sup\{|F(J)|, J \subset I', J \in \mathscr{I}(\tilde{E} \cap I)\}.$$

THEOREM 2.1. *Let* $(F, E, I)^*$ *be a convergent* COE-*net. Then*

(i) $(F, E, I_i)^*$ *converges for each* i;
(ii) $\sum_N^\infty \omega_F(I_i) < \infty$ *for* N *sufficiently large; and*
(iii) $\bar{F}(\tilde{E} \cap I) = \sum_1^\infty \bar{F}(I_i) = \lim(F, E, I)^*$.

Conversely if $(F, E, I)^*$ *is a* COE-*net for which* (i) *and* (ii) *hold then it is convergent and* (iii) *holds.*

Proof. We first note that for a COE-net and each i, $V^*(E, I_i)$ is the set of finite open intervals with closures in I_i. Thus (i) is equivalent to

(i′) $\lim_{\alpha \to a_i+, \beta \to b_i-} F(\alpha, \beta)$ exists for each i.

Assume that $(F, E, I)^*$ converges to k but (i') fails to hold for I_i. There then

https://doi.org/10.4153/CMB-1975-002-1 Published online by Cambridge University Press

exists $d>0$ with

$$\limsup_{\substack{\alpha\to a_i^+\\ \beta\to b_i^-}} F(\alpha,\beta)-\liminf_{\substack{\alpha\to a_i^+\\ \beta\to b_i^-}} F(\alpha,\beta) > d: \tag{2.2}$$

This implies that the net $(F, E, I)^*$ cannot eventually be in $(k-\delta, k+\delta)$ if $\delta<d/2$ giving a contradiction.

Assume next that $\sum_N^\infty \omega_F(I_i)=\infty$ for every $N\in\mathbb{N}$. There is then an infinite subsequence $\{i_n\}$ with

$$F(J_{i_n}) > \omega_F(I_{i_n})/2,\ J_{i_n}\subset I_{i_n},\ J_{i_n}\in\mathscr{I}(\tilde{E}\cap I)$$

and

$$\sum_{i_n>N}^{\infty} \omega_F(I_{i_n}) = \infty \quad \text{for every} \quad N\in N$$

and/or such a subsequence with $-F(J_{i_n})>\omega_F(I_{i_n})/2$ for each i_n. It is then easy to establish the existence of subnets with limits $+\infty$ and/or $-\infty$ contradicting the convergence of $(F, E, I)^*$.

Assume that (i') and (ii) hold and let $k=\sum_1^\infty \bar{F}(I_i)$. There then exists N_0 corresponding to $\varepsilon>0$ with

$$\sum_{N_0+1}^{\infty} \omega_F(I_i) < \varepsilon/3,$$

and intervals $J'\subset I_i$, $J'\in\mathscr{I}(\tilde{E}\cap I)$ with

$$\sum_1^{N_0} |\bar{F}(I_i)-F(J_i)| < \varepsilon/3,\ J_i'\subset J_i\subset I_i,\ J_i\in\mathscr{I}(\tilde{E}\cap I).$$

Then if $V_{a'}=\{J':i=1, 2, \ldots, N_0\}$; $V_{a'}\in V^*(E, I)$ and a standard calculation shows that $V_a\in(k-\varepsilon, k+\varepsilon)$ if $V_{a'}\leq V_a$. The proof that (i') and (ii) imply (iii) implies the converse part of the theorem.

Definition. For $I'\subset\tilde{E}\cap I$ define

$$V_F(I') = \sup\left\{\sum_{i=1}^{n} |F(J_i)|,\ J_i\subset I',\ J_i\in\mathscr{I}(\tilde{E}\cap I)\right\}.$$

Theorem 2.2. *Let (F, E, I) be a convergent* OE-*net. Then*

(i) *(F, E, I_i) converges for each i;*
(ii) *$\sum^\infty V_F(I_i)<\infty$ for N sufficiently large;*
(iii) *$\bar{F}(\tilde{E}\cap I)=\sum_1^\infty \bar{F}(I_i)=\lim(F, E, I)$.*

Conversely if (F, E, I) is an OE-*net in R and* (i) *and* (ii) *hold, then (F, E, I) converges to $\sum_1^\infty \bar{F}(I_i)$.*

Proof. We first note that (i) is equivalent to

(i') for each i, $\lim_{a\to a_i^+,\beta\to b_i^-} F(\alpha, \beta)$ exists in R and
(ii') $\lim_{\alpha\to a_i^+} V_F(a_i, \alpha)=\lim_{\beta\to b_i^-} V_F(\beta, b_i)=0$.

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Assuming that (i) holds, $(F, E, I_i)^*$ converges to $\bar{F}(I_i)=\lim(F, E, I_i)$ by (N2) and (i′) is then a consequence of (i′) in Theorem 2.1.

The assumption that $V_F(a_i, \alpha)$ does not approach zero as $\alpha\to a^+$ leads to a contradiction of the assumption that (F, E, I_i) converges by an argument similar to that in the first part of Theorem 2.1. Thus (i) implies both (i′) and (ii′).

Assume that (i′) holds and let $k=\lim(F, E, I_i)^*$. Then, given $\varepsilon>0$, we can fix (α', β') in I_i with

$$|F(\alpha, \beta)-k| < \varepsilon/2;\ a_i < \alpha \le \alpha' < \beta' \le \beta < b_i;$$

$$V_F(a_i, \alpha')+V_F(\beta', b_i) < \varepsilon/2.$$

Let $V_{a'}=(\alpha', \beta')$. Then $V_{a'} \in V(E, I_i)$ and if $V_{a'}\le V_a$, V_a contains an interval (α, β) with $(\alpha, \beta)\supset(\alpha', \beta')$. If V_a^* denotes V_a less the interval (α, β),

$$|F(V_a)-k| = |F(\alpha, \beta)+F(V_a^*)-k| < \varepsilon.$$

The proof of Theorem 2.2 is then completed by arguments similar to those used for Theorem 2.1.

We note that (ii) of Theorem 2.2 implies (ii) of Theorem 2.1 but the converse is not true. Note that (i′) and (ii′) for Theorem 2.2 are independent in the present context. The following example shows that (ii′) does not imply (i′).

EXAMPLE 2.1. Let $I=(-\infty, \infty)$, $E=\varnothing$; $f(x)=e^{-x^2}$; $g(x)=e^x$; $F(\alpha, \beta)=g(\beta)-g(\alpha)$ if $0 \in (\alpha, \beta)$; $=f(\beta)-f(\alpha)$ if $0 \notin (\alpha, \beta)$. Then $V_F(-\infty, \alpha)=e^{-\alpha^2}\to 0$ as $\alpha\to-\infty$, $V_F(\beta, \infty)=e^{-\beta^2}\to 0$ as $\beta\to\infty$ but $F(\alpha, \beta)\to\infty$ as $\alpha\to-\infty$, $\beta\to\infty$.

REMARK 2.1. Assume that for every pair of adjacent intervals (α_1, α_2), (α_2, α_3) of $\mathscr{I}(\tilde{E} \cap I)$, contained in I_i with $(\alpha_1, \alpha_3) \in \mathscr{I}(\tilde{E} \cap I)$,

$$F(\alpha_1, \alpha_3) = F(\alpha_1, \alpha_2)+F(\alpha_2, \alpha_3) \tag{2.3}$$

Then it can be shown that, if $\lim(V_F(a_i, \alpha);\ \alpha\to a_i^+)=0=\lim(V_F(\beta, b_i);\ \beta\to b_i^-)$, then $\lim(F(\alpha, \beta): \alpha\to a_i^+, \beta\to b_i^-)$ exists.

The convergence of (F, E, I) or $(F, E, I)^*$ implies the convergence of (F, E, I_i) $((F, E, I_i)^*)$ for every i. However if I' is an arbitrary open interval in $\tilde{E} \cap I$, $(F, E, I')((F, E, I')^*)$ may fail to converge even when $I' \in \mathscr{I}(\tilde{E} \cap I)$ and, for $I' \in \mathscr{I}(\tilde{E} \cap I)$, may converge but to a value different from $F(I')$.

EXAMPLE 2.2. $I=(-1, 1)$, $E=\varnothing$, $f(x)=1/x$, $x\ne 0$, $f(0)=0$; $F(\alpha, \beta)=f(\beta)-f(\alpha)$, $(\alpha, \beta) \in \mathscr{I}(I)$. Then (F, E, I) and $(F, E, I)^*$ converge to 2 whereas (F, E, I') and $(F, E, I')^*$ do not converge if $I'=(-a, 0)$ or $(0, a)$, $0<a<1$.

EXAMPLE 2.3. $I=(-1, 1)$, $E=\varnothing$, $f(x)=0$, $x<0$, $=1$, $x\ge 0$; $F(\alpha, \beta)=f(\beta)-f(\alpha)$. Then (F, E, I) and $(F, E, I)^*$ converge to 1 and converge for every $I' \in \mathscr{I}(\tilde{E} \cap I)$ but $\bar{F}(-a, 0)=0\ne F(-a, 0)=1$, $0<a\le 1$.

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DEFINITION. We call F *inner continuous on* $\mathscr{I}(\tilde{E} \cap I)$ if, for every $J=(a', b') \in \mathscr{I}(\tilde{E} \cap I)$,

$$F(J) = \lim_{\substack{\alpha \to a'^+ \\ \beta \to b'^-}} F(\alpha, \beta) \tag{2.4}$$

Noting that $J \in \mathscr{I}(\tilde{E} \cap I)$ implies that $\tilde{E} \cap J=J$, (ii) in Theorem 2.1 is trivially satisfied and $(F, E, J)^*$ converges if and only if (i′) holds. Thus in order that *$(F, E, J)^*$ converge to $F(J)$ for every $J \in \mathscr{I}(\tilde{E} \cap I)$ it is necessary and sufficient that F be inner continuous.*

If F is an arbitrary function on $\mathscr{I}(\tilde{E} \cap I)$, F is trivially finitely additive on $\mathscr{I}(\tilde{E} \cap I)$ since no open interval can be expressed as a union of two or more disjoint open intervals.

EXAMPLE 2.4. Let μ be an arbitrary finitely additive measure on the Borel subsets of $\mathbb{R}$. For arbitrary E and I define $F(J)=\mu(J)$, $J \in \mathscr{I}(\tilde{E} \cap I)$. If $J=(a', b')$ and $a'<c<b'$ then for every α, β, $a' \leq \alpha < c < \beta \leq b'$,

$$F(\alpha, \beta) = \mu(\alpha, \beta) = \mu(\alpha, c)+\mu(c, \beta)+\mu\{c\}.$$

This motivates the following

DEFINITION We say that F is *weakly additive* on $\mathscr{I}(\tilde{E} \cap I)$ if there exists a function k on $\tilde{E} \cap I$ such that for each $c \in \tilde{E} \cap I=\bigcup_1^\infty I_i$ and each i,

$$F(\alpha, \beta) = F(\alpha, c)+F(c, \beta)+k(c) \tag{2.5}$$

for every (α, β) with $a_i<\alpha<c<\beta<b_i$.

PROPOSITION 2.1. *Assume that F is inner continuous and weakly additive on $\mathscr{I}(\tilde{E} \cap I)=\mathscr{I}(I)$, $I=(a, b)$. Then if $(F, E, I)^*$ converges, $(F, E, (a, c))^*$ and $(F, E, (c, b))^*$ converge for every c, $a<c<b$, and* (2.5) *holds for $a \leq \alpha < c < \beta \leq b$.*

Proof. Since $(F, E, I)^*$ converges, $\bar{F}(a, b)=\lim_{\alpha\to a^+ \beta\to b^-} F(\alpha, \beta) \in \mathbb{R}$. For c fixed, (2.5) holds for $a<\alpha<c<\beta<b$ and, since α and β can be varied independently, it follows that $F(\alpha, c)$ and $F(c, \beta)$ converge as $\alpha \to a^+$, $\beta \to b^-$. Thus

$$\bar{F}(a, b) = \lim_{\alpha \to a^+} F(\alpha, c)+ \lim_{\beta \to b^-} F(c, \beta)+k(c) = K_1+K_2+k(c),$$

defining K_1 and K_2.

Fix c', $a<c'<c$. By the argument of the preceding paragraph $F(\alpha, c')$ and $F(c', \beta)$ converge as $\alpha \to a^+$, $\beta \to c^-$. Thus

$$\bar{F}(a, c) = \lim_{\substack{\alpha \to a^+ \\ \beta \to c^-}} F(\alpha, \beta) = \lim_{\alpha \to a^+} F(\alpha, c')+ \lim_{\beta \to c^-} F(c', \beta)+k(c'),$$

showing that $(F, E(a, c))^*$ converges. Similarly $F(c, b)=\lim(F, E, (c, b))^*$ exists.

Inner continuity can then be used to show that $K_1=F(a, c)$ and $K_2=F(c, b)$.

Let $\mathscr{I}^*(\tilde{E} \cap I)$ denote the set of *all* open intervals contained in $\tilde{E} \cap I$.

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COROLLARY 3.1. *Let F be inner continuous and weakly additive on $\mathscr{I}(\tilde{E} \cap I)$. Then if $(F, E, I)^*$ converges,* (i) *$\bar{F}$ is defined on $\mathscr{I}^*(\tilde{E} \cap I)$ and extends F from $\mathscr{I}(\tilde{E} \cap I)$ to $\mathscr{I}^*(\tilde{E} \cap I)$,* (ii) *where k is the function in the definition of weak additivity,* (2.5) *holds for every $(\alpha, \beta) \in \mathscr{I}^*(\tilde{E} \cap I)$ and* (iii) *$(F, E, I')^*$ converges for every open interval $I' \subset I$.*

To prove (iii) we note that $\mathscr{I}(\tilde{E} \cap I) \subset \mathscr{I}^*(\tilde{E} \cap I)$ and conditions (i) and (ii) of Theorem 2.1 follow from (i) above and Theorem 2.1 (ii) for $(F, E, I)^*$.

If f is an arbitrary real valued function on $\tilde{E} \cap I$, $F(\alpha, \beta)=f(\beta)-f(\alpha)$ defines a function on $\mathscr{I}(\tilde{E} \cap I)$. Conversely, given F on $\mathscr{I}(\tilde{E} \cap I)$ we can ask when there exists an f for which such a relation holds. We ask also that f be continuous.

DEFINITION. We say that F, defined on $(\tilde{E} \cap I)$ has a *continuous integral* on $\tilde{E} \cap I$ if there exists a continuous function $\mathbb{F}: \tilde{E} \cap I \to \mathbb{R}$ with

$$F(\alpha, \beta) = \mathbb{F}(\beta) - \mathbb{F}(\alpha)$$

for every $(\alpha, \beta) \in \mathscr{I}(\tilde{E} \cap I)$.

If both $\mathbb{F}_1$ and $\mathbb{F}_2$ are continuous integrals for F it is easy to verify that

$$\mathbb{F}_1 - \mathbb{F}_2 = \sum_i c_i \chi I_i.$$

If F has a continuous integral on $\tilde{E} \cap I$ then it is easily verified that F is inner continuous on $\mathscr{I}(\tilde{E} \cap I)$ and weakly additive with $k(x) \equiv 0$.

With each weakly additive F on $\mathscr{I}(\tilde{E} \cap I)$ we associate functions $\mathscr{F}$ as follows. In $I_i=(a_i, b_i)$ let $(\alpha_n, n \in \mathbb{N})$ be any sequence of points in I_i decreasing to a_i as limit. Define, in (α_n, b_i),

$$\begin{aligned} \mathscr{I}(x) &= F(\alpha_1, x), \\ &= F(\alpha_n, x) - \sum_2^n F(\alpha_i, \alpha_{i-1}) - \sum_1^{n-1} k(\alpha_i), \qquad n > 1. \end{aligned}$$

Note that if $x \in (\alpha_{n-1}, b_i)$, then using (2.5), it can be shown that

$$\mathscr{F}(x) = F(\alpha_{n-1}, x) - \sum_2^{n-1} F(\alpha_i, \alpha_{i-1}) - \sum_1^{n-2} k(\alpha_i) \tag{2.6}$$

so that $\mathscr{F}$ is defined inductively on all of I_i. A different choice of α_1 would change $\mathscr{F}$ on I_i by a constant function.

Given $x, y \in I_i$, with $x<y$, there exists $\alpha_n<x$ and

$$\mathscr{F}(y) - \mathscr{F}(x) = F(\alpha_n, y) - F(\alpha_n, x) = F(x, y) + k(x). \tag{2.7}$$

If there exists a continuous integral for F on I_i, $k(x) \equiv 0$ on I_i and $\mathscr{F}$ is a continuous integral for F on I_i.

If we assume that F is weakly additive and inner continuous on $\mathscr{I}(\tilde{E} \cap I)$ then, using Proposition 2.1 and keeping y fixed in (2.7), $\lim_{x \to y^-} F(\alpha_n, x) = F(\alpha_n, y)$ showing that $\mathscr{F}$ is left continuous on I_i. On the other hand, keeping x fixed and fixing α, $x<y<\alpha<b_i$,

$$\mathscr{F}(y) - \mathscr{F}(x) = F(x, y) + k(x) = F(x, \alpha) - F(y, \alpha) - k(y) + k(x).$$

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Since $F(y, \alpha) \to F(x, \alpha)$ as $y \to x^+$, $\lim_{y\to x^+} \mathscr{F}(y) = \mathscr{F}(x)$ if and only if k is right continuous at x. We have shown

PROPOSITION 2.2. *If F is weakly additive and inner continuous on $\mathscr{I}(\tilde{E} \cap I)$ then $\mathscr{F}$ is a continuous integral on $\tilde{E} \cap I$ if and only if k is right continuous on $\tilde{E} \cap I$.*

EXAMPLE 2.5. Let f and g be real valued functions on $\tilde{E} \cap I$. For each $(\alpha, \beta) \in \mathscr{I}(\tilde{E} \cap I)$ define $F(\alpha, \beta) = f(\alpha) + g(\beta)$. Then $k(x) = -f(x) - g(x)$ and F is weakly additive but cannot have a continuous integral unless $f(x) + g(x) = 0$ in $\tilde{E} \cap I$. Furthermore F is inner continuous if and only if f is right continuous, g left continuous.

Let g be continuous, f right continuous but not continuous on $\tilde{E} \cap I$. Then F is inner continuous and weakly additive and $\mathscr{F}$ is continuous on $\tilde{E} \cap I$ but F does not have a continuous integral on $\tilde{E} \cap I$.

It can be shown that if F is weakly additive on $\mathscr{I}(I)$ and $V_F(I) < \infty$ then $\sum_{x\in I} |k(x)| = \sup\{\sum_{i=1}^n k(x_i);\ x_i \in I\} < \infty$. This implies that, if $\mathscr{F}$ is continuous on $\tilde{E} \cap I$ and every $J' \in \mathscr{I}(\tilde{E} \cap I)$ contains a subinterval J with $V_F(J) < \infty$ then $k(x) \equiv 0$ and $\mathscr{F}$ is a continuous integral on $\tilde{E} \cap I$ [1].

PROPOSITION 2.3. *Let F have a continuous integral on $\tilde{E} \cap I$. Then if $(F, E, I)^*$ converges, $\omega_F(I_i) < \infty$ for every i and Theorem* 2.1 (ii) *can be replaced by*

$$\text{(ii)}' \qquad \sum_1^\infty \omega_F(I_i) < \infty.$$

If G is a continuous integral for F and $x_0 \in I_i$, $a_i < \alpha < x_0 < \beta < b_i$, then $G(\alpha) = G(x_0) - F(\alpha, x_0)$. It follows as in Proposition 2.1 that $F(\alpha, x_0)$ and thus also $G(\alpha)$ converge as $\alpha \to a_i^+$. Similarly $\lim_{\beta\to b_i^-} G(\beta)$ exists. Thus G is bounded and $\omega_F(I_i)$ is finite for each I_i and (ii′) follows.

Let A and B be closed sets with $B \subset A$. Let $\tilde{B} \cap I = \cup_i I_i$, $\tilde{A} \cap I = \cup_i \cup_j I_{ij}$ where $\tilde{A} \cap I \cap I_i = \cup_j I_{ij}$.

THEOREM 2.3. *Let F be inner continuous and weakly additive on $\mathscr{I}(\tilde{A} \cap I)$ and assume that $(F, A, I)^*$ converges. Let $G_A : \mathscr{I}(\tilde{B} \cap I) \to \mathbb{R}$ and define*

$$H(J) = G_A(J) + \bar{F}(\tilde{A} \cap J),\ J \in \mathscr{I}(\tilde{B} \cap I). \tag{2.8}$$

Then (i) *if $(G_A, B, I)^*$ converges, $(H, B, I)^*$ converges and*

$$H(B \cap I) = \bar{G}_A(\tilde{B} \cap I) + \bar{F}(\tilde{A} \cap I); \tag{2.9}$$

(ii) *if G_A is inner continuous, then H is inner continuous;*

(iii) *if G_A is weakly additive, then H is weakly additive; and*

(iv) *if F has a continuous integral on $\tilde{A} \cap I$ and G_A has a continuous integral on $\tilde{B} \cap I$, then H has a continuous integral on $\tilde{B} \cap I$.*

Proof. If $J \in \mathscr{I}(\tilde{B} \cap I)$, then $J \subset I$ and $(F, A, J)^*$ converges by Corollary 2.1 of Proposition 2.1. By definition $\bar{F}(\tilde{A} \cap J) = \lim(F, A, J)^*$. It follows that $H : \mathscr{I}(\tilde{B} \cap I) \to \mathbb{R}$.

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If $J=(\alpha, \beta) \in \mathscr{I}(\tilde{B} \cap I)$, then using Theorem 2.1 (iii), (2.8) becomes

$$(2.10) \qquad H(J) = G_A(J) + \sum_{I_{ij} \subset J} \bar{F}(I_{ij}) + \bar{F}(I') + \bar{F}(I''),$$

where I', I'' are partial intervals of intervals I_{ij} arising when α and/or β are in $\tilde{A} \cap I$.

We verify that (i') holds for each I_i. First assume that a_i and b_i are not limit points of $\tilde{A} \cap I_i$. Then for all α, β near a_i and b_i respectively, $\sum_{I_{ij} \subset J} \bar{F}(I_{ij})$ is constant and if $I'=(\alpha, b_{il})$, $I''=(a_{in'}, \beta)$,

$$\bar{F}(I') \to \bar{F}(a_i, b_{il}),\ \bar{F}(I'') \to \bar{F}(a_{in'}, b_i) \quad \text{as} \quad \alpha \to a_i^+,\ \beta \to b_i^-,$$

using Proposition 2.1. Since $G_A(\alpha, \beta) \to G_A(I_i)$ by Theorem 2.1, (i') holds for H and

$$\bar{H}(I_i) = \lim_{\alpha \to a_i^+, \beta \to b_i^-} H(J) = \bar{G}_A(I_i) + \sum_j \bar{F}(I_{ij}) = \bar{G}_A(I_i) + \bar{F}(\tilde{A} \cap I_i).$$

Next suppose that a_i and/or b_i is a limit point of $\tilde{A} \cap I_i$. Then as $\alpha \to a^+$, $\beta \to b^-$, $\sum_{I_{ij} \subset J} \bar{F}(I_{ij})$ does not remain constant but Theorem 2.1 (ii) implies that it differs from $\sum_j \bar{F}(I_{ij})$ by less than $\sum_k^\infty \omega_F(I_{ij})$ for some k where $k \to \infty$ as $\alpha \to a_i^+$, $\beta \to b^-$. It then follows that (2.9) holds.

Now

$$\omega_H(I_i) = \sup_{J \in \mathscr{I}(I_i)} |H(J)| = \sup_{J \in \mathscr{I}(I_i)} \left|G_A(J) + \sum_j \bar{F}(J \cap I_{ij})\right| < \omega_{G_B}(I_i) + \sum_j \omega_F(I_{ij});$$

$$\sum_N^\infty \omega_H(I_i) < \sum_N^\infty \omega_{G_B}(I_i) + \sum_{i=N}^\infty \sum_j \omega_F(I_{ij})$$

and (ii) of Theorem 2.1 for F and G_A implies that this is finite for N sufficiently large giving (ii) for $(H, B, I)^*$. Theorem 2.1 then implies that $(H, B, I)^*$ converges and (2.9) holds.

(ii) Let $J=(a', b') \in \mathscr{I}(\tilde{B} \cap I)$. Then $J \subset I_i$ for some i and, using arguments similar to those in (i) above, $\bar{F}(\tilde{A} \cap (\alpha, \beta)) \to \bar{F}(\tilde{A} \cap J)$ as $\alpha \to a'^+$, $\beta \to b'^-$. The assumption that G_A is inner continuous and (2.8) above imply that $H(\alpha, \beta) \to H(J)$ as $\alpha \to a_i^+$, $\beta \to b^-$.

(iii) The assumption that G_A is weakly additive on $\mathscr{I}(\tilde{B} \cap I)$ implies that there exists k_A on $\tilde{B} \cap I$ for which (2.5) holds with the present notation. The assumption that F is weakly additive on $\mathscr{I}(\tilde{A} \cap I)$ implies the existence of k on $\tilde{A} \cap I$ for which (2.5) holds for F. We extend k to $\tilde{B} \cap I$ by defining $k(x)=0$, $x \in A \cap \tilde{B} \cap I$. Using (2.10) it is then easy to verify that H is weakly additive on $\mathscr{I}(\tilde{B} \cap I)$ with $k_H(x)=k_A(x)+k(x)$, $x \in \tilde{B} \cap I$.

(iv) If G_A has a continuous integral it is weakly additive and inner continuous. Thus H is weakly additive and inner continuous by (ii) and (iii) above. Since the existence of continuous integrals for F and G_A imply that $k_A(x)=k(x)=0$, $x \in \tilde{B} \cap I$, $k_H(x)=0$ in $\tilde{B} \cap I$ and the point function $\mathscr{H}$ associated with H on $\tilde{B} \cap I$ is continuous on $\tilde{B} \cap I$ by Proposition 2.2. It follows from (2.8) that $\mathscr{H}$ is a continuous integral for H on $\tilde{B} \cap I$.

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3. **Applications.** Let $I=(a, b)$, $-\infty \leq a < b \leq \infty$. Let $\mathbf{L}(A)$ denote the set of Lebesgue integrable functions on A. A function f will be called locally Lebesgue integrable at x if there is a neighborhood of x over which f is Lebesgue integrable. Denote by $E_L=E_L(f)$ the points of $[a, b]$ at which f is not locally Lebesgue integrable. If $x \in \tilde{E}_L$ there exists $I_\delta=(x-\delta, x+\delta)$ with $f \in \mathbf{L}(I_\delta)$. Since $I_\delta \subset \tilde{E}_L$, E_L is closed. Corresponding definitions will be assumed for other types of integrals.

Let $f: I \to \mathbb{R}$ be arbitrary and define $F(G)=\int_J f\, d\mu$ on $\mathscr{I}(\tilde{E}_L \cap I)$, where μ denotes Lebesgue measure and the integral is the Lebesgue integral. Then (F, E_L, I) is an OE-net. Noting that $V_F(I')=\int_{I'} |f|\, d\mu$ if $I' \subset \tilde{E}_L \cap I$, Theorem 2.2 implies that (F, E_L, I) converges if and only if $f \in \mathbf{L}(\tilde{E}_L \cap I)$ and that convergence implies that $\bar{F}(\tilde{E}_L \cap I)=\int_{E_L \cap I} f\, d\mu$.

The classical Riemann integral is defined only for bounded functions on finite intervals. If $\mathbf{R}(I)$, $\mathbf{C}_{ae}(I)$ and $\mathbf{B}(I)$ denote respectively the spaces of Riemann integrable, continuous almost everywhere and bounded functions on the interval I then, for I finite,

$$\mathbf{R}(I) = \mathbf{C}_{ae}(I) \cap \mathbf{B}(I). \tag{3.1}$$

For $f \in \mathbf{B}(I)$ with I finite continuity almost everywhere plays a role for Riemann integration analogous to measurability for Lebesgue integration. If $f \in \mathbf{C}_{ae}(I) \cap \mathbf{B}(I)$ the Riemann integral of f $((R)\int_I f\, dx)$ exists and can be approximated arbitrarily closely from above and below by Riemann integrals of step functions (constant on intervals) whereas if f is measurable its Lebesgue integral exists and can be approximated arbitrarily closely from above and below by integrals of simple functions (constant on measurable sets of finite Lebesgue measure).

We next use OE-nets to extend the classical Riemann integral into $\mathbf{C}_{ae}(I)$ where I is finite or infinite and the functions need not be bounded.

If $f \in \mathbf{C}_{ae}$ the set E_R of points at which f is not locally Riemann integrable is closed and $\mu(E_R)=0$ since E_R is contained in the set of points of discontinuity of f. Define $F(J)=(R)\int_J f\, dx$ for $J \in \mathscr{I}(\tilde{E}_R \cap I)$. Then (F, E_R, I) is an OE-net. Call f generalized Riemann integrable on $I (f \in (\bar{\mathbf{R}}(I))$ if (F, E_R, I) converges and write $(R)\int_I f\, dx=\bar{F}(\tilde{E}_R \cap I)=\lim(F, E_R, I)$. We show that

$$\bar{\mathbf{R}}(I) = \mathbf{C}_{ae}(I) \cap \mathbf{L}(I). \tag{3.2}$$

Note first that for each $J \in \mathscr{I}(\tilde{E}_R \cap I)$, $F(J)=\int_J f\, d\mu$ and for each $I' \subset \tilde{E}_R \cap I$, $V_F(I')=\int_{I'} |f|\, d\mu$. Thus if (F, E_R, I) converges, (i′) and (ii′) of Theorem 2.2 imply that $\bar{F}(I_i)=\int_{I_i} f\, d\mu$ for each i. It then follows from (ii) that $\sum_i \bar{F}(I_i)$ converges and $f \in \mathbf{L}(\tilde{E}_R \cap I)$ and, since $\mu(E_R)=0$, that $f \in \mathbf{L}(I)$. Thus $\bar{\mathbf{R}}(I) \subset \mathbf{C}_{ae}(I) \cap \mathbf{L}(I)$. By (iii) of Theorem 2.2

$$(R)\int_I f\, dx = \lim(F, E_R, I) = \sum_i \bar{F}(E_i) = \int_I f\, d\mu.$$

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Assume now that $f \in \mathbf{C}_{ae}(I) \cap \mathbf{L}(I)$. Then for each i, $(F, E_R, I_i) = (F, E_L, I_i)$ which converges. Furthermore $\sum V_F(I_i) = \int_I |f| \, d\mu < \infty$. The convergence of (F, E_R, I) then follows from Theorem 2.2.

REMARK. The space $\bar{\mathbf{R}}(I)$ has most of the properties of $\mathbf{R}(I)$. In particular it is linear since $\mathbf{C}_{ae}(I)$ and $\mathbf{L}(I)$ are linear.

We next consider extensions using COE-nets and Theorems 2.1 and 2.3.

Define $F(J) = \int_J f \, d\mu$ for $J \in \mathscr{I}(\tilde{E}_L \cap I)$ and call f Cauchy-Lebesgue on $\tilde{E} \cap I$ if $(F, E_L, I)^*$ converges ($f \in \mathbf{CL}(\tilde{E}_L \cap I)$). Write

$$(CL)\int_{\tilde{E}_L \cap I} f \, d\mu = \lim(F, E_L, I)^* = \bar{F}(\tilde{E}_L \cap I) = \sum_i (CL)\int_I f \, d\mu.$$

If $(F, E_L, I)^*$ converges and $f \in \mathbf{L}(E_L)$ we say that f is Cauchy-Lebesgue integrable over $I (f \in \mathbf{CL}(I))$. We then write

$$(CL)\int_I f \, d\mu = \int_{E_L} f \, d\mu + \bar{F}(\tilde{E}_L \cap I) = \int_{E_L} f \, d\mu + \sum_i (CL)\int_{I_i} f \, d\mu.$$

In Theorem 2.3 set $A = E_L$, $B = \varnothing$,

$$G(J) = G_A(J) = \int_J f\chi_{E_L} \, d\mu, \qquad J \in \mathscr{I}(I),$$

$$H(J) = G(J) + \bar{F}(\tilde{E}_L \cap J),$$

then $(G, \varnothing, I)^*$ converges and thus $(H, \varnothing, I)^*$ converges and

$$\bar{H}(I) = \lim(H, \phi, I)^* = (CL)\int_I f \, d\mu. \tag{3.3}$$

Since F and G are inner continuous, weakly additive (with $k(x) = 0$) and have continuous integrals (on $\mathscr{I}(\tilde{E}_L \cap I)$ and $\mathscr{I}(I)$ respectively), H has these properties on $\mathscr{I}(I)$.

Consequences of the general theory include:

(i) If $f \in \mathbf{CL}(I)$, then $f \in \mathbf{CL}(I')$ for every $I' \subset I$ (Corollary 2.1 (iii)).

(ii) If $\mathscr{H}$ is a continuous integral and $I' = (a', b')$, $a \le a' < b' \le b$, then

$$(CL)\int_{I'} f \, d\mu = \mathscr{H}(b') - \mathscr{H}(a'),$$

i.e. a continuous integral for f corresponds to a primitive or indefinite integral.

(iii) $\mathbf{L}(I) \subset \mathbf{CL}(I)$ and if $f \in \mathbf{CL}(I)$ then $f \in \mathbf{L}(I)$ if and only if $|f| \in \mathbf{CL}(I)$. (Note that $E_L(f) = E_L(|f|)$.)

(iv) $\mathbf{CL}(I)$ is homogeneous but not additive and therefore not a vector space.

As an example of the lack of additivity first observe that the function $f_0(x) = d/dx(x^2 \sin x^{-2}) \in \mathbf{CL}(0, \pi)/\mathbf{L}(0, \pi]$. Furthermore if $I = (0, \infty)$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{f_0(x-n)\chi_{(n\pi, (n+1)\pi)}(x)}{2^n},$$

then $E_L(f)=\{n\pi; n=0, 1, \ldots\}$ and if $F(J)=\int_J f\,d\mu$, $J \in \mathscr{I}(\tilde{E}_L \cap I)$, then (i′) and (ii) of Theorem 2.1 hold so that $(F, E_L(f), I)^*$ converges. Since $f \in \mathbf{L}(E_L \cap I)$, $f \in \mathbf{CL}(I)$.

Let $g(x)=\sin x/x$, $x \in I$. Then $g \in \mathbf{CL}(I)\backslash\mathbf{L}(I)$, $E_L(g)=\varnothing$, $E_L(g+h)=E_L(h)$ and it is easy to verify that (ii) of Theorem 2.1 does not hold for $f+g$ showing that $f+g \notin \mathbf{CL}(I)$.

Let $\tilde{E}_L \cap I=\cup_i I_i$, write $E=E_L$ and let E' denote the points of $[a, b]$ for which there exists no $I_\delta=(x-\delta, x+\delta)$ with: (i) $f \in \mathbf{L}(\tilde{E} \cap I)$ and (ii) $\sum \{\omega_F(I_i): I_i \cap I_\delta \neq 0\} < \infty$. Then E' is a closed subset of $E=E_L$.

Assuming: (iii) $(F, E, I_i)^*$ converges for every i, we show that $H=\bar{H}$ as in (3.3) extends F from $\mathscr{I}(\tilde{E} \cap I)$ to $\mathscr{I}(\tilde{E}' \cap I)$. Let $J \in \mathscr{I}(\tilde{E}' \cap I)$. Then $\bar{J} \subset \tilde{E}' \cap I$ and to every point x of $\bar{J}$ corresponds an open interval $I_{\delta(x)}$ for which (i) and (ii) hold. By compactness J is covered by a finite subcollection which implies that (i), (ii) and (iii) hold for J. It then follows as above that $f \in \mathbf{CL}(J)$. A less general approach would have been obtained by assuming that $(F, E, I)^*$ converged which would have implied both (ii) and (iii).

The set $E'=E_{CL}$ is the set at which f is not locally Cauchy-Lebesgue integrable. We can now define a second order or CL^2-integral for functions f for which $(H, E', I)^*$ converges and $f \in \mathbf{L}(E' \cap I)$ by analogy with the CL-definition. This process can be continued by finite and transfinite induction and gives a scale of integrals that can be related to the restricted Denjoy integral [1].

Similar Cauchy type extensions can be based on the Riemann and generalized Riemann integrals rather than the Lebesgue integral.

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https://doi.org/10.4153/CMB-1975-002-1 Published online by Cambridge University Press