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ON BISIMPLE SEMIGROUPS GENERATED BY A FINITE NUMBER OF IDEMPOTENTS

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Abstract

Non-completely simple bisimple semigroups S which are generated by a finite number of idempotents are studied by means of Rees matrix semigroups over local submonoids eSe, $e = e^2 \in S$. If under the natural partial order on the set E_S of idempotents of such a semigroup S the sets $\omega(e) = \{f \in E_S: f \leqslant e\}$ for each $e \in E_S$ are well-ordered, then S is shown to contain a subsemigroup isomorphic to Sp_4, the fundamental four-spiral semigroup. A non-completely simple bisimple semigroup is constructed which is generated by 5 idempotents but which does not contain a subsemigroup isomorphic to Sp_4.

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1. Introduction

The generalization by D. Allen, Jr. [1] of the Rees theorem to a class of regular semigroups, and its improvement by D. B. McAlister [7] can be refined to obtain detailed information about the structure of bisimple semigroups which are generated by a finite number of idempotents. We use this approach to investigate an embedding question first raised in [3]: which non-completely simple bisimple idempotent-generated semigroups contain a subsemigroup isomorphic to Sp_4?

The fundamental four-spiral semigroup Sp_4 is presented by $\langle a, b, c, d \mid a = ba, ab = b = bc, cb = c = dc, cd = d = da \rangle$ [3] and may be represented as the Rees matrix semigroup $\mathfrak{M}(\mathcal{C}(p, q)); 2, 2; \left(\begin{smallmatrix} 1 & q \\ 1 & 1 \end{smallmatrix}\right)$, over the bicyclic semigroup $\mathcal{C}(p, q)$ [2]. It is an example of a non-completely simple bisimple idempotent generated semigroup which is the smallest such in the following sense: any E-chain linking

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distinct comparable idempotents of a bisimple idempotent-generated semigroup has even length of at least 4, whereas the E-chain $a \mathcal{R} b \mathcal{L} c \mathcal{R} d \mathcal{L} ad$, which links the distinct comparable idempotents a and ad of Sp_4, has length exactly 4.

The embedding question asks for an analogue of the result that any non-completely simple bisimple regular semigroup contains a subsemigroup isomorphic to $\mathcal{C}(p, q)$. In [3] it was shown that any bisimple idempotent-generated semigroup S in which $\omega(e) = \{f \in E_S: f \leqslant e\}$ is an ω-chain for each $e \in E_S$ contains a subsemigroup isomorphic to Sp_4. In [4] constructions of pseudosemilattices were used to provide examples of non-completely simple bisimple semigroups generated by infinitely many idempotents which fail to contain subsemigroups isomorphic to Sp_4.

For each non-completely simple bisimple semigroup S which is generated by a finite number of idempotents we shall construct in Section 2 a Rees matrix semigroup $\mathcal{M}(eSe; m, m; P)$, also generated by a finite number of idempotents, which has S as a homomorphic image. This Rees matrix cover will be used in Section 3 to show that if the sets $\omega(e)$ for $e \in E_S$ are well-ordered in such a semigroup S, then S contains a subsemigroup isomorphic to Sp_4. In Section 4 a Rees matrix semigroup over an inverse semigroup is constructed which is non-completely simple, bisimple, is generated by 5 idempotents, but fails to contain a subsemigroup isomorphic to Sp_4. The main results of this paper were announced at the Nebraska Conference on Semigroups, Lincoln, Nebraska in September 1980.

2. A Rees matrix cover

We first establish some results which will enable us to deduce properties of the Rees matrix cover. The submonoids eSe, $e \in E_S$, of a semigroup S will be called *local submonoids* of S.

PROPOSITION 2.1. *Let S be a regular semigroup, $e \in E_S$. If SeS is finitely generated, then the local submonoid eSe is finitely generated.*

PROOF. Suppose we can prove the result in the special case $S = SeS$. Then the general result follows, for S regular implies SeS regular, and since $SeS = SeS \cdot e$ SeS is finitely generated, we conclude that $e \cdot SeS \cdot e = eSe$ is finitely generated.

So suppose $S = SeS$. Let $x_1, x_2, \ldots, x_n$ be generators for S. For each x_i choose idempotents u_i, v_i such that $v_i \mathcal{L} x_i \mathcal{R} u_i$, and elements r_i, r_i', s_i, s_i' such that $r_i r_i' = u_i$, $r_i' r_i \leqslant e$, $s_i s_i' = v_i$, $s_i' s_i \leqslant e$. Let $y = e(\prod_{l=1}^{w} x_{i_l})e$ be an arbitrary element of eSe where $1 \leqslant i_l \leqslant n$, $l = 1, 2, \ldots, w$. Then

$$y = e\left(\prod_{l=1}^{w} u_{i_l} x_{i_l} v_{i_l}\right) e = e\left(\prod_{l=1}^{w} r_{i_l} r'_{i_l} x_{i_l} s_{i_l} s'_{i_l}\right) e$$

$$= er_{i_1}\left[\prod_{l=1}^{w-1} \left(r'_{i_l} x_{i_l} s_{i_l}\right)\left(s'_{i_l} r_{i_{l+1}}\right)\right]\left(r'_{i_w} x_{i_w} s_{i_w}\right) s'_{i_w} e.$$

Thus the elements er_i, $r'_i x_i s_i$, $s'_i r_j$, $s'_i e$, $i, j = 1, 2, \ldots, n$, of eSe generate eSe, so eSe is generated by at most $n^2 + 3n$ generators.

Let L denote the 2-element lattice $L = \{0, 1\}$ in which we write meet as $\cdot$, join as $+$, so that $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$, $0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0$, $1 \cdot 1 = 1$. Thus under multiplication L is the trivial group with 0 adjoined. The semigroup L_m of all $m \times m$ matrices over L, m a positive integer, may be interpreted as the semigroup of binary relations of an m-element set.

Proposition 2.2. *Let P be an $m \times m$ matrix over the lattice L having diagonal entries all 1 and let M denote the Rees matrix semigroup $\mathfrak{M}^\circ(1; m, m; P)$ over the trivial group with 0. Then $\mathfrak{M}^\circ \setminus \{0\}$ is generated by the m idempotents $(i, 1, i)$, $i = 1, 2, \cdots, m$ if and only if all entries of P^r equal 1 for some r.*

Proof. The symbols $*$ and Σ denote products in $\mathfrak{M}^\circ$. All other products are in L_m. Thus, for example $(i, 1, j) * (k, 1, l) = (i, 1, j)P(k, 1, l)$, where as usual $(i, 1, j)$ denotes the matrix with a 1 in the (i, j) position, all other entries 0. Suppose $A \in L_m$ has precisely one non-zero row, say row i, and that $B \in L_m$ has precisely one non-zero column, say column j. Then AB has at most one non-zero entry, the (i, j) entry. In view of the operations in L there exists some k such that (1) $AB = A(k, 1, k)B$. On the other hand, if, for some k, $A(k, 1, k)B \neq 0$, then (2) $A(k, 1, k)B = AB = (i, 1, j)$.

Suppose now that $\mathfrak{M}^\circ \setminus \{0\}$ is generated by the m idempotents $(i, 1, i)$. Then given $(i, 1, j) \in \mathfrak{M}^\circ \setminus \{0\}$ we can write

$$(i, 1, j) = \prod_{l=1}^{w} (i_l, 1, i_l) = (i_1, 1, i_1)P(i_2, 1, i_2)P \cdots P(i_w, 1, i_w)$$

which by repeated use of (2) equals $(i_1, 1, i_1)P^{w-1}(i_w, 1, i_w)$. Therefore entry (i, j) of P^{w-1} is non-zero, so there exists r such that all entries of P^r equal 1.

Conversely, suppose that all entries of P^r equal 1 for some r, and let $(i, 1, j) \in \mathfrak{M}^\circ \setminus \{0\}$. Then $(i, 1, i)P^r(j, 1, j) = (i, 1, j)$, so by repeated use of (1) there exist $i = i_1, i_2, \ldots, i_w = j$ such that $(i_1, 1, i_1)P(i_2, 1, i_2)P \cdots P(i_w, 1, i_w) = (i, 1, j)$. Thus $(i, 1, j) = \prod_{l=1}^{w}(i_l, 1, i_l)$, so $\mathfrak{M}^\circ \setminus \{0\}$ is generated by the m idempotents $(i, 1, i)$.

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If $P = (p_{ij})$ is an $m \times m$ matrix over a semigroup S^1 we define $\bar{P} = (\bar{p}_{ij})$ to be the $m \times m$ matrix over the lattice $L = \{0, 1\}$ such that

$$\bar{p}_{ij} = \begin{cases} 1 & \text{if } p_{ij} = 1, \\ 0 & \text{if } p_{ij} \neq 1. \end{cases}$$

PROPOSITION 2.3. *Let P be an $m \times m$ matrix over a semigroup S^1 with diagonal entries all 1 such that the entries of $\bar{P}^r$ are all 1 for some r. Then $\mathfrak{M}(S^1; m, m; P)$ is generated by the m idempotents $(i, 1, i)$ if and only if the entries of P generate S^1.*

PROOF. Suppose $\mathfrak{M}$ is generated by the m idempotents $(i, 1, i)$ and let $s \in S^1$. Then $(1, s, 1) = \prod_{l=1}^{w}(i_l, 1, i_l) = (i_1, \prod_{l=1}^{w} p_{i_l, i_{l+1}}, i_w)$ for some $1 \leq i_l \leq m$, $l = 1, 2, \ldots, w$, so s is a product of entries of P.

Conversely, suppose that the entries of P generate S^1. Let $(i, s, j) \in \mathfrak{M}$. Then there exists i_l, j_l, $l = 1, 2, \ldots, w$ such that $s = \prod_{l=1}^{w} p_{i_l, j_l}$. Thus

$$(i, s, j) = (i, 1, i_1)\left[\prod_{l=1}^{w-1} (i_l, 1, i_l)(j_l, 1, j_l)(j_l, 1, i_{l+1})\right](i_w, 1, i_w)(j_w, 1, j).$$

The partial function θ: $\mathfrak{M}^\circ(1; m, m; \bar{P}) \to \mathfrak{M}(S^1; m, m; P)$ defined by $(i, 1, j) \to (i, 1, j)$, $i, j = 1, 2, \ldots, m$, is a partial homomorphism in the sense that if $x, y, xy \in \mathfrak{M}^\circ \setminus \{0\}$, then $(x\theta)(y\theta) = (xy)\theta$. Thus by Proposition 2.2 each of the factors in the product above is a product of the idempotents $(i, 1, i)$, $i = 1, 2, \ldots, m$. We conclude that $\mathfrak{M}(S^1; m, m; P)$ is generated by these idempotents.

We observe that any matrix P over a semigroup S^1 whose first row, first column, and diagonal entries are all 1 satisfies the hypothesis of Proposition 2.3, as does any matrix whose tridiagonal entries (those on the main diagonal and the two adjacent diagonals) are all 1.

The following result is a refinement of McAlister's Local Isomorphism Theorem [7], which in turn draws heavily on the ideas of D. Allen [1]. We denote the identity element e of the local submonoid eSe by 1.

THEOREM 2.4. *Let S be a bisimple semigroup which is generated by a finite number of idempotents and let $e \in E_S$. Then S is a homomorphic image by rectangular bands of a Rees matrix semigroup $\mathfrak{M}(eSe; m, m; P)$ over the finitely generated bisimple monoid eSe where*

(1) *$\mathfrak{M}$ is generated by m idempotents, m a positive integer;*

(2) *the entries of P generate eSe and the tridiagonal entries of P are all 1;*

(3) *the homomorphism is an isomorphism when restricted to any subsemigroup $\{i\} \times eSe \times \{j\}$.*

PROOF. Let S be a bisimple semigroup which is generated by a finite number of idempotents $x_1, x_2, \ldots, x_n$ and let $e \in E_S$. It is easy to check that the monoid eSe is bisimple. By Proposition 2.1 eSe is finitely generated.

Since $S = \bigcup_{i=1}^{n} x_i S$ each $\mathcal{R}$-class of S is less than one of the finite number of maximal $\mathcal{R}$-classes of S, and similarly for $\mathcal{L}$-classes. Since S is bisimple and idempotent-generated the biordered set of S is connected [3]. Thus there exists a sequence $e = e_1, e_2, e_3, \ldots, e_m$ of not necessarily distinct idempotents of S with $e_1 \mathcal{R} e_2 \mathcal{L} e_3 \mathcal{R} \cdots e_m$ such that each x_i appears in the sequence. Let $I = \{e_1, e_2, \ldots, e_m\}$. Then I contains an idempotent from each maximal $\mathcal{R}$-class and from each maximal $\mathcal{L}$-class. Thus, given $g \in E_S$, there exist i, j such that $e_i g = g$, $g e_j = g$. To simplify notation in what follows we let $e_0 = e = e_1$. For $i = 1, 2, \ldots, m$ let

$$r_i = \begin{cases} e_i e_{i-2} \cdots e_1 & \text{if } i \text{ is odd,} \\ e_{i-1} e_{i-3} \cdots e_1 & \text{if } i \text{ is even;} \end{cases}$$

$$r_i' = \begin{cases} e_0 e_2 \cdots e_{i-1} & \text{if } i \text{ is odd,} \\ e_0 e_2 \cdots e_i & \text{if } i \text{ is even.} \end{cases}$$

The $m \times m$ matrix $P = (p_{ij})$ is defined by $p_{ij} = r_i' r_j$. Since $r_i r_i' = e_i$, $r_i' r_i = e$, $i = 1, 2, \ldots, m$, each $p_{ij} = (r_i' r_i) r_i' r_j (r_j' r_j)$ belongs to eSe. Since $e = r_i' r_{i+1} = r_{i+1}' r_i$ for $i = 1, 2, \ldots, m-1$, the tridiagonal entries of P are all 1. Since each generator x_i appears in I, any element of eSe can be written in the form

$$e\left(\prod_{l=1}^{w} e_{i_l}\right)e = e\left(\prod_{l=1}^{w} r_{i_l} r_{i_l}'\right)e = r_1' r_{i_1}\left(\prod_{l=1}^{w-1} r_{i_l}' r_{i_{l+1}}\right) r_{i_w}' r_1.$$

Thus the entries of P generate eSe.

Since eSe is bisimple and regular it is easy to check that $\mathfrak{M}$ is also. By Proposition 2.3 $\mathfrak{M}$ is generated by the m idempotents $(i, 1, i)$.

The mapping ϕ: $\mathfrak{M}(eSe; m, m; P) \to S$ by $(i, s, j) \to r_i s r_j'$ is a homomorphism since $[(i, s, j)(k, t, l)]\phi = (i, s p_{jk} t, l)\phi$ and it maps $\mathfrak{M}$ onto S since any $s \in S$ can be written as

$$s = ss'ss's = r_i r_i' ss' \cdot s \cdot s' s r_j r_j' = r_i\left(r_i' s r_j\right) r_j' = \left(i, r_i' s r_j, j\right)\phi$$

for some i, j. To show that ϕ is an isomorphism into S when restricted to the subsemigroup $\{i\} \times eSe \times \{j\}$ suppose that $(i, s, j)\phi = (i, t, j)\phi$. Then $r_i s r_j' = r_i t r_j'$, so $r_i' r_i s r_j' r_j = r_i' r_i t r_j' r_j$ and thus $ese = ete$ so $s = t$. To show that ϕ is a homomorphism by rectangular bands let $g \in E_S$ and suppose $(i, s, j)\phi = g$. Then $r_i s r_j' = g$, so $r_i s r_j' r_i s r_j' = r_i s r_j'$. Multiplying by r_i' on the left and r_j on the right gives $s r_j' r_i s = s$, so (i, s, j) is idempotent. If $(k, t, l)\phi = g$, then $r_k t r_l' = r_i s r_j'$, so $(i, s, j)(k, t, l)(i, s, j) = (i, s r_j' r_k t r_l' r_i s, j) = (i, s, j)$, so $g\phi^{-1}$ is a rectangular band, as required.

3. Embedding Sp_4 in certain bisimple idempotent-generated semigroups

The defining relations for the fundamental four-spiral semigroup Sp_4 imply that a, b, c, d, ad are idempotents with $ad \leqslant a$. Although $ad < a$ in Sp_4, Sp_4 has a least non-identity congruence, and this congruence identifies ad and a. Therefore a semigroup S contains a subsemigroup isomorphic to Sp_4 if and only if S contains idempotents a, b, c, d with $a\mathcal{R}b\mathcal{L}c\mathcal{R}d$ such that $da = d$ and $ad \neq a$.

LEMMA 3.1. *Let S and T be semigroups and let ϕ: $S \to T$ be a homomorphism from S onto T which does not identify distinct comparable idempotents of S. If S contains a subsemigroup isomorphic to Sp_4, then so does T.*

PROOF. The restriction of ϕ to the subsemigroup of S isomorphic to Sp_4 must induce the identity congruence, since distinct comparable idempotents are not identified.

LEMMA 3.2. *Let S be an inverse semigroup with natural partial order $\leqslant$. A Rees matrix semigroup $\mathfrak{M}(S; 2, 2; (\begin{smallmatrix} s & t \\ v & u \end{smallmatrix}))$ over S contains a subsemigroup isomorphic to Sp_4 if and only if there exist elements $a, b, c, d \in S$ such that*

(1) $a\mathcal{R}b\mathcal{L}c\mathcal{R}d$;

(2) $a \leqslant s^{-1}, b \leqslant v^{-1}, c \leqslant u^{-1}, d \leqslant t^{-1}$; *and*

(3) *either* (i) $dsa = d$, $atd < a$ *or* (ii) $dsa < d$, $atd = a$.

PROOF. Suppose that $\mathfrak{M}$ contains a subsemigroup isomorphic to Sp_4. Since any $\mathcal{R}$-class or $\mathcal{L}$-class of $\mathfrak{M}$ contains at most one idempotent which belongs to a subsemigroup $\{i\} \times S \times \{j\}$, there exist idempotents $(1, a, 1)\mathcal{R}(1, b, 2)\mathcal{L}(2, c, 2)\mathcal{R}(2, d, 1)$ such that either (i) $(2, d, 1)(1, a, 1) = (2, d, 1)$, $(1, a, 1)(2, d, 1) < (1, a, 1)$ or (ii) $(2, d, 1)(1, a, 1) < (2, d, 1)$, $(1, a, 1)(2, d, 1) = (1, a, 1)$. These conditions on idempotents of $\mathfrak{M}$ imply conditions (1), (2), (3) on the elements a, b, c, d of S.

Conversely, suppose a, b, c, d are elements of S such that (1), (2) and (3) hold. Then $(1, a, 1)$, $(1, b, 2)$, $(2, c, 2)$, $(2, d, 1)$ are idempotents of $\mathfrak{M}$ which generate a subsemigroup isomorphic to Sp_4.

We follow the usual convention of calling the semilattice E of idempotents of an inverse semigroup *well-ordered* if the reverse of the natural partial order on E is a well-ordering of E. Below $\leqslant$ denotes the usual order on the ordinals.

THEOREM 3.3. *Let S be a non-completely simple bisimple semigroup which is generated by a finite number of idempotents. If E_{eSe} is well-ordered for each $e \in E_S$, then S contains a subsemigroup isomorphic to Sp_4.*

PROOF. Suppose S is a non-completely simple bisimple semigroup which is generated by a finite number of idempotents in which E_{eSe} is well-ordered for each $e \in E_S$. Let $\mathfrak{M}(eSe; m, m; P)$ be a Rees matrix cover for S, as guaranteed by Theorem 2.4. To show that S contains a subsemigroup isomorphic to Sp_4 it suffices by Lemma 3.1 to show that $\mathfrak{M}$ does. The maximum idempotent-separating congruence μ on eSe induces an idempotent-separating homomorphism from $\mathfrak{M}(eSe; m, m; P)$ into $\mathfrak{M}(eSe/\mu; m, m; P\mu^\natural)$ where for $P = (p_{ij})$ we denote by $P\mu^\natural$ the matrix $(p_{ij}\mu^\natural)$. Let $E = E_{eSe}$. The bisimple semigroup eSe/μ is isomorphic to a full inverse subsemigroup of T_E, which since T_E is combinatorial (E is well-ordered) implies that eSe/μ is isomorphic to T_E. Thus $\mathfrak{M}(eSe/\mu; m, m; P\mu^\natural)$ is isomorphic to a Rees matrix semigroup $\mathfrak{M}(T_E; m, m; P')$ over T_E. Since $\mathfrak{M}(T_E; m, m; P')$ is an idempotent-separating homomorphic image of $\mathfrak{M}(eSe; m, m; P)$, to prove the theorem it suffices to show that $\mathfrak{M}(T_E; m, m; P')$ contains a subsemigroup isomorphic to Sp_4.

Since E is uniform, E is isomorphic to an ordinal power of ω[5], [10] say $E = \omega^r$, and since S is non-completely simple, $r \geqslant 1$. Given ordinals $a, b < \omega^r$, Ea *will denote the principal ideal* $\{x\colon a \leqslant x < \omega^r\}$ of E generated by a, and the unique principal ideal isomorphism from Ea to Eb is given by $a + x \to b + x$ for $0 \leqslant x < \omega^r$ (usual addition of ordinals). Below we will use the fact, which follows from the normal form for ordinals [9], that for ordinals x and a, $x = \omega^a + x \Rightarrow x \geqslant \omega^{a+1}(*)$.

We claim that some entry of P' has no fixed point. Suppose to the contrary that each of the finitely many entries of P' has a fixed point, and let $\bar{x}$ be their supremum. Then each entry of P' is the identity on $\bar{X} = \{x\colon \bar{x} \leqslant x < \omega^r\}$. Thus, since the entries of P' generate T_E, each element of T_E is the identity on $\bar{X}$. But this is impossible, since if r is not a limit ordinal then the principal ideal isomorphism $E0 \to E\omega^{r-1}$ has no fixed point by $(*)$, while if r is a limit ordinal, then $E0 \to E\omega^k$, where k is chosen so that $\bar{x} < \omega^k$, does not fix w^k (again by $(*)$). This establishes the claim.

Of those entries of P' without fixed points let η be one which is closest to the main diagonal of P'. If η appears above the diagonal it belongs to a 2×2 submatrix of P' of the form

	i	$i+1$
j	α	η
$j+1$	β	γ

where α, β, γ have fixed points (the case where η lies below the diagonal is entirely similar). Thus there exist elements $u, v \in E$ which are fixed by α, β, γ

such that $u\eta = v$. Since the principal ideal isomorphisms $Ev \to Ev$, $Ev \to Ev$, $Ev \to Ev$, $Ev \to Eu$ satisfy the conditions (1), (2), (3) on a, b, c, d of Lemma 3.2, $\mathfrak{M}(T_E; m, m; P')$ contains a subsemigroup isomorphic to Sp_4, as required.

4. The counterexample

Motivated by the Rees matrix cover described in Theorem 2.4 we construct a semigroup which yields a negative answer to the following question, posed as problem B2 in [8]: does every non-completely simple bisimple semigroup which is generated by a finite number of idempotents contain a subsemigroup isomorphic to Sp_4, the fundamental four-spiral semigroup?

EXAMPLE 4.1. Let S denote the P-semigroup $P(G, \mathcal{X}, \mathcal{Y})$ [6] where $G = \mathbf{Z} \times \mathbf{Z}$ is the direct product of two copies of the group of integers under addition, $\mathcal{X} = \mathbf{Z} \times \mathbf{Z}$ is the direct product of two copies of the semilattice of integers under the usual order, and $\mathcal{Y} = \mathbf{Z}^- \times \mathbf{Z}^-$ is the subsemilattice and ideal of $\mathcal{X}$ consisting of all elements of $\mathcal{X}$ whose components are both $\leqslant 0$. Let G act on $\mathcal{X}$ by order automorphisms as follows: if $g = (e, f)$, $A = (m, n)$, then $gA = (e + m, f + n)$. Under the multiplication $(A, g) \cdot (B, h) = (A \wedge gB, gh)$, $S = \{(A, g) \in \mathcal{Y} \times G\colon g^{-1}A \in \mathcal{Y}\}$ becomes an E-unitary inverse semigroup with semilattice $\mathcal{Y}$ and maximum group homomorphic image $S/\sigma \cong G$. The natural partial order on S is given by $(A, g) \leqslant (B, h)$ if and only if $A \leqslant B$ and $g = h$. We will denote the element (A, g) in S, where $A = (m, n)$, $g = (e, f)$ by $(m, n; e, f)$.

Let $p = (0, 0; 1, 0)$, $r = (-1, 0; -1, 1)$. Then $p^{-1} = (0, 0; -1, 0)$, $r^{-1} = (0, -1; 1, -1)$ and S is generated as a semigroup by p, r, p^{-1}, r^{-1}. Let $\mathfrak{M}(S; 5, 5; P)$ be the Rees matrix semigroup over S with matrix

$$P = \begin{pmatrix} 1 & 1 & r & r & p \\ 1 & 1 & 1 & r & r \\ r^{-1} & 1 & 1 & 1 & r \\ r^{-1} & r^{-1} & 1 & 1 & 1 \\ p^{-1} & r^{-1} & r^{-1} & 1 & 1 \end{pmatrix}.$$

As usual, 1 denotes the identity element of S, so $1 = (0, 0; 0, 0)$. By Proposition 2.3 $\mathfrak{M}$ is generated by the 5 idempotents $(i, 1, i)$, $i = 1, 2, 3, 4, 5$. Since S is bisimple but not completely simple, the same is true of $\mathfrak{M}$.

We claim that $\mathfrak{M}$ does not contain a subsemigroup isomorphic to Sp_4. Suppose to the contrary that $\mathfrak{M}$ does. Then there exist i, j, k, l such that the subsemigroup

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$\{i, j\} \times S \times \{k, l\}$ of $\mathfrak{M}$ contains a subsemigroup isomorphic to Sp_4. Thus there exists a submatrix $P' = \binom{s\ t}{v\ u}$ of P and elements $a = (A, g), b = (B, h), c = (C, k), d = (D, l)$ of S satisfying (1), (2), (3) of Lemma 3.2.

The $\mathfrak{R}$ and $\mathfrak{L}$ relations of (1) imply that $A = B$, $h^{-1}B = k^{-1}C$, and $C = D$, hence $A = hk^{-1}D$. If (3)(i) holds, then $dsa = d$ so $da^{-1}a = d$ and thus $d^{-1}d \leqslant a^{-1}a$. But since also $atd < a$, and thus $ad^{-1}d < a$, we conclude $d^{-1}d < a^{-1}a$, so $l^{-1}D < g^{-1}A$ and thus $D < lg^{-1}hk^{-1}D$. If (3)(ii) holds, then similarly we obtain $lg^{-1}hk^{-1}D < D$. Let $\alpha = lg^{-1}hk^{-1}$. Then in either case D and αD are distinct and comparable. By (2), $\alpha = (t^{-1}sv^{-1}u)\sigma^{\natural}$.

Let $s\sigma^{\natural} = (w_1, w_2)$, $t\sigma^{\natural} = (x_1, x_2)$, $u\sigma^{\natural} = (y_1, y_2)$, $v\sigma^{\natural} = (z_1, z_2)$. Then

$$P'\sigma^{\natural} = \begin{pmatrix} s\sigma^{\natural} & t\sigma^{\natural} \\ v\sigma^{\natural} & u\sigma^{\natural} \end{pmatrix} = \left(\begin{pmatrix} w_1 & x_1 \\ z_1 & y_1 \end{pmatrix}, \begin{pmatrix} w_2 & x_2 \\ z_2 & y_2 \end{pmatrix} \right)$$

and $\alpha = (w_1 - x_1 + y_1 - z_1, w_2 - x_2 + y_2 - z_2)$. Since D and αD are distinct and comparable, the components of α are not both zero and either both are $\geqslant 0$ or both are $\leqslant 0$. We will be helpful to call the quantity $w - x + y - z$ associated with the 2×2 matrix $W = \binom{w\ x}{z\ y}$ of integers the *increment* of W. Let

$$P_1 = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

be the matrices of first and second components, respectively, of the elements of $P\sigma^{\natural}$. To obtain the contradiction it suffices to show that if a 2×2 submatrix of P_1 has positive (negative) increment, then the corresponding 2×2 submatrix of P_2 has negative (positive) increment. This is clear for any pair of corresponding 2×2 submatrices which do not contain the (1, 5) or (5, 1) positions, for then the increments are negatives of each other. It is true by default for the pair of 2×2 submatrices consisting of the corner entries (increments both 0) and is easily checked for the 2×2 submatrices in the upper right and lower left corners. Any other pair of corresponding 2×2 submatrices must contain exactly one of the positions (1, 5), (5, 1), we may assume (1, 5) by symmetry, so has the form

$$\begin{pmatrix} w & 1 \\ z & y \end{pmatrix} \quad \begin{pmatrix} -w & 0 \\ -z & -y \end{pmatrix}$$

where $w \leqslant 0$, $y \leqslant 0$, $z \geqslant 0$. The increment of the first is $\leqslant 0$, that of the second is $\geqslant 0$. This contradicts the existence of α, and forces us to conclude that $\mathfrak{M}$ contains no subsemigroup isomorphic to Sp_4.

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