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LOCAL ENDOMORPHISM NEAR-RINGS

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The purpose of this paper is to study the consequences of an endomorphism near-ring of a finite group being a local near-ring and the existence of such near-rings. As we shall see in Section 2, an endomorphism near-ring of a finite group being local gives us some information about both the structure of the group (Theorem 2.2) and the automorphisms of the group lying in the near-ring (Theorem 2.3). Existence of local endomorphism near-rings of finite groups is considered in Section 3 where we obtain as our main result that any p-group of automorphisms of a p-group containing the inner automorphisms always generates a local endomorphism near-ring. In particular, we get as a corollary that the endomorphism near-ring of a finite group G generated by the inner automorphisms of G is local if and only if G is a p-group. The third section concludes with a discussion of endomorphism near-rings of dihedral 2-groups and generalized quaternion groups.

1. Preliminaries

We shall follow the conventions of [7] with regard to notation and terminology, while our basic reference on local near-rings is [6] suitably modified to the aforementioned specifications. In particular, this means that a near-ring R is *local* if the set

$$L=\{r\in R \mid r \text{ does not have a right inverse}\}$$

is a right R-subgroup of R.

Throughout, G will denote a finite group written additively (but not necessarily abelian). The sets of inner automorphisms, automorphisms, and endomorphisms of G will be denoted $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$, and $\mathrm{End}(G)$, respectively. If S is a semigroup of endomorphisms of G, S generates the *endomorphism near-ring* R under pointwise addition and composition of functions which is a distributively generated (d.g.) near-ring. The endomorphism near-rings generated by $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$, and $\mathrm{End}(G)$ will be respectively denoted $I(G)$, $A(G)$, and $E(G)$.

As in [7], we shall say that an endomorphism near-ring R generated by a semigroup of endomorphisms S is *tame* when $\mathrm{Inn}(G)\leqq S$. The reader should keep in mind that the notion of a subgroup H of G being an R-submodule (or R-subgroup) coincides with the notion of H being an R-ideal when R is tame [7, Lemma 10.7]. Moreover, it is easy to see that this equivalence extends to quotients of G by R-submodules:

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Lemma 1.1. *If R is a tame endomorphism near-ring of G, H is an R-ideal of G, and $\bar{G}=G/H$, then a subgroup $\bar{K}$ of $\bar{G}$ is an R-submodule if and only if $\bar{K}$ is an R-ideal of $\bar{G}$.*

Finally, the standard radicals ($J_2(R)$, $J_1(R)$, and $J_0(R)$) are all the same when R is a tame endomorphism near-ring and we shall denote the radical in this setting by $J(R)$.

If H and K are subgroups of a group G, we will use $[H,K]$ to denote the subgroup of G generated by the commutators $[h,k]$, $h\in H$, $k\in K$. The proof of Lemma 10.17 of [7] can be easily modified to obtain:

Lemma 1.2. *Let (R,S) be a* d.g. *near-ring and G be an (R,S)-group. If H and K are (R,S)-subgroups of G, then $[H,K]$ is an (R,S)-subgroup of G.*

In particular, we have that commutators of R-ideals in G are R-ideals when R is a tame endomorphism near-ring of G.

From time to time we will make use of the following notions from group theory: An element α in $\mathrm{Aut}(G)$ is said to *stabilize* a series

$$0=G_0\leqq G_1\leqq G_2\leqq\cdots\leqq G_n=G$$

of subgroups of G if $(g+G_i)\alpha=g+G_i$ for all $g\in G_{i+1}$ for all $i=0,1,\ldots,n-1$. In this case, $1-\alpha$ annihilates the series and is nilpotent of degree n in any endomorphism near-ring containing 1 and α. A subgroup A of $\mathrm{Aut}(G)$ is called a *stability group* of G if there is a series of subgroups

$$0=G_0\leqq G_1\leqq\cdots\leqq G_n=G$$

stabilized by each element of A. Using $\pi(G)$ to denote the set of primes dividing $|G|$, a basic and easily proved result about stability groups is the following (or see [**8**, Lemma 5] which is an even better result):

Lemma 1.3. *If A is a stability group of G, then $\pi(A)\leqq\pi(G)$.*

2. Consequences of localness

Let R be a tame endomorphism near-ring of a group G that is local. We begin this section by noting that $L=J(R)$ [**6**, Theorem 2.10] and proceed to obtain some results about G and R/L. Let

$$0=G_0\leqq G_1\leqq G_2\leqq\cdots\leqq G_n=G$$

be an R-principal series (that is, a maximal series of R-submodules) of G.

Lemma 2.1. *Let R be a tame endomorphism near-ring of G that is local and let $\bar{g}$ be any nonzero element of G_i/G_{i-1}. Then*

$$\mathrm{Ann}_R(\bar{g})=\mathrm{Ann}_R(G_i/G_{i-1})=J(R)$$

for all $i=1,2,\ldots,n$.

Proof. Clearly we have

$$\mathrm{Ann}_R(\bar{g}) \geqq \mathrm{Ann}_R(G_i/G_{i-1}) \geqq J(R) = L$$

Since L is the unique maximal R-subgroup of R [6, Theorem 2.2], $L \geqq \mathrm{Ann}_R(\bar{g})$ and the result follows.

Theorem 2.2. *If R is a tame endomorphism near-ring of G that is local, then:*

(i) *$G_i/G_{i-1} \simeq R/L$ as R-modules for $i = 1, 2, \ldots, n$.*

(ii) *G_i/G_{i-1} is an elementary abelian p-group for $i = 1, 2, \ldots, n$.*

(iii) *G is a p-group.*

(iv) *R/L is a finite field of characteristic p.*

(v) *The series $0 = G_0 \leqq G_1 \leqq \cdots \leqq G_n = G$ is a central series.*

Proof. (i) Let $\bar{g}$ be a nonzero element of G_i/G_{i-1}. As $\bar{g}R = G_i/G_{i-1}$, $G_i/G_{i-1} \simeq R/\mathrm{Ann}_R(\bar{g}) = R/L$ and R-modules.

(ii) This follows because R/L is a near-field [6, Corollary 2.11] and because the additive group of a finite near-field is an elementary abelian p-group.

(iii) is now immediate.

(iv) Since $R/L \leqq \mathrm{End}(G_i/G_{i-1})$, $\mathrm{End}(G_i/G_{i-1})$ is a ring, and R/L is a near-field, R/L must be a finite division ring which is a field and the characteristic is p by (i) and (ii).

(v) Since G is a p-group and G_i is a normal subgroup of G, $[G_i/G_{i-1}, G/G_{i-1}] < G_i/G_{i-1}$. Thus $[G_i/G_{i-1}, G/G_{i-1}] = 0$ since G_i/G_{i-1} is a simple R-module and the result follows.

We next obtain some information about the group of automorphisms lying in a tame endomorphism near-ring that is local.

Theorem 2.3. *Suppose R is a tame endomorphism near-ring of G that is local. Let $A = \mathrm{Aut}(G) \cap R$, $(R/L)^* = R/L - \{0\}$, and $\delta: A \to (R/L)^*$ be the multiplicative homomorphism obtained by restricting the natural projection from R onto R/L to A.*

(i) *$\ker \delta$ is the p-Sylow subgroup P of A where p is the prime dividing $|G|$.*

(ii) *P has a complement K in A which is cyclic and $|K|$ divides $p^n - 1$ where $p^n = |R/L|$.*

Proof. (i) First note that if $\alpha \in A$ has order a power of p, then $\alpha\delta = 1$ since $|(R/L)^*| = p^n - 1$ is relatively prime to p. Conversely suppose $\alpha \in \ker \delta$ and consider an R-principal series $0 = G_0 \leqq G_1 \leqq \cdots \leqq G_n = G$ of G. Since $1 - \alpha \in L$ and $L = \mathrm{Ann}_R(G_i/G_{i-1})$, α stabilizes this series and hence $|\alpha|$ is a power of p by Lemma 1.3. The result now follows.

(ii) This follows from the Schur–Zassenhaus Theorem [1, p. 221] and from the fact that a multiplicative subgroup of finite order in a field is cyclic.

3. Some local endomorphism near-rings

We begin this section with an elementary group theory result whose proof we include for the sake of completeness.

Lemma 3.1. *Let G be a p-group, A a p-group of automorphisms of G, and H a minimal A-invariant subgroup of G. Then A acts trivially on H.*

Proof. Let us momentarily switch to the usual group theory conventions of writing G as a multiplicative group and indicating the action of an automorphism α of G by exponentiation (i.e., $g\alpha = g^{\alpha}$ for $g \in G$). Let $[H, A]$ be the subgroup of G generated by the commutators $[h, \alpha] = h^{-1}h^{\alpha}$, $h \in H$, $\alpha \in A$. Since $[h, \alpha]^{\beta} = [h^{\beta}, \alpha^{\beta}]$ for $\beta \in A$, it follows that $[H, A]$ is an A-invariant subgroup of G. Viewing $[H, A]$ in the semidirect product GA which is also a p-group, we have $[H, A] < H$ and hence $[H, A] = 1$ since H is a minimal A-invariant subgroup.

We now come to the main result of this section. In the proof of this result we shall make use of the socle series of G for a tame endomorphism near-ring R on G which is obtained as follows: The socle of G, Soc(G), is the sum of the minimal R-subgroups of G and the socle series is defined by letting $\mathrm{Soc}_1(G) = \mathrm{Soc}(G)$ and $\mathrm{Soc}_k(G)$ be the inverse image of $\mathrm{Soc}(G/Soc_{k-1}(G))$ in G for $k > 1$. By Theorem 10.37 of [7] we have that $\mathrm{Soc}_n(G) = G$ for some positive integer n.

Theorem 3.2. *Let A be a p-group of automorphisms of a p-group G with* $Inn(G) \leqq A$. *If R is the endomorphism near-ring of G generated by A, then R is local.*

Proof. Let

$$0 < \mathrm{Soc}_1(G) < \mathrm{Soc}_2(G) < \cdots < \mathrm{Soc}_n(G) = G$$

be the socle series of G. We first show that A stabilizes this series. By induction on $|G|$, it suffices to show that A acts trivially on Soc(G) since $R/\mathrm{Ann}_R(G/\mathrm{Soc}(G))$ will be an endomorphism near-ring of the same type on $G/\mathrm{Soc}(G)$. But this needed trivial action on Soc(G) follows from the previous lemma since Soc(G) is the direct sum of the minimal R-subgroups of G.

We now have that $1 - \alpha$ annihilates the socle series of G for all $\alpha \in A$ and hence $1 - \alpha \in J(R)$ by Lemma 2.5 of [4]. Also, $p \cdot 1$ annihilates the socle series since the socle summands are elementary abelian p-groups [7, Theorem 10.30]. Thus it follows that $R/J(R) \simeq Z_p$ and $J(R)$ is a maximal right R-subgroup of R.

By Theorem 2.8 of [6], the proof will be complete if we can show that $J(R)$ is the unique maximal R-subgroup of R. Let M be a maximal R-subgroup of R. If p^k is the exponent of G, then $p^k r = 0$ for all $r \in R$ and hence the additive group of R, $(R, +)$, is a p-group. Let N denote the normal closure of $(M, +)$ in $(R, +)$; that is, N is the smallest normal subgroup of $(R, +)$ containing M and N is generated by elements of the form $-r + m + r$ where $r \in R$, $m \in M$. We have that $N < R$ since the normal closure of a proper subgroup of a p-group is proper (apply Theorem 4.3.2. of [2], for example). Also if $\alpha \in A$, $r \in R$, and $m \in M$,

$$(-r + m + r)\alpha = -r\alpha + m\alpha + r\alpha \in N$$

and it follows that N is an R-subgroup of R. Thus $N = M$ and M is a right ideal of R [7, Corollary 9.22]. Hence we have that $J(R) \leqq M$ (Theorem 5.17 of [7]) and consequently $J(R) = M$ completing the proof.

Using Theorems 2.2 and 3.2, we can now characterize those groups G for which $I(G)$ is local. For if $I(G)$ is local, then G is a p-group by Theorem 2.2. Conversely, if G is a p-group, then $I(G)$ is local by Theorem 3.2. Thus we have:

Corollary 3.3. *$I(G)$ is local if and only if G is a p-group.*

We conclude this section by noting that our results unify some of the work that has already been done on endomorphism near-rings. Specifically, when G is a dihedral 2-group or a generalized quaternion group we have an alternative approach for obtaining some of the results in [3] and [5].

If G is a dihedral group of order 2^n $(n>2)$,

$$G=\langle a,b \mid 2^{n-1}a=2b=0,\ -b+a+b=-a\rangle,$$

then Aut(G) is a 2-group (Lemma 2.2 of [3]) and so it follows that $A(G)$ and $I(G)$ are both local by Theorem 3.2. Moreover, we see from the proof of Theorem 3.2 that $R/J(R)\simeq Z_2$ in both cases, thereby obtaining the result of Theorem 3.4 of [3] in this setting. We also remark that $E(G)$ is not local since the projection from G onto the cyclic subgroup generated by b is an idempotent endomorphism of G and local near-rings have no nontrivial idempotents [6, Theorem 4.2].

Similar results hold for $A(G)$ and $I(G)$ if

$$G=\langle a,b \mid 2^{n-1}a=0,\ 2^{n-2}a=2b,\ -b+a+b=-a\rangle$$

is a generalized quaternion group of order 2^n when $n>3$ since Aut(G) is a 2-group [5, Theorem 2]. Moreover these results extend to $E(G)$ since $A(G)=E(G)$ as shown in [5] (Theorem 5) or which can be seen by examining the endomorphisms of G as follows: End (G) has three nontrivial endomorphisms that are not automorphisms whose kernels are the normal subgroups of G generated by a, b and $a+b$ and whose images are the subgroup generated by $2^{n-2}a$ which is the centre of G. (See the proof of Theorem 2 of [5]). If β is such an endomorphism, it is easily checked that the mapping α defined by $g\alpha=g(1+\beta)$ is an automorphism of G and hence $E(G)=A(G)$.

Finally, we point out that $E(G)$ and $A(G)$ are not local if G is the quaternion group of order 8 (a case not covered in [5]) where we will still have $E(G)=A(G)$. This lack of localness follows because G has an automorphism α of order 3 [9, p. 148] which acts trivially on $\langle 2a\rangle$. Hence, if $E(G)$ were local, $1-\alpha\in J(R)$ by Lemma 2.1 which violates part (i) of Theorem 2.3. This example brings up the question of whether we might have a converse to Theorem 3.2; that is, if G is a p-group and A is a group of automorphisms of G with $\mathrm{Inn}(G)\leqq A$ which generates a local near-ring R, is A a p-group? As of this writing we have been unable to resolve this question.

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