A NOTE ON THE COMPOSITIONS OF AN INTEGER

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1. Partial ordering of the r-compositions of n.

Given an integer n, we define an r-composition of n as follows:

An r-composition of n, $(t_1, \ldots, t_r)$, is a set of t_i where $t_i \geqslant 1$ is an integer for $i = 1, \ldots, r$ such that

$$t_1 + \ldots + t_r = n.$$

If r is an integer such that $1 \leqslant r \leqslant n$, we have, obviously, $\binom{n-1}{r-1}$ distinct r-compositions of n.

We shall say that an r-composition of n, $(t_1, \ldots, t_r)$, "dominates" the r-composition of n, $(t_1', \ldots, t_r')$, if and only if

$$\begin{aligned} t_1 &\geqslant t_1' \\ t_1 + t_2 &\geqslant t_1' + t_2' \\ &\vdots \\ t_1 + \ldots + t_{r-1} &\geqslant t_1' + \ldots + t_{r-1}' \end{aligned} \qquad \text{(A)}$$

Evidently

$$t_1 + \ldots + t_r = t_1' + \ldots + t_r' = n.$$

The relation of domination defined by (A) is reflexive, transitive and anti-symmetric. It thus represents a partial ordering of the r-compositions of n.

We shall now make a transformation on the r-compositions of n, suggested by the relations (A). After this transformation,

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we can decide immediately whether any two given r-compositions of n satisfy the relation of domination or not. Given an r-composition of n, $(t_1, \ldots, t_r)$, we associate with it the vector of r-elements $(T_1, \ldots, T_r)$ obtained as follows:

$$T_1 = t_1$$

$$T_2 = t_1 + t_2$$

$$\vdots$$

$$T_{r-1} = t_1 + \ldots + t_{r-1}$$

$$T_r = t_1 + \ldots + t_r = n.$$

We notice that the T_i are integers and

$$0 < T_1 < T_2 < \ldots < T_r = n. \qquad \text{(B)}$$

Evidently, given the composition $(t_1, \ldots, t_r)$, we can obtain the vector $(T_1, \ldots, T_r)$ and conversely, given the vector $(T_1, \ldots, T_r)$, (satisfying, of course, the relations (B)), we could obtain the r-composition $(t_1, \ldots, t_r)$. There are thus $\binom{n-1}{r-1}$ "composition-vectors" $(T_1, \ldots, T_r)$ and we may, without fear of confusion, talk either of the r-composition $(t_1, \ldots, t_r)$ or the associated vector $(T_1, \ldots, T_r)$.

If the composition $(t_1, \ldots, t_r)$ dominates $(t'_1, \ldots, t'_r)$ we shall find it convenient to say that the corresponding vector $(T_1, \ldots, T_r)$ dominates the corresponding vector $(T'_1, \ldots, T'_r)$. In the event that of two vectors, $(T_1, \ldots, T_r)$, $(T'_1, \ldots, T'_r)$, neither is dominated by the other, we shall say that they are <u>incomparable</u>.

It can be proved by mathematical induction that the number of r-compositions of n which are dominated by a particular r-composition, whose vector is $(T_1, \ldots, T_r)$, is given by D_{r-1} in the following formula:

$$D_k = \binom{T_k}{1} D_{k-1} - \binom{T_{k-1}+1}{2} D_{k-2} + \binom{T_{k-2}+2}{3} D_{k-3} - \ldots \ldots$$
$$\ldots + (-1)^{k-1} \binom{T_1+k-1}{k} D_0, \text{ where } D_0 = 1.$$

2. Lattice formed by the r-compositions of n.

Given two vectors, $T = (T_1, \ldots, T_r)$, $T' = (T'_1, \ldots, T'_r)$ corresponding to the r-compositions of n, $(t_1, \ldots, t_r)$, $(t'_1, \ldots, t'_r)$ respectively, let

$$M_i = \max(T_i, T'_i) \quad \text{for all } i = 1, \ldots, r$$
$$N_i = \min(T_i, T'_i)$$

$$(M_r = N_r = n)$$

The vectors

$$M = (M_1, \ldots, M_r)$$

$$N = (N_1, \ldots, N_r)$$

are easily seen to correspond to r-compositions of n, and we can prove easily that

(i) M dominates both T and T'.

(ii) If V dominates T and if V dominates T', then V dominates M.

Thus M is the l.u.b. of T and T', and similarly N is the g.l.b. of T and T'.

Let $T = (T_1, \ldots, T_r)$, $T' = (T'_1, \ldots, T'_r)$ and $T'' = (T''_1, \ldots, T''_r)$ be the composition-vectors corresponding to any three r-compositions of n. Utilising the standard notation of lattice theory, we can easily prove that

$$(T \cup T') \cap T'' = (T \cap T'') \cup (T' \cap T'');$$

for, this is equivalent to proving that, for all $i = 1, \ldots, r$, $\min[\max(T_i, T'_i), T''_i] = \max[\min(T_i, T''_i), \min(T'_i, T''_i)]$ which is established by considering all possible relations between T_i, T'_i, T''_i such as:

$$T_i < T'_i < T''_i$$
$$T_i = T'_i < T''_i$$
$$T_i < T'_i = T''_i \quad \text{etc.}$$

We see that:

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THEOREM 1. The r-compositions of an integer n form a distributive lattice. $(1 \leq r \leq n)$

3. An anti-isomorphism and an application.

Let $T = (T_1, \ldots, T_r)$ be the vector corresponding to an r-composition of n. Deleting the integers $T_1, \ldots, T_{r-1}$ from the set of positive integers $(1, \ldots, n)$ in their natural order, we have a set of $(n-r+1)$ integers which corresponds to the $(n-r+1)$ - composition vector $T' = (T_1', \ldots, T_{n-r+1}' = n)$. It is clear that, if we start with the $(n-r+1)$-composition vector $T' = (T_1', \ldots, T_{n-r+1}' = n)$ and follow the above procedure, we arrive at the r-composition vector $T = (T_1, \ldots, T_r)$.

We have thus defined a one-to-one correspondence between the r-compositions and the $(n-r+1)$-compositions of n.

Let us consider the vectors $T^{(1)} = (T_1^{(1)}, \ldots, T_r^{(1)})$ and $T^{(2)} = (T_1^{(2)}, \ldots, T_r^{(2)})$ associated with two distinct r-compositions of n, and the corresponding $(n-r+1)$-composition vectors $T^{(1)'} = (T_1^{(1)'}, \ldots, T_{n-r+1}^{(1)'})$ and $T^{(2)'} = (T_1^{(2)'}, \ldots, T_{n-r+1}^{(2)'})$. It is obvious that $T^{(2)'}$ dominates, is dominated by or is incomparable with $T^{(1)'}$ according as $T^{(1)}$ dominates, is dominated by or is incomparable with $T^{(2)}$, and hence:

THEOREM 2. The one-to-one correspondence between the r-compositions of n and the $(n-r+1)$-compositions of n is an anti-isomorphism.

Let a(n) and b(n) be the set of all compositions of n with elements ≤ 2 and ≥ 2 respectively.

If an r-composition of n involves the integers 1 and 2 only, the elements of the associated vector, $(T_1, \ldots, T_r)$, will be such that

$$T_i - T_{i-1} = 1 \text{ or } 2, \text{ for } i = 2, \ldots, r,$$

and $$T_1 = 1 \text{ or } 2.$$

Obviously, the elements of the corresponding $(n-r+1)$-composition vector, $(T_1', \ldots, T_{n-r+1}')$, will be such that

$$T_i' - T_{i-1}' \geq 2, \text{ for } i = 2, \ldots, (n-r),$$

while $$T_{n-r+1}' - T_{n-r}' \geq 1$$

and $$T_1' \geq 1.$$

To ensure that all elements of our (n-r+1)-composition are $\geqslant$ 2, we add one to the first and last elements, giving us an (n-r+1)-composition belonging to b(n+2). Clearly, starting with an (n-r+1)-composition of b(n+2) and applying the above procedure in reverse, we obtain an r-composition of a(n). Thus the anti-isomorphism of theorem 2 yields a one-to-one correspondence between the compositions belonging to a(n) and the compositions belonging to b(n+2).

A simple procedure for obtaining the composition of b(n+2) corresponding to the composition of a(n) is due to L.E. Bush. It can easily be seen that his procedure will give us the same one-to-one correspondence between the compositions belonging to a(n) and the compositions belonging to b(n+2).

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References

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L.E. Bush, Amer. Math. Monthly, G4 (1957), 649-654.

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The correspondence given by L.E. Bush is in his report of solutions to problems on the Putnam examination. Such a correspondence was given earlier by K. Bush.

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