p-INJECTIVITY OF SIMPLE PRE-TORSION MODULES

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Introduction. V-rings and their generalisations have been studied extensively in recent years [2], [3], [5], [6], [7]. All the rings we consider will be associative rings with $1 \neq 0$ and all the modules considered will be unitary left R-modules. All the concepts will be left-sided unless otherwise mentioned. Thus by an ideal in R we mean a left ideal of R. A ring R is called a V-ring (respectively a GV-ring) if every simple (resp. simple, singular) module is injective. An R-module M is called p-injective if any homomorphism $f: I \to M$ with I a principal left ideal of R can be extended to a homomorphism $g: R \to M$. A ring R is called a p-V-ring (resp. a p-V′-ring) if every simple (resp. simple, singular) module over R is p-injective. The object of the present paper is to introduce torsion theoretic generalizations of p-V-rings and prove results similar to those obtained by Yue Chi Ming about p-V-rings and p-V′-rings [6], [7]. For any $M \in R$-mod, $J(M)$ will denote the Jacobson radical of M and $Z(M)$ the singular submodule of M. For any $\lambda \in R$, we denote the left annihilator $\{r \in R \mid r\lambda = 0\}$ of λ in R by $l(\lambda)$.

In what follows we will follow the terminology from [4] regarding torsion theories. σ will denote a left exact pre-radical in R-mod, $\mathbf{T}_\sigma = \{M \in R\text{-mod} \mid \sigma(M) = M\}$ the associated hereditary pretorsion class, $\mathcal{F}_\sigma = \{I \subset R \mid R/I \in \mathbf{T}_\sigma\}$ the associated left linear topology on R.

LEMMA 1. *Suppose every simple module S in $\mathbf{T}_\sigma$ is* p-*injective. Let λ be any element of R. Let $I \in \mathcal{F}_\sigma$ satisfy $I \supset R\lambda R + l(\lambda)$. Then $I = R$.*

Proof. Suppose if possible that $I \neq R$. Then there exists a maximal left ideal L of R with $I \subset L$. Since $I \in \mathcal{F}_\sigma$, it follows that $L \in \mathcal{F}_\sigma$ and hence R/L is a simple module in $\mathbf{T}_\sigma$. Define $g: R\lambda \to R/L$ by $g(r\lambda) = r + L$. Observe that g is well-defined. Since R/L is p-injective, there exists an extension $f: R \to R/L$ of g. Let $f(1) = c + L$. Then $1 + L = g(\lambda) = f(\lambda) = \lambda c + L$. But $\lambda c \in R\lambda R \subset I \subset L$. This implies that $1 \in L$, a contradiction. This contradiction proves that $I = R$. ■

THEOREM 1. *Suppose every simple module S in $\mathbf{T}_\sigma$ is* p-*injective. Then*
(1) *any $I \in \mathcal{F}_\sigma$ is idempotent,*
(2) *for any $0 \neq I \in \mathcal{F}_\sigma$ there exists a simple quotient of I,*
(3) $J(R) \cap \sigma(R) = 0$,
(4) *if c is any element of R satisfying $l(c) = 0$ and $RcR \in \mathcal{F}_\sigma$ then $R = RcR$.*

Proof. (1) Suppose $I \neq I^2$. Let $a \in I$ satisfy $a \notin I^2$. Using Zorn's lemma choose a left ideal L of R with $I^2 \subset L \subset I$ and maximal with respect to the property $a \notin L$. It is well-known and easy to see that $(Ra + L)/L$ is simple. But $(Ra + L)/L \simeq Ra/(L \cap Ra)$.

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Hence $Ra/(L \cap Ra)$ is simple. Let $\eta : Ra \to Ra/(L \cap Ra)$ denote the canonical quotient map and $\bar{a} = \eta(a)$. Then $R\bar{a}$ is simple; moreover $l(\bar{a}) = (L \subset Ra : a) = (L : a)$. From $Ia \subset I^2 \subset L$ we get $l(\bar{a}) \supset I$, hence $l(\bar{a}) \in \mathcal{F}_\sigma$ yielding $R\bar{a} \in \mathbf{T}_\sigma$. It follows that $R\bar{a}$ is p-injective. Hence there exists an extension $f : R \to Ra/L \cap Ra$ of η. If

$$f(1) = \lambda a + L \cap Ra,$$

then $a + L \cap Ra = \eta(a) = f(a) = a\lambda a + L \cap Ra$. Hence $a - a\lambda a \in L \cap Ra$. But $a\lambda a \in RaRa \subset L \cap Ra$. It follows that $a \in L \cap Ra$, contradicting the fact that $a \notin L$ by the very choice of L. Hence $I = I^2$.

(2) We will actually show that if $0 \neq I \in \mathcal{F}_\sigma$, then $I \not\subset J(R)$. This will prove (2), because if M is a maximal left ideal of R with $I \not\subset M$, then $I \cap M$ is a maximal submodule of I. Now, suppose on the contrary $I \subseteq J(R)$. Let $0 \neq a \in I$. Let L be a submodule of I maximal with respect to the property $a \notin L$. Then as in (1), $Ra/(L \cap Ra)$ is simple. We claim that $Ia \subseteq L$. If $Ia \not\subset L$ then $Ia + L \cap Ra = Ra$, yielding $a = \lambda a + x$ with $\lambda \in I$, $x \in L \cap Ra$. Thus $(1 - \lambda)a = x \in L$. From $\lambda \in I \subseteq J(R)$ we see that $(1 - \lambda)$ is a unit. Hence $a \in L$, a contradiction to the choice of L. Hence $Ia \subseteq L \cap Ra$. This implies $Ra/(L \cap Ra) \in \mathbf{T}_\sigma$ as in (1). Hence $Ra/(L \cap Ra)$ is injective. As in (1) this again yields an element $r \in R$ with $a - ara \in L$. Thus $(1 - ar)a \in L$. From $a \in I \subseteq J(R)$ we see that $(1 - ar)$ is a unit. Hence $a \in L$, leading to a contradiction. This contradiction proves that $I \not\subset J(R)$.

(3) Let $\lambda \in J(R) \cap \sigma(R)$. From $\lambda \in \sigma(R)$ we see that $R\lambda \in \mathbf{T}_\sigma$, hence $l(\lambda) \in \mathcal{F}_\sigma$. In particular $R\lambda R + l(\lambda) \in \mathcal{F}_\sigma$. From Lemma 1 we get $R\lambda R + l(\lambda) = R$. Now $R\lambda R \subseteq J(R)$. Since $J(R)$ is small in R we get $l(\lambda) = R$, hence $\lambda = 0$.

(4) Since $l(c) = 0$ we get $RcR = RcR + l(c)$. If $RcR \in \mathcal{F}_\sigma$, by Lemma 1 we get $R = RcR + l(c)$. Hence $R = RcR$. ■

REMARKS. (a) Let $\sigma(M) = M$ for all $M \in R$-mod. Then σ is a left exact radical with $\mathcal{F}_\sigma = \{$all the left ideals I in $R\}$ and $\mathbf{T}_\sigma = R$-mod. In this case Theorem 1 yields the following.

COROLLARY 1. *Let R be a* p-V-*ring. Then*
(1) *every left ideal of R is idempotent,*
(2) *every non-zero left ideal of R has a simple quotient,*
(3) $J(R) = 0$,
(4) *$R = RcR$ for every $c \in R$ with $l(c) = 0$.*

This slightly strengthens Lemma 1 of [**6**].

(b) Let $\sigma = z$, the singular left exact pre-radical. Then $\mathcal{F}_z = \{I \mid I$ is an essential left ideal in $R\}$. Given any $\lambda \in R$ we can choose a left ideal K of R with $(R\lambda R + l(\lambda)) \cap K = 0$ and $(R\lambda R + l(\lambda)) \oplus K$ essential in R. Thus in this case Lemma 1 yields the following.

COROLLARY 2. *Let R be a* p-V′-*ring. Then for any $\lambda \in R$ there exists a left ideal K in R with*

$$(R\lambda R + l(\lambda)) \cap K = 0 \quad \text{and} \quad (R\lambda R + l(\lambda)) \oplus K = R.$$

This is Lemma 1 of [**7**].

Also in this case Theorem 1 yields the following:

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COROLLARY 3. *Let R be a* p-V′*-ring. Then*

(1) *every essential left-ideal of R is idempotent,*

(2) *every essential left ideal of R has a simple quotient,*

(3) $J(R) \cap Z(R) = 0$,

(4) $R = RcR$ *for every non-zero divisor c in R* (*i.e.* $l(c) = 0 = r(c)$).

Here $r(c)$ is the right annihilator of c in R.

Actually (1), (2), (3) follow from (1), (2), (3) of Theorem 1. As for (4), from Corollary 2 we get $K \subset R$ with $RcR \oplus K = R$. Now $cK \subset RcR \cap K = 0$. Since $r(c) = 0$ we get $K = 0$. Hence $R = RcR$.

Corollary 3 slightly strengthens Proposition 3 of [**7**].

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