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THE DUNFORD-PETTIS PROPERTY ON VECTOR-VALUED CONTINUOUS AND BOUNDED FUNCTIONS

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Let X be a completely regular space, E a Banach space, $C_b(X, E)$ the space of all continuous, bounded and E-valued functions defined on X, $M(X, \mathcal{L}(E, F))$ the space of all $\mathcal{L}(E, F)$-valued measures defined on the algebra generated by zero subsets of X. Weakly compact and β_0-continuous operators defined from $C_b(X, E)$ into a Banach space F are represented by integrals with respect to $\mathcal{L}(E, F)$-valued measures. The strict Dunford-Pettis and the Dunford-Pettis properties are established on $(C_b(X, E), \beta_i)$, where β_i denotes one of the strict topologies β_0, β or β_1, when E is a Schur space; the same properties are established on $(C_b(X, E), \beta_0)$, when E is an AM-space or an AL-space.

1. NOTATIONS AND DEFINITIONS

Let X be a completely regular space, E a Banach space, and $C_b(X, E)$ the space of all continuous, bounded and E-valued functions defined on X (if $E = \mathbb{R}$, then $C_b(X, \mathbb{R}) = C_b(X)$). We use $B(X)$ ($Ba(X)$) to denote the smallest algebra (σ-algebra) containing the zero-sets, the so-called Baire algebra (Baire σ-algebra), and $M(X)$ to denote the space of all zero-set regular measures defined on $B(X)$. We define $M(X, E')$ to be the space of all finitely additive vector measures $\mu\colon B(X) \to E'$ such that

(1) for each $x \in E$, μ_x, defined by $\mu_x(B) = \langle \mu(B), x\rangle$, is in $M(X)$

(2) $|\mu|(X) < \infty$, where $|\mu|(B) = \sup\{|\Sigma\langle \mu(B_i), s_i\rangle| : \{B_i\}$ is a finite $B(X)$-partition of B and $\{s_i\}$ is a finite collection from the unit ball of $E\}$.

It is known that $C_b(X, E)' = M(X, E')$.

For $B \in B(X)$, $\mu \in M(X, E')$ and a μ-measurable function f, we define $\int_B f\,d\mu = \lim \Sigma\mu(B_i)f(x_i)$, where the limit is taken over the directed set of all finite $B(X)$-partitions of B and $x_i \in B_i$.

Let F be another Banach space. We define $M(X, \mathcal{L}(E, F))$ to be the space of all finitely additive vector measures $\mu\colon Ba(X) \to \mathcal{L}(E, F)$ such that

(1) for each $x' \in F'$, $x'\mu$, defined by $x'\mu(B) = x'(\mu(B))$, is in $M(X, E')$,

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(2) $\|\mu\|(X) < \infty$, where for $B \in B(X)$, $\|\mu\|(B) = \sup\{|x'\mu|(B) \colon \|x'\| \leqslant 1\}$.

Let $\mu \in M(X, \mathcal{L}(E, F))$ and let f be a μ-measurable function from X into E. We say that f is μ-integrable in $B \in B(X)$ if

(i) for each $x' \in F'$, the integral $\int_B f d(x'\mu)$ exists .

(ii) there exists a vector in F, denoted by $\int_B f d\mu$, such that for all $x' \in F'$ we have $x'\left(\int f d\mu\right) = \int f d(x'\mu)$.

If f is μ-integrable over all B in $B(X)$, we say that f is μ-integrable.

We shall pay careful attention [**8**] to three classes of Baire measures described as follows:

Let μ be a Baire measure. μ is called a σ-additive measure if $\mu(E_n) \to 0$ for every sequence $\{E_n\}$ in $B(X)$ such that $E_n \downarrow \emptyset$. μ is called τ-additive if $\mu(E_\alpha) \to 0$ for every net $\{E_\alpha\}$ in $B(X)$ such that $E_\alpha \downarrow \emptyset$. μ is called tight if, given $\varepsilon > 0$, there exists a compact subset K of X such that $|\mu|(X \setminus K) < \varepsilon$. It is known that each of these measures can be extended to $Ba(X)$ [**11**].

We shall denote by $M_\sigma(X)$, $M_\tau(X)$ and $M_t(X)$ the space of all σ-additive, τ-additive and tight measures respectively. We shall understand for $M_\sigma(X, E')$ the space of all vector measures $\mu \in M(X, E')$ such that $\mu_x \in M_\sigma(X)$, for all $x \in E$. Similar meanings for $M_\tau(X, E')$ and $M_t(X, E')$. Finally, we shall understand by $M_\sigma(X, \mathcal{L}(E, F))$ the space of all vector measures $\mu \in M(X, \mathcal{L}(E, F))$ such that $y'\mu \colon B(X) \to E'$ belongs to $M_\sigma(X, E')$, for all $y' \in F'$. Similar meanings for $M_\tau(X, \mathcal{L}(E, F))$ and $M_t(X, \mathcal{L}(E, F))$. Clearly, we have that $M_t(X, \mathcal{L}(E, F)) \subset M_\tau(X, \mathcal{L}(E, F)) \subset M_\sigma(X, \mathcal{L}(E, F))$.

We shall define three locally convex topologies on $C_b(X, E)$, denoted by β_1, β and β_0, as follows:

Let Ω and Ω_1 be, respectively, the class of all compact and all zero sets in $\beta X \setminus X$, where βX denotes the Stone-Čech compactification of X. Let $Q \in \Omega(\Omega_1)$. We define β_Q as the locally convex topology generated by the family of semi-norms $f \to \|fg\|$, where $g \in B_Q = \{h \in C_b(X) \colon \widehat{h} \equiv 0 \text{ on } Q\}$ ($\widehat{h}$ denotes its extension to βX). $\beta(\beta_1)$ is the inductive limit of the topologies β_Q as Q ranges over $\Omega(\Omega_1)$. β_0 is defined as the finest locally convex topology which coincides on the norm bounded sets with the compact-open topology.

It is known that $(C_b(X, E), \beta_i)' = M_i(X, E')$, where β_i is any one of the above topologies.

The following characterisation of β_0-equicontinuous will be used [**3**].

Lemma 1. *A subset H of $M_t(X, E')$ is β_0-equicontinuous if and only if*

(a) *H is norm-bounded, and*

(b) *for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|\mu|(X \setminus K) < \varepsilon$ for all $\mu \in H$.*

Recall that a linear operator T from a topological vector space A into another B is weakly compact if it maps bounded subsets of A into relatively weakly compact subsets of B.

Katsara and Liu [7] showed the following theorem:

THEOREM 2. *Let F be another Banach space. If T is a continuous weakly compact operator from $C_b^{rc}(X, E) = \{f \in C_b(X, E) \colon f(X)$ is relatively compact in $E\}$ into F, then there exists a unique $m \in M(X, \mathcal{L}(E, F))$ such that*

(1) *every $f \in C_b^{rc}(X, E)$ is m-integrable and $T(f) = \int f dm$;*
(2) $\|T\| = \|m\|(X)$;
(3) *for every $x' \in F$, we have $T'x' = x'm$;*
(4) *for every bounded set S in E, the set $V_{mS} = \{\sum m(G_i)s_i \colon \{G_i\}$ is a finite B-partition of X, $s_i \in S\}$ is relatively weakly compact.*

Conversely, if $m \in M(X, \mathcal{L}(E, F))$ is such that (4) holds, then every f in $C_b^{rc}(X, E)$ is m-integrable and the operator $T(f) = \int f dm$ is norm-continuous and weakly compact.

The following theorem is a characterisation of a β_0-continuous and weakly compact operator T defined from $C_b^{rc}(X, E)$ into F. It is known that $C_b^{rc}(X, E)$ is β_0-dense in $(C_b(X, E), \beta_0)$.

THEOREM 3. *Let T be a weakly compact operator defined from $C_b^{rc}(X, E)$ into F. The following statements are equivalent:*

(1) *T is β_0-continuous.*
(2) *$T_{|B}$ is compact-open-continuous, where B is the unit ball of $C_b^{rc}(X, E)$.*
(3) *$(\forall \varepsilon > 0)(\exists K \subset X$, K compact) $(|x'm|(X \setminus K) < \varepsilon$ uniformly for $\|x'\| \leqslant 1)$, where m is the associated vector measure given in the above theorem.*

PROOF: Since β_0 is the finest locally convex topology agreeing with the compact-open topology on bounded subsets of $C_b(X, E)$, we have (1) $\Leftrightarrow$ (2).

(3) $\Rightarrow$ (1). Let $\{f_\alpha\}$ be a compact-open null convergent net. Hence, for a given $\varepsilon > 0$, there exists a compact subset K of X such that $\|m\|(X \setminus K) = \sup\{|x'm|(X \setminus K) \colon \|x'\| \leqslant 1, x' \in F'\} < \varepsilon$, which implies that $|x'm|(X \setminus K) < \varepsilon$ uniformly for $x' \in F'$, $\|x'\| \leqslant 1$. Therefore, $\{x'm \colon x' \in F', \|x'\| \leqslant 1\}$ is a β_0-equicontinuous subset of $M_t(X, E')$. Since $x'm = T'x'$ and $(C_b^{rc}(X, E), \beta_0)' \cong M_t(X, E')$, we have $\{T'x' \colon x' \in F', \|x'\| \leqslant 1\}$ is also β_0-equicontinuous.

Finally, since $\|Tf\| = \sup\{|T'x'(f)| : x' \in F', \|x'\| \leqslant 1\}$, we conclude that $Tf_\alpha \to 0$ in norm.

(1) $\Rightarrow$ (3). Since $\{T'x' : x' \in F', \|x'\| \leqslant 1\}$ is β_0-equicontinuous is equivalent to $\{x'm : x' \in F', \|x'\| \leqslant 1\}$ is β_0-equicontinuous, the arguments are similar to these given above. $\square$

THEOREM 4. *Let T be a β_0-continuous and weakly compact linear operator defined on $C_b(X, E)$. Then there exists a unique $m \in M(X, \mathcal{L}(E, F))$ such that*

(1) *every $f \in C_b(X, E)$ is m-integrable and $T(f) = \int f dm$;*
(2) $\|T\| = \|m\|(X)$;
(3) *for every $x' \in F$, we have $T'x = x'm$;*
(4) *for every bounded set S in E, the set $V_{mS} = \{\sum m(G_i)s_i : \{G_i\}$ is a finite B-partition of X, $s_i \in S\}$ is relatively weakly compact.*

PROOF: Since $C_b^{rc}(X, E)$ is β_0-dense in $(C_b(X, E), \beta_0)$, T is the unique β_0-continuous extension of $T|_{C_b^{rc}(X,E)}$ to $C_b(X, E)$. By Theorem 3, $T|_{C_b^{rc}(X,E)}$ has a unique vector measure m associated to it which satisfies (1), (2), (3) and (4).

By the fact that if norm T and its restriction are the same and their respective transposes coincide, we only have to prove that each $f \in C_b(X, E)$ is m-integrable and $T(f) = \int f dm$.

Let $f \in C_b(X, E)$; hence, by the definition of the topology β_0 and the density of $C_b^{rc}(X, E)$, there exists a net $\{f_\alpha\} \in C_b^{rc}(X, E)$ inside the ball of radius $\|f\|$ such that $f_\alpha \to f$ uniformly on compact subsets of X. Take any $G \in Ba(X)$; since the dual of $(C_b(X, E), \beta_0)$ is $M_t(X, E')$, we have that, for any $x' \in F'$, $\int_G f dx'm$ is defined. At the same time, it is not difficult to see that $\{\int_G f dm\}$ is a norm-Cauchy net in F and then it is convergent to some $v_G \in F$.

We claim that $\int_G f_\alpha dx'm \to \int_G f dx'm$. In fact, given $\varepsilon > 0$, there exists a compact subset K of X, such that $|x'm|(G \setminus K) \leqslant |x'm|(X \setminus K) < \varepsilon$ uniformly for $\|x'\| \leqslant 1$. Also, there exists α_0 such that, for $\alpha \geqslant \alpha_0$,

$$\sup\{\|f_\alpha(x) - f(x)\| : x \in K\} < \varepsilon.$$

The claim follows from

$$\begin{aligned} \left|\int_G (f_\alpha - f) dx'm\right| &\leqslant \int_G \|f_\alpha - f\| \, d|x'm| \\ &\leqslant \int_{G\cap K} \|f_\alpha - f\| \, d|x'm| + \int_{G\setminus K} \|f_\alpha - f\| \, d|x'm|. \end{aligned}$$

Since $x'\left(\int_G f_\alpha dm\right) = \int_G f_\alpha dx'm \to x'(v_G)$ and $\int_G f_\alpha dx'm \to \int_G f dx'm$, we conclude that $\int_G f dx'm = x'(v_G)$. Defining $v_G = \int_G f dm$, we have, for each $G \in Ba(X)$, $\int_G f_\alpha dm \to \int_G f dm$; in particular, $T(f_\alpha) = \int_X f_\alpha dm \to \int_X f dm$. Therefore, by the continuity of T, we have $\int_X f dm = T(f)$. $\square$

We shall adopt the following definitions:

A locally convex space E is said to have the Dunford-Pettis (respectively strict Dunford-Pettis) property if for any Banach space F, any weakly compact and continuous operator T from E into F transforms absolutely convex weakly compact subsets (respectively weakly Cauchy sequences) of E into relatively compact subsets (respectively convergent ones) of F [6]. It is known that both definitions are equivalent in Banach spaces.

The next theorem was proved in [9] and it establishes some relation between the Dunford-Pettis (D-P) and the strict Dunford- Pettis (strict D-P) properties.

THEOREM 5. *If E', the topological dual of E, has a σ-compact dense subset in the $\sigma(E', E)$-topology and E has the strict D-P property then E has the D-P property.*

2. THE STRICT DUNFORD-PETTIS AND THE DUNFORD-PETTIS PROPERTIES

In this section, we shall discuss, first, the strict D-P and the D-P properties on $(C_b(X, E), \beta_i)$ with E a Schur space. After that, we shall discuss the same properties on $(C_b(X, E), \beta_0)$ with some special space E.

It is known that in $C_b(X, E)$, β_i-bounded subsets coincide with normed-bounded subsets.

Let T be an F-valued, weakly compact and β_i-continuous operator defined on $C_b(X, E)$. Since the strict topologies are coarser than the norm topology and have the same bounded subsets, we have that T is also $\|.\|$-continuous and weakly compact. Therefore by Theorem 2, $T|_{C_b^{rc}(X, E)}$ has an associated measure m of $M(X, \mathcal{L}(E, F))$.

LEMMA 6. *Let T be an operator as above. Then there exists a nonnegative finite real-valued, σ-additive measure μ such that, for any $\varepsilon > 0$, there exists $\delta > 0$, so that for each $A \in Ba(X)$ with $\mu(A) < \delta$, we have $|x'm|(A) < \varepsilon$, uniformly for $\|x'\| \leqslant 1$.*

PROOF: It is easy to see that each $x'm$ in $M_i(X, E')$ can be extended to $Ba(X)$ and that $M_i(X, E')$ is $\|.\|$-closed in $M(X, E')$.

Since T is weakly compact, we have that $T'\colon F' \to C_b(X, E)' = M(X, E')$ is weakly compact. Also, by the β_i-continuity of T, we have that $T'x' \in M_i(X, E')$. Therefore, if $B_{F'}$ denotes the unit ball of F', then $\overline{T'(B_{F'})}^{\|\cdot\|}$ is convex and weakly closed contained in $M_i(X, E')$. Now, since $\sigma\big(M_i(X, E'), M_i(X, E')'\big)$ coincides with the induced $\sigma\big(M(X, E'), M(X, E')'\big)$ on $M_i(X, E')$, we have that $T'(B_{F'})$ is relatively weakly compact in $M_i(X, E')$.

Thus, $\{|x'm| : \|x'\| \leqslant 1\}$ is uniformly σ-additive and then there exists a nonnegative finite real-valued σ-additive measure μ such that for any $\varepsilon > o$, there exists $\delta > 0$, so that for each $A \in Ba(X)$ with $\mu(A) < \delta$, we have $|x'm|(A) < \varepsilon$, uniformly for $\|x'\| \leqslant 1$ [4].

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The measure μ, given in the above theorem, is called a control measure of $\{|x'm| : \|x'\| \leqslant 1\}$. ∎

A Banach space E is a Schur space if weakly convergent sequences of E are norm convergent on it.

Theorem 7. *If E is a Schur space, then $(C_b(X, E), \beta_i)$ has the strict D-P property.*

Proof: Let T be a weakly compact and β_i-continuous operator defined from $C_b(X, E)$ into a Banach space F. We shall denote by m its associated vector measure.

Since T is weakly compact and, in particular, β_1-continuous, we have that $\{|x'm| : \|x'\| \leqslant 1\}$ has a control measure μ ([Lemma 6]). Let $\{f_n\}$ be a weakly Cauchy sequence in $(C_b(X, E), \beta_i)$; hence $\{f_n\}$ is uniformly bounded and, for each $x \in X$, $\{f_n(x)\}$ is a weakly Cauchy sequence in E (this follows from the fact that $\delta_x \otimes e' \in M_i(X, E')$, where $\delta_x \otimes e'(f) = e'(f(x))$ and $e' \in E$). Using the fact that E is a Schur space, for each $x \in X$, $\{f_n(x)\}$ is norm-Cauchy. By Egoroff's Theorem, given $\varepsilon > 0$, there exists a compact subset K of X, such that $\mu(X \setminus K) < \varepsilon$ and $\{\|f_n\|\}$ converges uniformly on K. Now, since

$$\|T(f_n - f_m)\| = \sup\{|x'T(f_n - f_m)| : \|x'\| \leqslant 1\} = \sup\left\{\left|\int f\,dx'm\right| : \|x'\| \leqslant 1\right\}$$

and
$$\left|\int f\,dx'm\right| \leqslant \int_K \|f\|\,d|x'm| + \int_{X\setminus K} \|f\|\,d|x'm| < \varepsilon\,|x'm|\,(K) + 2M\varepsilon$$

uniformly for $\|x'\| \leqslant 1$, where M is the uniform bound of $\{f_n\}$, we have that $\{T(f_n)\}$ is convergent in F. This proves the theorem. ∎

Theorem 8. *If E is a Schur space and X is σ-compact, then $(C_b(X, E), \beta_i)$ has the Dunford-Pettis property.*

Proof: The proof is exactly the same as in Theorem 3.2 [1]. ∎

Lemma 9. *If Y is a compact space, then $(C_b(X, C(Y)), \beta_0)$ is β_0-homeomorphic to $(C_b(X \times Y), \beta_0)$. Consequently, $(C_b(X, C(Y)), \beta_0)$ has both properties, the strict and the Dunford-Pettis properties [9].*

Proof: The function $\Psi\colon C_b(X, C(Y)) \to C_b(X \times Y)$, define by $\Psi(f)(x, y) = f(x)(y)$, is one-one and onto with inverse function $\Psi\colon C_b(X \times Y) \to C_b(X, C(Y))$ defined by $\Psi(g)\colon X \to C(Y)$ where $\Psi(g)(x)(y) = g(x, y)$.

Note that $|\Psi(f)(x, y)| = |f(x)(y)| \leqslant \|f(x)\|$ which implies Ψ is continuous in the β_0-topology.

On the other hand, if $\{f_\alpha\}$ is a net in $C_b(X \times Y)$, converging to 0 in the β_0-topology, and K is a compact subset of X, then, by the compactness of Y, $f_\alpha \to 0$ uniformly on $K \times Y$; therefore $\Phi(f_\alpha) \to 0$ uniformly on K. ∎

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An AM-space is a Banach lattice which satisfies the axiom $\|\sup\{x, y\}\| = \sup\{\|x\|, \|y\|\}$. It is known that every AM-space E with unit is isomorphic to $C(K)$, where K is some compact space [10].

THEOREM 10. *If E is a AM-space, then $(C_b(X, E), \beta_0)$ has both properties, the strict and the Dunford-Pettis properties.*

PROOF: It follows from Lemma 9 and the fact that every AM-space is isomorphic to a $C(Y)$, for some compact space Y. ∎

An AL-space is a Banach lattice which satisfies $\|x + y\| = \|x\| + \|y\|$ for $x \geqslant 0$, $y \geqslant 0$. It is known that every AL-space is isomorphic to $L^1(\mu)$, where μ is a Radon measure defined on a locally compact space.

THEOREM 11. *If E is an AL-space, then $(C_b(X, E), \beta_0)$ has the strict Dunford-Pettis Property.*

PROOF: By the above remark, it is enough to prove that $C_b(X, L^1)$ has the strict Dunford-Pettis Property. Let T be a weakly compact and β_0-continuous linear operator from $C_b(X, L^1)$ into a Banach space F and let m be its associated vector measure. Take an arbitrary weakly Cauchy sequence $\{f_n\}$ in $C_b(X, L^1)$ and denote by M its uniform bound.

By Theorem 3, given $\varepsilon > 0$, there exists a compact subset K of X such that $|x'm|(X \setminus K) < \varepsilon/4M$, uniformly for $x' \in F'$, $\|x'\| \leqslant 1$.

We claim that $\{f_{n|_K}\}$ is weakly Cauchy in $\left(C(K, L^1), \|.\|\right)$. In fact, let λ be a continuous linear functional on $C(K, L^1)$ and let us define $\lambda_1\colon C_b(X, L^1) \to \mathbb{R}$ by $\lambda_1(f) = \lambda(f_{|_K})$. Clearly, λ_1 is $\|.\|$ and β_0-continuous and then, $\lambda(f_{n|_K} - f_{m|_K}) = \lambda_1(f_n - f_m) \to 0$ as $n, m \to \infty$.

Now, we define $L\colon C(K, L^1) \to F$ by $L(f) = \int_K f\,dm$. L is weakly compact and $\|.\|$-continuous. Therefore, since $C(K, L^1)$ has the strict Dunford-Pettis or simply the Dunford-Pettis property [2], we have that, for the $\varepsilon > 0$ given above, there exists $N \in \mathbb{N}$ such that for $n, m \geqslant N$,

$$\|L(f_n - f_m)\| = \left\|\int_K (f_n - f_m)dm\right\| < \varepsilon/2$$

and then

$$\left|\int_K (f_n - f_m)dx'm\right| < \varepsilon/2 \quad \text{uniformly for} \quad \|x'\| \leqslant 1$$

Now, let $n, m \geqslant N$. Since

$$\begin{aligned}\|T(f_n - f_m)\| &= \sup\{|x'(T(f_n - f_m))| : x' \in F,\ \|x'\| \leqslant 1\} \\ &= \sup\left\{\left|\int (f_n - f_m)dx'm\right| : x' \in F,\ \|x'\| \leqslant 1\right\},\end{aligned}$$

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we have

$$\left|\int (f_n - f_m)dx'm\right| \leqslant \left|\int_K (f_n - f_m)dx'm\right| + \int_{X\setminus K} \|f_n - f_m\|\, d\,|x'm|$$
$$< \varepsilon/2 + 2M(\varepsilon/4M) = \varepsilon$$

uniformly for $\|x'\| \leqslant 1$. Therefore,

$$\|T(f_n - f_m)\| = \sup\left\{\left|\int (f_n - f_m)dx'm\right| : x' \in F,\ \|x'\| \leqslant 1\right\} < \varepsilon$$

for $n, m \geqslant N$.

This proves the theorem. □

THEOREM 12. *If E is an AL-space, then $(C_b(X, E), \beta_0)$ has the Dunford-Pettis property.*

PROOF: By the isomorphism between E and L^1, it is enough to assume that $E = L^1$.

We shall first suppose that X has a σ-compact dense subset X. Then, developing the same argument given in [9, Theorem 3, p.363], we prove that $\left(M_t(X, L^1), w^*\right)$ contains a σ-compact dense. Therefore, by Theorem 5 and Theorem 11 we conclude that $\left(C_b(X, L^1), \beta_0\right)$ has the Dunford-Pettis property.

Suppose now that X is not necessarily as above and let T be a weakly compact, β_0-continuous operator defined from $C_b(X, L^1)$ into F with associated vector measure m.

By Theorem 3, given $\varepsilon = 1/n$, there exists a compact subset K_n of X such that $|x'm|(X \setminus K_n) < 1/n$, uniformly for $x' \in F'$, $\|x'\| \leqslant 1$. Defining X_0 as the closure of $\bigcup K_n$, we have that X_0 is a σ-compact subset of X and $|x'm|(X \setminus X_0) = 0$, uniformly for $x' \in F'$, $\|x'\| \leqslant 1$.

We define $L\colon C_b(X_0, L^1) \to F$ by $L(f) = \int_{X_0} f\,dm$. We claim that L is weakly compact and $\|.\|$-continuous. By Katsara [7], it is enough to prove that, for all bounded S in E, the sets V_S and $V_{0_S} = \{\Sigma m(G_i)s_i\colon \{G_i\}$ is a finite B-partition of X, $s_i \in S\}$ are equal. In fact, we note first that, for any $s \in E$, $\|s\| \leqslant 1$, $|x'm(X \setminus X_0)s| \leqslant |x'm|(X \setminus X_0) = 0$ and then $\|m(X \setminus X_0)s\| = 0$, which implies that $m(X \setminus X_0) \equiv 0$. By the same argument, we show that, for any $G \in B(X)$, $m(G \cap X \setminus X_0) \equiv 0$. Thus,

$$m(G)s = m((G \cap X \setminus X_0) \cup (G \cap X_0))s = m(G \cap X \setminus X_0)s + m(G \cap X_0)s = m(G \cap X_0)s$$

and then $V_S \subset V_{0_S}$. On the other hand, let $\Sigma m(G_i)s_i \in V_{0_S}$, where $X_0 = \bigcup G_i$ and $s_i \in S$. Hence, taking the $B(X)$-partition $G_1, \ldots, G_n, X \setminus X_0$ of X and any $s \in S$, we have that $\Sigma m(G_i)s_i = \Sigma m(G_i)s_i + m(X \setminus X_0)s \in V_S$.

Now, since V_S is relatively weakly compact in F, so is V_{0_S}. Then L is weakly compact and $\|.\|$-continuous [Theorem 4].

The operator $\phi\colon C_b(X, L^1) \to C_b(X_0, L^1)$ defined by $\phi(f) = f_{|X_0}$ is clearly β_0-continuous, since X_0 is closed. Therefore, ϕ is weak-weak continuous. Also, $T = L_0\phi$, since $m(X \setminus X_0) \equiv 0$. Therefore, if B is an absolutely convex, weakly compact subset of $C_b(X, L^1)$, then $\phi(B)$ is an absolutely convex and relatively weakly compact subset of $C_b(X_0, L^1)$. Since $(C_b(X, L^1), \beta_0)$ has the D-P property, $L(\phi(B))$ is relatively compact in F, and since $T(B) = L(\phi(B))$, we have that T transforms absolutely convex and weakly compact subsets of $C_b(X, L^1)$ into compact subsets of F. This proves the theorem. ∎

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