Forum of Mathematics, Sigma (2025), **Vol. 13**:e144 1–14
doi:10.1017/fms.2024.149

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Boundedness of slc degenerations of polarized log Calabi–Yau pairs

Junpeng Jiao

Yau Mathematical Sciences Center, Tsinghua University, Beijing, China; E-mail: jiao_jp@tsinghua.edu.cn.

Received: 17 January 2024; **Revised:** 2 October 2024; **Accepted:** 16 November 2024

2020 Mathematical Subject Classification: *Primary* – 14E30; *Secondary* – 14E05

Abstract

Given a family of pairs over a smooth curve whose general fiber is a log Calabi–Yau pair in a fixed bounded family, suppose there exists a divisor on the family whose restriction on a general fiber is ample with bounded volume. We show that if the total space of the family has relatively trivial log canonical divisor and the special fiber has slc singularities, then every irreducible component of the special fiber is birationally bounded.

Contents

Throughout this paper, we work over the complex number field $\mathbb{C}$.

1. Introduction

A family of projective pairs of dimension d over a smooth curve (possibly non-proper) is an object

$$f : (X, \Delta) \to C,$$

consisting of a morphism of schemes $f : X \to C$ and an effective $\mathbb{Q}$-divisor Δ satisfying the following properties,

- f is projective, flat, of finite type, of relative dimension d, with reduced fibers,
- every irreducible component $D_i \subset \mathrm{Supp}(\Delta)$ dominates C, and all fibers of $\mathrm{Supp}(D)$ have pure dimension $d - 1$, and

- f is smooth at generic points of $X_s \cap \mathrm{Supp}(D)$ for every $s \in C$, where $X_s := f^{-1}(s)$ denotes the fiber over s.

Usually, fibers of a family of projective pairs behave wildly. [13, 1.43] gives several examples of families of projective surfaces whose special fibers are canonically polarized and general fibers are even not of general type. [13] also shows that such jumps of Kodaira dimension happen when the canonical class of the total space is not $\mathbb{Q}$-Cartier. Thus, it is natural to consider the case when $K_X + \Delta$ is $\mathbb{Q}$-Cartier. And in this case, $K_{X_s} + \Delta_s = (K_X + \Delta)|_{X_s}$ is also $\mathbb{Q}$-Cartier for all closed point $s \in C$ according to the adjunction formula.

In general, a family of projective pairs $f : (X, \Delta) \to C$ over a smooth curve such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X_s, Δ_s) is an slc pair for every closed point $s \in C$ is called locally stable. The notion of locally stable morphisms has been verified to be a very important definition in the moduli of varieties and satisfies many nice properties – for example, the plurigenera are constant; see [13, Theorem 5.11] (see Definition 2.1 for the definition of slc pairs).

For a family of projective pairs $f : (X, \Delta) \to C$ which is locally stable over $0 \in C$, we call (X_0, Δ_0) an slc degeneration of $\{(X_s, \Delta_s), s \neq 0\}$. In this paper, we study the birational boundedness of slc degenerations of polarized log Calabi–Yau fibrations. The boundedness of polarized log Calabi–Yau pairs is studied in [6]. The first result shows that for a family of projective pairs over a curve, suppose the log canonical divisor of the total space is relatively trivial, general fibers are in a fixed bounded family of polarized log Calabi–Yau pairs and the special fiber is an slc degeneration. Then every irreducible component of the slc degeneration is bounded up to birationally equivalence (see Definition 2.4 for the definition of boundedness).

Theorem 1.1. *Fix a natural number d and positive rational numbers c, v. Let X be a quasi-projective normal variety, $f : (X, \Delta) \to C$ a family of projective pairs of dimension d over a smooth curve C and $0 \in C$ a closed point. Suppose*

- $K_X + \Delta \sim_{\mathbb{Q},C} 0$,
- (X_0, Δ_0) *is an slc pair, and*
- *there is a divisor N on X such that a general fiber $(X_g, \Delta_g), N_g$ is a (d, c, v)-polarized log Calabi–Yau pair (see Definition 4.1).*

Then every irreducible component of X_0 is birationally bounded.

Note $K_X + \Delta \sim_{\mathbb{Q},C} 0$ implies $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Because the discrepancy is a lower semi-continuous function (see [12, Corollary 4.10]), then (X_0, Δ_0) is an slc pair implies (X_s, Δ_s) is an slc pair for every s in an open neighborhood of 0, and it means $(X, \Delta) \to C$ is locally stable over an open neighborhood of 0.

Fano varieties naturally have polarized log Calabi–Yau structures. The following corollary is an application of Theorem 1.1 to families of log Fano pairs.

Corollary 1.2. *Fix a natural number d and positive rational numbers c, ϵ. Let X be a normal quasi-projective variety, $f : (X, \Delta) \to C$ a family of projective pairs of dimension d over a smooth curve C and $0 \in C$ a closed point. Suppose*

- $-(K_X + \Delta)$ *is ample over C,*
- (X_0, Δ_0) *is an slc pair,*
- *a general fiber (X_g, Δ_g) is ϵ-lc, and*
- $\mathrm{coeff}(\Delta) \subset c\mathbb{N}$.

Then every irreducible component of X_0 is birationally bounded.

Note that in Theorem 1.1, we only assume the existence of polarization on general fibers, the slc degeneration has no assumption on positivity, and hence, it does not have a polarized log Calabi–Yau pair structure. Example 2.5 shows that boundedness up to birational equivalence is the best result one can hope for. Example 2.6 shows that the number of irreducible components of the slc degeneration cannot be bounded either.

After the paper has been completed, Birkar informed the author that he and Santai Qu [8] obtained Theorem 1.1 and Corollary 1.2 independently.

Sketch of the proof of Theorem 1.1. The main tools used in this paper are the moduli space of polarized log Calabi–Yau pairs [6] and the weak semistable reduction [1]; see also [2]. With the same notation as in Theorem 1.1, because a general fiber $(X_g, \Delta_g), N_g$ is a (d, c, v)-polarized log Calabi–Yau pair, there exists a moduli map $C \setminus 0 \to \mathcal{S}$, where $\mathcal{S}$ is the moduli space of (d, c, v)-polarized log Calabi–Yau pairs. Because $\mathcal{S}$ is proper, the moduli map extends to a morphism $C \to \mathcal{S}$, and after a finite cover, it will define a new fibration $(X', \Delta'), N' \to C$ whose fibers are (d, c, v)-polarized log Calabi–Yau pairs. In particular, (X'_0, Δ'_0) is log bounded. Because $(X, \Delta) \to C$ and $(X', \Delta') \to C$ are both log Calabi–Yau fibrations over C with the same generic fiber, then they are crepant birationally equivalent over C. Therefore, any irreducible component of X_0 is an lc place of (X', Δ'). Note that an lc center of (X', Δ') contained in X'_0 is also an lc center of (X'_0, Δ'_0) by adjunction which is in a bounded family. The main difficulty is to use the boundedness of lc centers to prove the birational boundedness of lc places since the contraction from an exceptional divisor to its image can not be controlled. We use weak semistable reduction to make singularities toric, and for toric cases, such contraction is well understood according to [10].

2. Preliminary

2.1. Notations and basic definition

We will use the same notation as in [15] and [17].

A *sub-log pair* (X, Δ) consists of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We call (X, Δ) a *log pair* if in addition, Δ is effective. If $g : Y \to X$ is a birational morphism and E is a divisor on Y, the *discrepancy* $a(E, X, \Delta)$ is $-\mathrm{coeff}_E(\Delta_Y)$, where $K_Y + \Delta_Y := g^*(K_X + \Delta)$. A sub-log pair (X, Δ) is called *sub-klt* (respectively *sub-lc*) if for every birational morphism $Y \to X$ as above, $a(E, X, \Delta) > -1$ (respectively ≥ -1) for every divisor E on Y. A log pair (X, Δ) is called *klt* (respectively *lc*) if (X, Δ) is sub-klt (respectively sub-lc) and (X, Δ) is a log pair.

Let $(Y, \Delta_Y), (X, \Delta)$ be two sub-log pairs and $h : Y \to X$ a projective birational morphism. We say $(Y, \Delta_Y) \to (X, \Delta)$ is a *crepant birational morphism* if $K_Y + \Delta_Y \sim_{\mathbb{Q}} h^*(K_X + \Delta)$, two sub-log pairs $(X_i, \Delta_i), i = 1, 2$ are *crepant birationally equivalent* if there is a sub-log pair (Y, Δ_Y) and two crepant birational morphisms $(Y, \Delta_Y) \to (X_i, \Delta_i), i = 1, 2$.

Let (X, Δ) be a sub-log pair. We say a divisor P over X is a *log place* of (X, Δ) if the discrepancy $a(P, X, \Delta) < 0$. A closed subvariety $W \subset X$ is called a *log center* of (X, Δ) if W is the image of a log place of (X, Δ) on X. In particular, a log place P of a sub-log pair (X, Δ) such that $a(P, X, \Delta) \leq -1$ is called a *nklt place*, respectively, a *nklt center* is the image of a nklt place. When (X, Δ) is sub-lc, a nklt place (respectively, a nklt place) is also called an *lc place* (respectively, an *lc center*).

A *contraction* is a projective morphism $f : X \to Z$ of quasi-projective varieties with $f_*\mathcal{O}_X = \mathcal{O}_Z$. If X is normal, then so is Z, and the fibers of f are connected. A *fibration* is a contraction $f : X \to Z$ of normal quasi-projective varieties with $\dim X > \dim Z$.

For a scheme X, a *stratification* of X is a disjoint union $\coprod_i X_i$ of finitely many locally closed subschemes $X_i \hookrightarrow X$ such that the corresponding morphism $\coprod_i X_i \to X$ is both monomorphism and surjective.

Definition 2.1. A *semi-pair* (X, Δ) consists of a reduced quasi-projective scheme of pure dimension and a $\mathbb{Q}$-divisor $\Delta \geq 0$ on X satisfying the following conditions:

- X is S_2 with nodal codimension one singularities,
- no component of $\mathrm{Supp}(\Delta)$ is contained in the singular locus of X, and
- $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

We say that (X, Δ) is *semi-log canonical (slc)* if, in addition, we have

- if $\pi : X^\nu \to X$ is the normalization of X and Δ^ν is the sum of the birational transform of Δ and the conductor divisor of π, then every irreducible component of (X^ν, Δ^ν) is lc. We call (X^ν, Δ^ν) the normalization of (X, Δ).

For an slc pair (X, Δ) with normalization (X^ν, Δ^ν), we say a divisor P over X is a *log place* (respectively, an *lc place*) of (X, Δ) if P is a log place (respectively, an lc place) of an irreducible component of (X^ν, Δ^ν), and the image of P on X is called a log center (respectively, an *lc center*) of (X, Δ).

The following is the definition of locally stable morphisms defined in [13, Chapter 4]. In this paper, we only work on the case when the base is smooth, and in this case, the definition of locally stable morphisms is much more simple; see Lemma 2.3.

Definition 2.2. Let S be a reduced scheme and n a natural number. *A projective family of pairs* of dimension n over S is an object

$$f : (X, \Delta) \to S,$$

consisting of a morphism of schemes $f : X \to S$ and an effective $\mathbb{Q}$-divisor Δ satisfying the following properties:

- f is projective, flat, of finite type, of pure relative dimension n, with geometrically reduced fibers,
- every irreducible component $D_i \subset \operatorname{Supp}(\Delta)$ dominates an irreducible component of S, and all nonempty fibers of $\operatorname{Supp}(\Delta) \to S$ have pure dimension $n - 1$. In particular, $\operatorname{Supp}(\Delta)$ does not contain any irreducible component of any fiber of f, and
- the morphism f is smooth at generic points of $X_s \cap \operatorname{Supp}(D)$ for every $s \in S$.

We say a projective family of pairs $f : (X, \Delta) \to S$ is *well-defined* if further,

- there exists an open subset $U \subset X$ such that
  - codimension of $X_s \setminus U_s$ is ≥ 2 for every $s \in S$, and
  - $\Delta|_U$ is $\mathbb{Q}$-Cartier.

Let $f : (X, \Delta) \to S$ be a well-defined projective family of pairs over a reduced scheme S. We say f is *locally stable* if it satisfies the following conditions:

- $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, and
- (X_s, Δ_s) is an slc pair for every $s \in S$.

We say f is *stable* if further,

- $K_{X/S} + \Delta$ is ample over S.

According to [13, Theorem-Definition 4.3], when S is normal, a family of projective family of pairs is naturally well-defined.

Lemma 2.3 [13, Corollary 4.55]. *Let S be a smooth scheme and $f : (X, \Delta) \to S$ a morphism. Then f is locally stable if and only if the pair $(X, \Delta + f^*D)$ is slc for every snc divisor $D \subset S$.*

Definition 2.4. We say that a set $\mathcal{X}$ of varieties is *bounded* (respectively, *birationally bounded*) if there is a projective morphism $\mathcal{W} \to \mathcal{T}$, where $\mathcal{T}$ is of finite type, such that for every $X \in \mathcal{X}$, there is a closed point $t \in \mathcal{T}$ and an isomorphism $\mathcal{W}_t \to X$ (respectively, a birational map $\mathcal{W}_t \dashrightarrow X$).

Example 2.5. [11, Corollary 1.2] shows that for any $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that $\alpha^2 + \beta^2 + \gamma^2 = 3\alpha\beta\gamma$, there exists a morphism $X \to C$ over a smooth curve germ $0 \in C$, such that $K_X \sim_{\mathbb{Q},T} 0$, X_t is isomorphic to $\mathbb{P}^2$ for $t \neq 0$ and X_0 is isomorphic to the weighted projective space $\mathbb{P}(\alpha, \beta, \gamma)$. Write $U := C \setminus 0$ and $X_U := X \times_C U$. Because $-K_{X_U}$ is very ample over U, let $\Delta_U, N_U \in |-K_{X_U/U}|$ be two general elements. Then a general fiber of $(X_U, \Delta_U), N_U \to U$ is a $(2,1,9)$-polarized log Calabi–Yau pair. Since the function $\alpha^2 + \beta^2 + \gamma^2 = 3\alpha\beta\gamma$ has infinitely many positive integer solutions, the special fiber is not bounded, while they are all birationally equivalent to $\mathbb{P}^2$.

Example 2.6. Let $f : (X, \Delta) \to C$ be a family of pairs over a smooth curve satisfying the assumptions in Theorem 1.1. Suppose X_0 has more than two irreducible components. Note that every irreducible component of X_0 is an lc center of $(X, \Delta + X_0)$. Then $(X, \Delta + X_0)$ has infinitely many lc places over 0.

Fix a positive integer $m \gg 0$. Suppose $(Y, \Delta_Y) \to (X, \Delta + X_0)$ is a crepant birational morphism that only extracts at least m lc places of $(X, \Delta + X_0)$ over c. Denote the morphism $Y \to C$ by f_Y. Then Δ_Y is equal to the strict transform of Δ plus $\mathrm{red}(f_Y^* 0)$. Suppose l is the least common multiple of the set of coefficients of $f_Y^* 0$. Let $\pi : C' \to C$ be a ramified cover whose ramified index along $0'$ is l, where $0'$ is a closed point of $\pi^{-1}(0)$. Let Y' be the normalization of $Y \times_C C'$. Then by [13, Lemma 2.53], $f_{Y'} : Y' \to C'$ has reduced fibers.

Denote the morphism $Y' \to Y$ by π_Y. By the Hurwitz's formula, there is a $\mathbb{Q}$-divisor Δ'_Y on Y' such that $K_{Y'} + \Delta'_Y \sim_{\mathbb{Q}} \pi_Y^*(K_Y + \Delta_Y)$. Because $\Delta_Y \geq \mathrm{red}(f_Y^* 0)$, by [12, 2.42], $\Delta'_Y \geq \mathrm{red}(f_{Y'}^* 0') = f_{Y'}^* 0'$. Then by the definition, $(Y', \Delta'_Y - f_{Y'}^* 0') \to C'$ is a locally stable morphism satisfying the assumptions in Theorem 1.1, and Y'_0 has at least m irreducible components.

3. Almost semistable reduction and toroidal embedding

3.1. Toric varieties

Let $N', N \cong \mathbb{Z}^n$ be lattices, Σ', Σ be fans in N', N respectively. A map between fans, in notation $\psi : \Sigma' \to \Sigma$, is a homomorphism $\psi : N' \to N$ of lattices that satisfies the condition: For each $\sigma' \in \Sigma'$, there exists a $\sigma \in \Sigma$ such that $\psi(\sigma') \subset \sigma$. Such ψ determines a morphism $\tilde{\psi} : X_{\Sigma'} \to X_{\Sigma}$. A morphism between toric varieties that arises in this way is called a *toric morphism*.

Let Σ'_σ be the set of cones in Σ' whose interior is mapped to the interior of $\sigma \in \Sigma$. Pick $\sigma' \in \Sigma'_\sigma$. The image $\psi(N'/N'_{\sigma'})$ in N/N_σ is independent of the choice of σ' in Σ'_σ. We define the index $[N/N_\sigma : \psi(N'/N'_\sigma)]$ to be the index of $\tilde{\psi}$ over O_σ, and denote it by $Ind(\sigma)$.

Let $\tau' \in \Sigma'_\sigma$ and $\{\sigma'_1, \sigma'_2, \ldots\}$ be the set of cones in Σ'_σ that contains τ' as a face. Then each σ'_i determines a cone $\bar{\sigma'_i}$ in $\psi^{-1}((N_\sigma)_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}}$, defined by

$$\bar{\sigma'_i} = (\sigma'_i + (N'_{\tau'})_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}}.$$

Note that $\sigma'_i + (N'_{\tau'})_{\mathbb{R}}$ is contained in $\psi^{-1}(N_\sigma)_{\mathbb{R}}$ since $\tau', \sigma'_i \in \Sigma'_\sigma$. Thus, $\{\bar{\sigma'_1}, \bar{\sigma'_2}, \ldots\}$ defines a fan in $\psi^{-1}((N_\sigma)_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}}$. The fan in $\psi^{-1}((N_\sigma)_{\mathbb{R}})/(N'_{\tau'})_{\mathbb{R}}$ constructed above will be called the *relative star* of τ' over σ and will be denoted by $Star_\sigma(\tau')$

A cone $\tau' \in \Sigma'_\sigma$ is called *primitive* with respect to ψ if none of the faces of τ' are in Σ'_σ.

Let X_Σ be a toric variety. We call the divisor $D_\Sigma := X_\Sigma \setminus T$ the *toric boundary* of X_Σ, where T is the big torus.

Theorem 3.1. [10, Proposition 2.1.4] *Let $\tilde{\psi} : X_{\Sigma'} \to X_\Sigma$ be a toric morphism induced by a map of fans $\psi : \Sigma' \to \Sigma$. Then,*

- *The image $\tilde{\psi}(X_{\Sigma'})$ of $\tilde{\psi}$ is a subvariety of X_Σ. It is realized as the toric variety corresponding to the fan $\Sigma_\psi := \Sigma \cap \psi(N'_{\mathbb{R}})$.*
- *The fiber of $\tilde{\psi}$ over a point $y \in X_{\Sigma_\psi}$ depends only on the orbit $O_\sigma, \sigma \in \Sigma_\psi$, that contains y. Denote this fiber by F_σ. Then it can be described as follows.*

  Define Σ'_σ to be the set of cones σ' in Σ', whose interior is mapped to the interior of σ. Let $Ind(\sigma)$ be the index of $\tilde{\psi}$ over O_σ. Then $\psi^{-1}(y) = F_\sigma$ is a disjoint union of $Ind(\sigma)$ identical copies of connected reducible toric variety F^c_σ, whose irreducible components $F^{\tau'}_\sigma$ are the toric variety associated to the relative star $Star_\sigma(\tau')$ of the primitive elements τ' in Σ'_σ.
- *For $\sigma \in \Sigma_\psi, \tilde{\psi}^{-1}(O_\sigma) = \tilde{O}_\sigma \times F^c_\sigma$, where $\tilde{O}_\sigma$ is a connected covering space of O_σ of order $Ind(\sigma)$.*

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

Remark 3.2. Here, the term reducible toric variety means a reducible variety obtained by gluing a collection of toric varieties along some isomorphic toric orbits.

Theorem 3.3. [10, Remark 2.1.12] *If ψ is surjective, then $Ind(\sigma) = 1$ for all $\sigma \in \Sigma$.*

For any toric variety X_Σ, it is well-known that there is a refinement $\psi : \Sigma' \to \Sigma$, that is, each cone of Σ is a union of cones in Σ', such that $\tilde{\psi} : X_{\Sigma'} \to X_\Sigma$ is a resolution of singularities.

Theorem 3.4. *Let $\tilde{\psi} : X_{\Sigma'} \to X_\Sigma$ be the resolution defined by a refinement $\psi : \Sigma' \to \Sigma$. Suppose V is a prime divisor of $X_{\Sigma'} \setminus T_N$. Then P is birationally equivalent to $\mathbb{P}^r \times \tilde{\psi}(P)$, where $r = \dim V - \dim \tilde{\psi}(P)$.*

Proof. Because $\tilde{\psi}$ is a toric morphism, every prime divisor of $X_{\Sigma'} \setminus T_N$ corresponds to a 1-dimension cone of Σ'. Fix a cone $\sigma \in \Sigma$ of dimension ≥ 2, and suppose $\sigma_1', \sigma_2', \ldots \in \Sigma_\sigma'$ are the 1-dimensional cones that map to the interior of σ, which are clearly primitive. Let $\sigma_1'', \sigma_2'', \ldots \in \Sigma_\sigma'$ be other primitive cones. By Theorem 3.3 and Theorem 3.1, $\tilde{\psi}^{-1}(O_\sigma) = O_\sigma \times F_\sigma^c$, and the irreducible components of F_σ^c correspond to the cones $\{\sigma_1', \sigma_2', \ldots\} \cup \{\sigma_1'', \sigma_2'', \ldots\}$.

By comparing the dimension of exceptional locus, it is easy to see that the codimension 1 components of $\tilde{\psi}^{-1}(O_\sigma)$ are equal to $O_\sigma \times F_{\sigma'}^c$, where the irreducible components of $F_{\sigma'}^c$ are the toric variety associated to the relative stars $\{Star_\sigma(\sigma_1'), Star_\sigma(\sigma_2'), \ldots\}$. Suppose P is the divisor defined by σ_1'. Then $P \subset \tilde{\psi}^{-1}(O_\sigma)$ is a codimension 1 component and birational equivalent to $O_\sigma \times F_{\sigma_1'}^c$, where $F_{\sigma_1'}^c$ is the toric variety associated to the relative star $Star_\sigma(\sigma_1')$. Because every irreducible toric variety is birationally equivalent to $\mathbb{P}^r$ for some $r \in \mathbb{N}$, the result follows. □

3.2. *Toroidal embeddings*

Given a normal variety X and an open subset $U_X \subset X$, the embedding $U_X \subset X$ is called *toroidal* if for every closed point $x \in X$, there exist a toric variety X_σ, a point $s \in X_\sigma$, and an isomorphism of complete local k-algebras

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_\sigma, s},$$

such that the ideal of $X \setminus U_X$ maps isomorphically to the ideal of $X_\sigma \setminus T_\sigma$, where T_σ is the big torus.

Given a normal variety X and a reduced divisor D on X, we call (X, D) a *toroidal pair* if $U_X := X \setminus D \subset D$ is a toroidal embedding.

In this paper, we will assume that every irreducible component of $X \setminus U_X$ is normal – that is, $U_X \subset X$ a strict toroidal embedding.

Proposition 3.5 [14, Page 195]. *Let $U \subset X$ be a toroidal embedding of varieties and x a closed point of X. Then there exists an affine toric variety X_σ and an étale morphism ψ from an open neighborhood of $x \in X$ to X_σ, such that locally at x (for the Zariski topology), we have $U = \psi^{-1}(T)$, where T is the big torus.*

A dominant morphism $f : (Y, D_Y) \to (X, D)$ of toroidal pairs is called toroidal if for every closed point $x \in X$, there exist local models (X_σ, s) at x, (X_τ, t) at $f(x)$ and a toric morphism $g : X_\sigma \to X_\tau$ so that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{X_\sigma, s} \\ {\scriptstyle \hat{f}^\#}\uparrow & & \uparrow{\scriptstyle \hat{g}^\#} \\ \hat{\mathcal{O}}_{B, f(x)} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{X_\tau, t}, \end{array}$$

where $\hat{f}^\#$ and $\hat{g}^\#$ are the algebra homomorphisms induced by f and g.

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

Corollary 3.6 [1, Corollary 1.6]. *If $f : (X, D) \to (Y, D_Y)$ and $g : (Y, D_Y) \to (Z, D_Z)$ are toroidal morphisms, then $g \circ f : (X, D) \to (Z, D_Z)$ is a toroidal morphism.*

Definition 3.7 [1, Section 8.2]. Let $f : (X, D) \to (Z, B)$ be a projective toroidal morphism between toroidal pairs with connected fibers. We say f is *almost semistable* if

- the morphism f is equidimensional,
- all the fibers of the morphism f are reduced,
- Z is smooth, and
- X has quotient singularities.

Theorem 3.8 (Almost Semistable Reduction). *Let $X \to Z$ be a projective morphism between projective normal varieties and $D \subset X$ be a divisor. Then there exists a proper, surjective, generically finite morphism of irreducible varieties $b : Z' \to Z$, a projective birational morphism of irreducible varieties $a : X' \to (X \times_Z Z')^m$, where $(X \times_Z Z')^m$ is the main component of the fiber product $X \times_Z Z'$, and divisors $B' \subset Z'$, $D' \subset X'$, such that*

- $a^{-1}(D \times_Z Z') \cup f'^{-1}(B') \subset D'$, *and*
- *the morphism $f' : (X', D') \to (Z', B')$ is almost semistable.*

Proof. This is a direct result of [1, Theorem 2.1], [1, Proposition 4.4], [1, Proposition 5.1] and [1, Section 8.2]. □

Lemma 3.9 [1, Lemma 6.2]. *Let $f : (X, D) \to (Z, B)$ be an almost semistable morphism, $g : C \to Z$ a morphism such that C is nonsingular and $g^{-1}(B)$ is a normal crossing divisor. Define $X_C = C \times_Z X$ and let $g_C : X_C \to X$, $f_C : X_C \to C$ be the two projections.*

Denote $B_C = g^{-1}(B)$ and $D_C = g_C^{-1}(D)$. Then (C, B_C) and (X_C, D_C) are toroidal pairs, and $f_C : (X_C, D_C) \to (C, B_C)$ is an equidimensional toroidal morphism with reduced fibers.

Lemma 3.10. *Let X be a projective normal variety, and D a reduced divisor on X such that (X, D) is a toroidal pair. Suppose $\Delta \leq D$ is a $\mathbb{Q}$-divisor such that (X, Δ) is sub-lc.*

If P is an lc place of (X, Δ), then P is birational equivalent to $\mathbb{P}^r \times V$, where V is the image of P in X and $r = \dim X - \dim V - 1$.

Proof. Let P be an lc place of (X, Δ), and suppose x is a general point of the image of P on X. For the rest of the proof, we consider Zariski locally near x by replacing X with an open neighborhood of x.

Let X_σ be the affine toric variety defined in Proposition 3.5 and $\sigma \subset \sigma_1$ a subdivision such that $h_\sigma : X_{\sigma_1} \to X_\sigma$ is a resolution. Because π is étale, $X_1 := X_{\sigma_1} \times_{X_\sigma} X$ is a log resolution of (X, D). We have the following diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi_1} & X_{\sigma_1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_\sigma} \\ X & \xrightarrow[\pi]{} & X_\sigma \end{array}$$

Let D_1 be the strict transform of D on X_1 plus the h-exceptional divisor. Then (X_1, D_1) is log smooth, and $h : (X_1, D_1) \to (X, D)$ is a toroidal morphism. By an easy computation of discrepancies on log smooth pairs, it is easy to see that P can be obtained by a sequence of blow-ups along strata of (X_1, D_1). We will show that such morphism is étale locally equal to a toric morphism between toric varieties.

Suppose we have a sequence of blow-ups $h_i : X_{i+1} \to X_i$, $1 \leq i \leq k-1$ along a strata V_i of (X_i, D_i), where D_{i+1} is the strict transform of D_i plus the h_i-exceptional divisor, so that P is a divisor on X_k. Next, we show that there is a Cartesian diagram

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

$$\begin{array}{ccc} X_j & \xrightarrow{\pi_j} & X_{\sigma_j} \\ {\scriptstyle g_j}\downarrow & & \downarrow \\ X_1 & \xrightarrow{\pi_1} & X_{\sigma_1}, \end{array}$$

where

- ∘ the horizontal arrows are étale morphisms,
- ∘ σ_j is a refinement of σ_1,
- ∘ $X_{\sigma_j} \to X_{\sigma_1}$ is the corresponding toric morphism, and
- ∘ near any closed point of $g_j^{-1}x_1$, we have $U_j := X_j \setminus D_j = \pi_j^{-1}T_j$, where T_j is the big torus of X_{σ_j},

for all $1 \le j \le k$.

Suppose it is true for $j = i$. Let $X_{\sigma_{i+1}} \to X_{\sigma_i}$ be the toric morphism determined by blowing up X_i along the image of V_i on X_{σ_i}. Because blowing up is uniquely determined by local equations and both $X_{i+1} \to X_i$ and $X_{\sigma_{i+1}} \to X_{\sigma_i}$ are obtained by blowing up the same subvariety étale locally, then there is a natural étale morphism $\pi_{i+1} : X_{i+1} \to X_{\sigma_{i+1}}$ such that near any closed point of $g_j^{-1}x$, we have $U_{i+1} = \pi_{i+1}^{-1}T_{i+1}$, where T_{i+1} is the big torus of $X_{\sigma_{i+1}}$. Because the composition of $X_{\sigma_{i+1}} \to X_{\sigma_i}$ and $X_{\sigma_i} \to X_{\sigma_1}$ is a toric morphism, the claim is true for $j = i + 1$.

Now we have the following Cartesian diagram

$$\begin{array}{ccc} X_k & \xrightarrow{\pi_k} & X_{\sigma_k} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\sigma} \\ X & \xrightarrow[\pi]{} & X_{\sigma}. \end{array}$$

By assumption, P is a divisor on X_k and π_k is étale near a general point of P. Then P is equal to the pullback of a divisor P_{σ_k} on X_{σ_k}. Because $\sigma_k \to \sigma$ is a refinement, by Theorem 3.4, $f_\sigma|_{P_{\sigma_k}}$ is birationally equivalent to a $\mathbb{P}^r$-bundle. Because the diagram is Cartesian, $f|_P$ is also birationally equivalent to a $\mathbb{P}^r$-bundle. Then P is birationally equivalent to $V \times \mathbb{P}^r$. □

4. Moduli of polarized log Calabi–Yau pairs

In this section, we recall some definitions and results on the moduli of stable pairs and polarized log Calabi–Yau pairs, see [13], [16], [5] and [6]. We fix a natural number d and positive rational numbers c, v.

Definition 4.1. A log Calabi–Yau pair is an slc pair (X, Δ) such that $K_X + \Delta \sim_{\mathbb{Q}} 0$.

A polarized log Calabi–Yau pair consists of a log Calabi–Yau pair (X, Δ) and an effective ample integral divisor $N \ge 0$ such that $(X, \Delta + uN)$ is slc for any sufficiently small positive real number $u \ll 1$.

A (d, c, v)-polarized log Calabi–Yau pair is a polarized log Calabi–Yau pair $(X, \Delta), N$ such that $\dim X = d, \Delta = cD$ for some integral divisor D, and $\mathrm{vol}(N) = v$.

Let $f : X \to S$ be a flat morphism of schemes with S_2 fibers of pure dimension. A closed subscheme $D \subset X$ is a relative Mumford divisor over S if there is an open subset $U \subset X$ such that

- ∘ codimension of $X_s \setminus U_s$ is ≥ 2 for every $s \in S$,
- ∘ $D|_U$ is a Cartier divisor,
- ∘ $\mathrm{Supp}(D|_U)$ does not contain any irreducible component of any fiber U_s,
- ∘ D is the closure of $D|_U$, and
- ∘ $X \to S$ is smooth at the generic points of $X_s \cap D$ for every $s \in S$.

Definition 4.2. Let S be a reduced scheme. A (d, c, v)-polarized log Calabi–Yau family over S consists of a projective morphism $f : X \to S$ of schemes, a $\mathbb{Q}$-divisor Δ and an integral divisor N on X such that

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

- $(X, \Delta + uN) \to S$ is a stable family for some rational number $u > 0$ with fibers of pure dimension d,
- $\Delta = cD$, where $D \geq 0$ is a relative Mumford divisor,
- $N \geq 0$ is a relative Mumford divisor,
- $K_{X/S} + \Delta \sim_{\mathbb{Q},S} 0$, and
- for any fiber X_s of f, $\mathrm{vol}(N|_{X_s}) = v$.

Lemma 4.3. *There exist a positive rational number t and a natural number r both depending only on d, c, v such that $rc, rt \in \mathbb{N}$ satisfying the following. Assume $(X, \Delta), N$ is a (d, c, v)-polarized log Calabi–Yau pair. Then*

- *$(X, \Delta + tN)$ is an slc pair,*
- *$\Delta + tN$ uniquely determines Δ, N and*
- *$r(K_X + \Delta + tN)$ is very ample with*

$$h^j(mr(K_X + \Delta + tN)) = 0$$

for $m, j > 0$.

Proof. This is Lemma 7.7 in the first arXiv version of [6]. □

The following definition comes from Chapter 7 in the first arXiv version of [6].

Definition 4.4. Let t, r be as in Lemma 4.3. To simplify notation, let $\Theta = (d, c, v, t, r)$. A strongly embedded Θ-polarized log Calabi–Yau family over a reduced scheme S is a (d, c, v)-polarized log Calabi–Yau family $f : (X, \Delta), N \to S$ together with a closed embedding $g : X \to \mathbb{P}^n_S$ such that

- $n = h^0(r(K_{X_s} + \Delta_s + tN_s))$ for a closed point $s \in S$,
- $(X, \Delta + tN) \to S$ is a stable family,
- $f = \pi g$, where π denotes the projection $\mathbb{P}^n_S \to S$,
- letting $\mathcal{L} := g^*\mathcal{O}_{\mathbb{P}^n_S}(1)$, we have $R^q f_*\mathcal{L} \cong R^q \pi_* \mathcal{O}_{\mathbb{P}^n_S}(1)$ for all q, and
- for every $s \in S$, we have

$$\mathcal{L}_s \cong \mathcal{O}_{X_s}(r(K_{X_s} + \Delta_s + tN_s)).$$

We denote the family by $f : (X \subset \mathbb{P}^n_S, \Delta), N \to S$.

Define the functor $\mathcal{E}^s\mathcal{PCY}_\Theta$ on the category of reduced schemes by setting

$$\mathcal{E}^s\mathcal{PCY}_\Theta(S) = \{\text{strongly embedded } \Theta\text{-polarized log Calabi–Yau families over } S\}.$$

Proposition 4.5. *The functor $\mathcal{E}^s\mathcal{PCY}_\Theta$ is represented by a reduced separated scheme $\mathcal{S} := E^sPCY_\Theta$ together with a universal family $(\mathcal{X} \subset \mathbb{P}^n_{\mathcal{S}}, \mathcal{D}), \mathcal{N} \to \mathcal{S}$.*

Proof. This is Proposition 7.8 in the first arXiv version of [6]. □

5. Boundedness of log places

The main result in this section is the following.

Theorem 5.1. *Fix a natural number d and positive rational numbers c, v. Then there exist a natural number l and a bounded family of projective varieties $\mathcal{W} \to \mathcal{T}$ both depending only on d, c, v, such that:*

Suppose X is a normal quasi-projective variety, (X, Δ) is an lc pair, and $f : X \to C$ is a projective morphism of relative dimension d over a smooth curve C (possibly non-proper), such that

- *$K_X + \Delta \sim_{\mathbb{Q},C} 0$, and*
- *there is a divisor N on X such that a general fiber $(X_g, \Delta_g), N_g$ is a (d, c, v)-polarized log Calabi–Yau pair.*

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

Let $0 \in C$ *be a closed point and* P *an lc place of* $(X, \Delta + \mathrm{lct}(X, \Delta; f^*0) f^*0)$. *Then there is a closed point* $t \in \mathcal{T}$ *and a rational map* $\mathcal{W}_t \dashrightarrow P$ *which is a finite cover over the generic point of* P *with degree less or equal to* $\min\{l, \mathrm{mult}_P f^*0\}$.

Lemma 5.2. *Let* $(\mathcal{X}, \mathcal{D}') \to \mathcal{S}$ *be a locally stable morphism over a smooth variety* $\mathcal{S}$, *and* $\mathcal{D}$ *be a* $\mathbb{Q}$*-divisor such that* $\mathcal{D} \le \mathcal{D}'$ *and* $K_{\mathcal{X}} + \mathcal{D}$ *is* $\mathbb{Q}$*-Cartier. Then the set*

$$\{V \mid V \text{ is a log center of } (\mathcal{X}_s, \mathcal{D}_s) \text{ for some closed point } s \in \mathcal{S}\}$$

is bounded.

Proof. After passing to a stratification of $\mathcal{S}$, we may assume that $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ has a fiberwise log resolution $\xi : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{S}$ is smooth. Define $\mathcal{D}_{\mathcal{Y}}$ by $K_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}} \sim_{\mathbb{Q}} \xi^*(K_{\mathcal{X}} + \mathcal{D})$. Then we have

$$K_{\mathcal{Y}_s} + \mathcal{D}_{\mathcal{Y}_s} \sim_{\mathbb{Q}} \xi^*(K_{\mathcal{X}_s} + \mathcal{D}_s)$$

for any closed point $s \in \mathcal{S}$. It is easy to see that every log center of $(\mathcal{X}_s, \mathcal{D}_s)$ is dominated by a log center of $(\mathcal{Y}_s, \mathcal{D}_{\mathcal{Y}_s})$.

By the construction, $(\mathcal{Y}, \mathrm{Supp}(\mathcal{D}_{\mathcal{Y}}))$ is log smooth over $\mathcal{S}$, and we denote its strata by $\mathcal{V}_i, i \in I$. Then $\mathcal{V}_i \to \mathcal{S}$ is smooth for all $i \in I$. Because $(\mathcal{Y}_s, \mathrm{Supp}(\mathcal{D}_{\mathcal{Y}_s}))$ is log smooth for all $s \in \mathcal{S}$, then any log center of $(\mathcal{Y}_s, \mathcal{D}_{\mathcal{Y}_s})$ is $V_i|_{\mathcal{Y}_s}$ for some $i \in I$. Then any log center of $(\mathcal{X}_s, \mathcal{D}_s)$ is isomorphic to $\xi(V_i)|_{\mathcal{X}_s}$ for some $i \in I$, and the set of families $\xi(V_i) \to \mathcal{S}, i \in I$ parametrizes all log center of $(\mathcal{X}_s, \mathcal{D}_s)$. The result follows. □

Lemma 5.3. *Let* $f : X \to T$ *be a flat morphism from a normal variety to a smooth curve* T. *Let* $\pi : S \to T$ *be a ramified cover and* $Y \to X \times_T S$ *the normalization of the main component, and denote the projection* $Y \to S$ *by* f_Y.

Fix a closed point $t \in T$, *and let* $s \in \pi^{-1}t$ *be a closed point. Suppose* P *is an irreducible component of* f^*t *and* Q *is an irreducible component of the preimage of* P *in* Y *such that* $f_Y(Q) = s$. *Denote the ramified index of* π *along* s *by* r_S, *the multiplicity of* f^*t *along* P *by* m_P. *Then the degree of the finite morphism* $\pi_Q : Q \to P$ *is less or equal to* $\min\{r_S, m_P\}$.

Proof. By assumption, we have the following diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\pi_Q} & P \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi_Y} & X \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[\pi]{} & T. \end{array}$$

Denote the ramified index of π_Y along the generic point of Q by r_Q and $\mathrm{mult}_Q f_Y^* s$ by m_Q.

Next, we calculate the multiplicity of $\pi_Y^* f^* t$ along Q. By the definition of the ramified index, we have

$$\mathrm{mult}_Q \pi_Y^* f^* t = m_P \mathrm{mult}_Q \pi_Y^* P = m_P r_Q.$$

However, since $\pi_Y f = f_Y \pi$, we have

$$\mathrm{mult}_Q f_Y^* \pi^* t = r_S \mathrm{mult}_Q f_Y^* s = r_S m_Q.$$

Then we have $m_P r_Q = r_S m_Q$.

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

Choose a general point $x \in P$. The degree of π_Q is equal to the number of points in $\pi_Q^{-1}(x)$. By comparing the preimages of x in Y (with multiplicity), we have

$$\deg(\pi_Q) r_Q \le r_S.$$

After multiplying both sides by m_Q, we have

$$\deg(\pi_Q) r_Q m_Q \le r_S m_Q = r_Q m_P.$$

Then we have $\deg(\pi_Q) m_Q \le m_P$. Since r_Q, m_Q are positive integers, $\deg(\pi_Q)$ is less or equal to $\min\{r_S, m_P\}$. □

Proof of Theorem 5.1. Suppose $(X, \Delta) \to C$ is a fibration, and N is a divisor on X such that

- $K_X + \Delta \sim_{\mathbb{Q},C} 0$, and
- a general fiber $(X_g, \Delta_g), N_g$ is a (d, c, v)-polarized log Calabi–Yau pair.

By Lemma 4.3, there exist a positive rational number t and a natural number r such that $(X_g, \Delta_g + tN_g)$ is an slc pair and $r(K_{X_g} + \Delta_g + tN_g)$ is very ample without higher cohomology. By [9, §3, Theorem 12.11], $r(K_{X_U} + \Delta_U + tN_U)$ is very ample over an open subset $U \subset C$, and it defines a closed embedding $g : X_U \hookrightarrow \mathbb{P}^n_U$, where $n = h^0(r(K_{X_g} + \Delta_g + tN_g))$. Also because $(X_U, \Delta_U + tN_U) \to U$ is a stable family, then $f_U : (X_U \subset \mathbb{P}^n_U, \Delta_U), N_U \to U$ is a strongly embedded polarized log Calabi–Yau family over U. Since $\mathcal{E}^s\mathcal{PCY}_\Theta$ has a fine moduli space $\mathcal{S}$ with the universal family $(\mathcal{X} \subset \mathbb{P}^n_{\mathcal{S}}, \mathcal{D}), \mathcal{N} \to \mathcal{S}$ according to Proposition 4.5, we have $(X_U, \Delta_U) \cong (\mathcal{X}, \mathcal{D}) \times_{\mathcal{S}} U$, where $U \to \mathcal{S}$ is the moduli map defined by f_U. We denote this moduli map by ϕ_U. Note $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ is a (d, c, v)-polarized log Calabi–Yau family over $\mathcal{S}$. In particular, $K_{\mathcal{X}/\mathcal{S}} + \mathcal{D} \sim_{0,\mathcal{S}} 0$ is $\mathbb{Q}$-Cartier.

After replacing $\mathcal{S}$ by a dense open subset, we may assume that $\mathcal{S}$ is smooth and there is a fiberwise log resolution $\xi : (\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{D})$ over $\mathcal{S}$, where $\mathcal{D}_{\mathcal{Y}}$ is the $\mathbb{Q}$-divisor such that $K_{\mathcal{Y}/\mathcal{S}} + \mathcal{D}_{\mathcal{Y}} \sim_{\mathbb{Q}} \xi^*(K_{\mathcal{X}/\mathcal{S}} + \mathcal{D})$. Then $\mathcal{Y}$ is smooth. Let $\mathcal{S}'$ be a smooth compactification of $\mathcal{S}$ such that $\mathcal{S}' \setminus \mathcal{S}$ is a divisor and $(\mathcal{S}', \mathcal{S}' \setminus \mathcal{S})$ is log smooth, $\mathcal{Y}'$ a smooth compactification of $\mathcal{Y}$ such that $\mathcal{Y} \to \mathcal{S}$ extends to a projective morphism $\mathcal{Y}' \to \mathcal{S}'$ and $\mathcal{Y}' \setminus \mathcal{Y}$ is pure of codimension 1. Let $\mathcal{R}'$ be the reduced divisor whose support is equal to the sum of $\mathcal{Y}' \setminus \mathcal{Y}$ and the closure of $\operatorname{Supp}(\mathcal{D}_{\mathcal{Y}})$ in $\mathcal{Y}'$. Then $(\mathcal{Y}', \mathcal{R}') \times_{\mathcal{S}'} \mathcal{S} \cong (\mathcal{Y}, \operatorname{Supp}(\mathcal{D}_{\mathcal{Y}}))$. Because $(\mathcal{Y}, \operatorname{Supp}(\mathcal{D}_{\mathcal{Y}}))$ is log smooth over $\mathcal{S}$, then $(\mathcal{Y}', \mathcal{R}') \to \mathcal{S}'$ is almost semistable over $\mathcal{S}$.

By Theorem 3.8, there is a generically finite cover $\tau : \bar{\mathcal{S}} \to \mathcal{S}'$ and a birational morphism $\psi : \bar{\mathcal{Y}} \to \mathcal{Y}' \times_{\mathcal{S}'} \bar{\mathcal{S}}$ such that

$$\chi : (\bar{\mathcal{Y}}, \bar{\mathcal{R}}) \to (\bar{\mathcal{S}}, \bar{\mathcal{B}})$$

is an almost semistable morphism, where $\bar{\mathcal{B}} \supset \bar{\mathcal{S}} \setminus \tau^{-1}\mathcal{S}$ and $\bar{\mathcal{R}} \supset \chi^{-1}\bar{\mathcal{B}} \cup \psi^{-1}(\mathcal{D}' \times_{\mathcal{S}'} \bar{\mathcal{S}})$ are reduced divisors. Perhaps after replacing $\mathcal{S}$ by a dense open subset, we may assume there is a $\mathbb{Q}$-divisor $\bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}$ on $\bar{\mathcal{Y}}$ such that

- τ is a finite cover over $\mathcal{S}$,
- every component of $\operatorname{Supp}(\bar{\mathcal{D}}'_{\bar{\mathcal{Y}}})$ is horizontal over $\bar{\mathcal{S}}$,
- $(K_{\bar{\mathcal{Y}}} + \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}})|_{\bar{\mathcal{Y}}_{\bar{\mathcal{S}}^o}} \sim_{\mathbb{Q},\bar{\mathcal{S}}^o} 0$, where $\bar{\mathcal{S}}^o := \tau^{-1}\mathcal{S}, \bar{\mathcal{Y}}_{\bar{\mathcal{S}}^o} := \bar{\mathcal{Y}} \times_{\bar{\mathcal{S}}} \bar{\mathcal{S}}^o$, and
- for every point $\bar{p} \in \bar{\mathcal{S}}^0$ (not necessarily closed), the fiber of $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}) \to \bar{\mathcal{S}}$ over $\bar{p}$ is crepant birationally equivalent to the base change of the fiber of $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ over $\tau(\bar{p})$.

Because $K_{\mathcal{Y}/\mathcal{S}} + \mathcal{D}_{\mathcal{Y}} \sim_{\mathbb{Q}} \xi^*(K_{\mathcal{X}/\mathcal{S}} + \mathcal{D})$, a general fiber of $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ is slc, $(\mathcal{Y}, \operatorname{Supp}(\mathcal{D}_{\mathcal{Y}}))$ is log smooth over $\mathcal{S}$, and $\bar{\mathcal{R}} \supset \chi^{-1}\bar{\mathcal{B}} \cup \psi^{-1}(\mathcal{D}' \times_{\mathcal{S}'} \bar{\mathcal{S}})$. Then we have $\bar{\mathcal{D}}'_{\bar{\mathcal{Y}}} \le \bar{\mathcal{R}}$. Since $\chi : (\bar{\mathcal{Y}}, \bar{\mathcal{R}}) \to (\bar{\mathcal{S}}, \bar{\mathcal{B}})$ is almost semistable, $\bar{\mathcal{Y}}$ is $\mathbb{Q}$-factorial according to [15, Proposition 5.15]. Also because every component of $\operatorname{Supp}(\bar{\mathcal{D}}'_{\bar{\mathcal{Y}}})$ is horizontal over $\bar{\mathcal{S}}$, then $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}},\ge 0}) \to \bar{\mathcal{S}}$ is locally stable.

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press

Note we replace $\mathcal{S}$ by a dense open subset. Then after applying the same argument on the complementary set, we get a stratification of $\mathcal{S}$. We also replace U by an open subset so that $\phi_U : U \to \mathcal{S}$ is still a morphism.

Let $\bar{C}$ be the closure of $\bar{U} := U \times_{\mathcal{S}} \bar{\mathcal{S}}^o$. Then there is a finite cover $\pi : \bar{C} \to C$. We choose $\bar{0}$ to be a closed point of $\pi^{-1}(0)$. Because $\bar{\mathcal{S}}$ is proper, the moduli map $\phi_U : U \to \mathcal{S}$ defines a morphism $\bar{\phi} : \bar{C} \to \bar{\mathcal{S}}$. Define $\bar{Y} := \bar{\mathcal{Y}} \times_{\bar{\mathcal{S}}} \bar{C}$ and $\bar{D}' = \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}} \times_{\bar{\mathcal{S}}} \bar{C}$. It is easy to see that $\bar{f}' : (\bar{Y}, \bar{D}') \to \bar{C}$ is the base change of $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}) \to \bar{\mathcal{S}}$ via $\bar{\phi} : \bar{C} \to \bar{\mathcal{S}}$. Because $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}},\geq 0}) \to \bar{\mathcal{S}}$ is locally stable, then $\bar{f}' : (\bar{Y}, \bar{D}'_{\geq 0}) \to \bar{C}$ is also locally stable. Let $\bar{R}$ be the base change of $\bar{\mathcal{R}}$ on $\bar{Y}$; by Lemma 3.9, $(\bar{Y}, \bar{R}) \to (\bar{C}, \bar{0})$ is a also toroidal morphism with reduced fibers. Because $\bar{\mathcal{D}}'_{\bar{\mathcal{Y}}} \leq \bar{\mathcal{R}}$, we have $\bar{D}' \leq \bar{R}$.

Define $(\bar{Y}_{\bar{U}}, \bar{D}'_{\bar{U}}) := (\bar{Y}, \bar{D}') \times_{\bar{C}} \bar{U}$. Then $(\bar{Y}_{\bar{U}}, \bar{D}'_{\bar{U}})$ is equal to the pull-back of $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}) \times_{\bar{\mathcal{S}}} \bar{\mathcal{S}}^o$ via $\bar{\phi}|_{\bar{U}} : \bar{U} \to \bar{\mathcal{S}}^o$. Because $(K_{\bar{\mathcal{Y}}} + \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}})|_{\bar{\mathcal{Y}}_{\bar{\mathcal{S}}^o}} \sim_{\mathbb{Q},\bar{\mathcal{S}}^o} 0$, then there is a $\mathbb{Q}$-divisor $\bar{D}$ on $\bar{Y}$ such that

- ∘ $\bar{D}|_{\bar{Y}_{\bar{U}}} = \bar{D}'_{\bar{U}}$,
- ∘ $\bar{D} \leq \bar{D}'$,
- ∘ $\operatorname{Supp}(\bar{D})$ does not contain the whole fiber $\bar{Y}_{\bar{0}}$ and
- ∘ $K_{\bar{Y}} + \bar{D} \sim_{\mathbb{Q},\bar{C}} 0$.

It is easy to see that $\bar{D}$ is the largest $\mathbb{Q}$-divisor on $\bar{Y}$ such that $\bar{D} \leq \bar{D}'$ and $K_{\bar{Y}} + \bar{D} \sim_{\mathbb{Q},\bar{C}} 0$.

Because $\bar{f}' : (\bar{Y}, \bar{D}'_{\geq 0}) \to \bar{C}$ is locally stable and $\bar{D} \leq \bar{D}'$, then $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ is sub-lc and $K_{\bar{Y}} + \bar{D} + \bar{Y}_{\bar{0}} \sim_{\mathbb{Q},\bar{C}} K_{\bar{Y}} + \bar{D} + (\bar{f}')^*\bar{0} \sim_{\mathbb{Q},\bar{C}} 0$.

Let $\bar{X}$ be the normalization of the main component of $X \times_C \bar{C}$, π_X denote the projection $\bar{X} \to X$ and $\bar{f}$ denote the projection $\bar{X} \to \bar{C}$. We replace Δ by $\Delta + \operatorname{lct}(X, \Delta; f^*0) f^*0$. Then we may assume $\operatorname{lct}(X, \Delta; f^*0) = 0$. By the Hurwitz's formula, there is a $\mathbb{Q}$-divisor $\bar{\Delta}$ such that

$$K_{\bar{X}} + \bar{\Delta} \sim_{\mathbb{Q}} \pi_X^*(K_X + \Delta).$$

Note we only add a $\mathbb{Q}$-divisor which is vertical over C. Then the generic fiber of $(X, \Delta) \to C$ is unchanged.

Suppose P is an lc place of (X, Δ). Let $X' \to X$ be a dlt modification of (X, Δ) such that P is a divisor on X', $\bar{X}'$ the normalization of the main component of $X' \times_C \bar{C}$, and $\bar{P}$ an irreducible component of the preimage of P on $\bar{X}'$. By [12, 2.41], $\bar{P}$ is an lc place of $(\bar{X}, \bar{\Delta})$.

By Lemma 5.3, we have

$$\deg(\bar{P} \to P) \leq \min\{\deg(\pi), \operatorname{mult}_P f^*0\}.$$

By the definition of $\bar{C}$, $\deg(\pi)$ is equal to the degree of the finite cover $\bar{\mathcal{S}} \to \mathcal{S}$. Let l be the degree of the finite morphism $\bar{\mathcal{S}} \to \mathcal{S}$. Then $\min\{\deg(\pi), \operatorname{mult}_P f^*0\}$ is less or equal to $\min\{l, \operatorname{mult}_P f^*0\}$. Thus, we only need to prove that $\bar{P}$ is birationally bounded.

Note $(\bar{Y}_{\bar{U}}, \bar{D}'_{\bar{U}})$ is equal to the pull-back of $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}) \times_{\bar{\mathcal{S}}} \bar{\mathcal{S}}^o$ via $\bar{\phi}|_{\bar{U}} : \bar{U} \to \bar{\mathcal{S}}$. Because $(X_U, \Delta_U) \cong (\mathcal{X}, \mathcal{D}) \times_{\mathcal{S}} U$ and $\bar{X}$ is the normalization of the main component of $X \times_C \bar{C}$, then $(\bar{X}, \bar{\Delta}) \times_{\bar{C}} \bar{U}' \to \bar{U}'$ is isomorphic to the pull-back of $(\mathcal{X}, \mathcal{D}) \times_{\mathcal{S}} \bar{\mathcal{S}}^o$ via $\bar{\phi}|_{\bar{U}'} : \bar{U}' \to \bar{\mathcal{S}}$ for an open subset $\bar{U}' \subset \bar{C}$. Also because the fiber of $(\bar{\mathcal{Y}}, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}}) \to \bar{\mathcal{S}}$ over $\bar{p}$ is crepant birationally equivalent to the base change of the fiber of $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ over $\tau(\bar{p})$ for every point $\bar{p} \in \bar{\mathcal{S}}^0$, then the generic fiber of $\bar{f} : (\bar{X}, \bar{\Delta}) \to \bar{C}$ is crepant birationally equivalent to the generic fiber of $\bar{f}' : (\bar{Y}, \bar{D} + \bar{Y}_{\bar{c}}) \to \bar{C}$. Since $K_{\bar{X}} + \bar{\Delta} \sim_{\mathbb{Q},\bar{C}} 0$, $K_{\bar{Y}} + \bar{D} + \bar{Y}_{\bar{0}} \sim_{\mathbb{Q},\bar{C}} 0$, then there is a $\mathbb{Q}$-divisor F on $\bar{C}$ such that $(\bar{X}, \bar{\Delta} + \bar{f}^*F)$ is crepant birationally equivalent to $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$. Since both $(\bar{X}, \bar{\Delta})$ and $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ have an lc place dominating $\bar{0}$, then $\bar{0} \notin \operatorname{Supp}(F)$. After replacing $\bar{C}$ by an open neighborhood of $\bar{0}$, we may assume that $(\bar{X}, \bar{\Delta})$ is crepant birationally equivalent to $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$. In particular, a divisor $\bar{P}$ is an lc place of $(\bar{X}, \bar{\Delta})$ if and only if it is an lc place of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$.

Recall that $(\bar{Y}, \bar{R}) \to (\bar{C}, \bar{0})$ is a toroidal morphism and $\bar{D} \leq \bar{R}$. Since $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ is sub-lc, by Lemma 3.10, $\bar{P}$ is birationally equivalent to $V \times \mathbb{P}^r$, where V is the image of $\bar{P}$ on $\bar{Y}$. Because $\bar{P}$ is an lc place of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$, then V is an lc center of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$.

To prove $\bar{P}$ is birationally bounded, we only need to show that all lc centers of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ are bounded. Let W be the normalization of an irreducible component of $\bar{Y}_{\bar{0}}$ such that V is contained in the image of W on $\bar{Y}_{\bar{0}}$.

If V has codimension 1 in $\bar{Y}$, then $\bar{P}$ is just W. Since $\bar{Y}_{\bar{0}}$ is in a bounded family $\bar{\mathcal{Y}} \to \bar{\mathcal{S}}$ and W is the normalization of an irreducible component of $\bar{Y}_{\bar{0}}$, P is birationally bounded.

If V has codimension ≥ 2 in $\bar{Y}$, by applying the adjunction on $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$, we have

$$(K_{\bar{Y}} + \bar{D} + \bar{Y}_{\bar{0}})|_W = K_W + \bar{D}_W.$$

By the inverse of adjunction (see [12, Theorem 4.9]), an lc center of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ intersecting W corresponds to an lc center of $(W, \bar{D}_W)$, hence also an lc center of $(\bar{Y}_{\bar{0}}, \bar{D}'_{\bar{Y},\bar{0}})$. Let $s \in \bar{\mathcal{S}}$ be the image of $\bar{C}$ in $\bar{\mathcal{S}}$. By the definition of $\bar{Y}$, we have the isomorphism

$$(\bar{Y}_{\bar{0}}, \bar{D}'_{\bar{Y},\bar{0}}) \cong (\bar{\mathcal{Y}}_s, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}_s}).$$

By Lemma 5.2, all lc centers of $(\bar{\mathcal{Y}}_s, \bar{\mathcal{D}}'_{\bar{\mathcal{Y}}_s})$ are in a bounded family. Then all lc centers of $(\bar{Y}, \bar{D} + \bar{Y}_{\bar{0}})$ are in a bounded family. □

6. Proof of main theorems

Lemma 6.1. *Fix a natural number d and positive rational numbers $\epsilon, c \in (0, 1)$. Suppose (X, Δ) is an ϵ-lc pair of dimension d, $-(K_X + \Delta)$ is big and nef and* $\operatorname{coeff}(\Delta) \geq c$. *Then (X, Δ) is log bounded.*

Proof. By the main theorem of [4], X is bounded. Then there exist a natural number n, two constants V_1, V_2 depending only on d and ϵ, and a very ample divisor H on X defining an embedding $X \subset \mathbb{P}^n$ such that $H^d \leq V_1$ and $H^{d-1} \cdot K_X \geq -V_2$. Because $\operatorname{coeff}(\Delta) \geq c$, we have

$$\begin{aligned} cH^{d-1} \cdot \operatorname{Supp}(\Delta) &\leq H^{d-1} \cdot \Delta \\ &= H^{d-1} \cdot (K_X + \Delta) - H^{d-1} \cdot K_X \\ &\leq -H^{d-1} \cdot K_X \\ &\leq V_2. \end{aligned}$$

By the boundedness of the Chow variety, both X and $\operatorname{Supp}(\Delta)$ are parametrized by a subscheme of the Hilbert scheme. Then (X, Δ) is log bounded. □

Proof of Theorem 1.1. Because f has reduced fibers, then $X_0 = f^*0$, and we have $K_X + \Delta + X_0 \sim_{\mathbb{Q},C} 0$. By adjunction, we have $K_{X_0} + \Delta_0 \sim_{\mathbb{Q}} (K_X + \Delta + X_0)|_{X_0}$. Because (X_0, Δ_0) is slc, then its normalization (X_0^ν, Δ_0^ν) is lc. Also because X is a normal variety, by inverse of adjunction, $(X, \Delta + f^*0)$ is lc over an open neighborhood of $0 \in C$. It is easy to see that every irreducible component of X_0 is an lc place of $(X, \Delta + f^*0)$. After replacing C by an open neighborhood of 0, we may assume $(X, \Delta + f^*0)$ is lc.

Because $\operatorname{mult}_P f^*0 = 1$ for every irreducible component $P \subset X_0$, by Theorem 5.1, there exists a bounded family $\mathcal{W} \to \mathcal{T}$ and a finite dominant rational map $\mathcal{W}_t \dashrightarrow P$ whose degree is a factor of $\min\{l, \operatorname{mult}_P f^*0\} = 1$. Then $\mathcal{W}_t \dashrightarrow P$ is a birational map, which means P is birationally bounded. □

Proof of Corollary 1.2. By the proof of Theorem 1.1, $(X, \Delta + X_0)$ is lc over an open neighborhood of $0 \in C$, and every irreducible component of X_0 is an lc place of $(X, \Delta + X_0)$. After replacing C by an open neighborhood of 0, we may assume $(X, \Delta + X_0)$ is lc. Also because $X_0 = f^*0$ is a Cartier divisor, then (X, Δ) is lc, and its lc centers are not contained in X_0. After replacing C by an open neighborhood of 0, we may assume every lc center of (X, Δ) dominates C.

Because a general fiber (X_g, Δ_g) of f is ϵ-lc, by inverse of adjunction, $(X, \Delta + X_g)$ is plt in a neighborhood of X_g. Also because every lc center of (X, Δ) dominates C, then (X, Δ) is klt. Because $-(K_X + \Delta)$ is ample over C, let $B \in |-(K_X + \Delta)|_{\mathbb{Q}/C}$ be a general member. Then $(X, \Delta + B)$ is klt and $K_X + \Delta + B \sim_{\mathbb{Q},C} 0$. Thus, X is Fano type over C.

Because $-(K_X+\Delta+X_0) \sim_{\mathbb{Q},C} -(K_X+\Delta)$ is ample over C, X is Fano type over C and $\mathrm{coeff}(\Delta+X_0) \subset (c\mathbb{N} \cap [0,1]) \cup \{1\}$ is in a finite set, by [3, Theorem 1.8]. After replacing C by an open neighborhood of 0, there exist a natural number l depending only on d, c and a $\mathbb{Q}$-divisor Λ on X such that

- ◦ $\Lambda \geq \Delta + X_0$,
- ◦ $l(K_X + \Lambda) \sim_C 0$ and
- ◦ (X, Λ) is lc.

Because $\Lambda \geq \Delta + X_0$, then every irreducible component of X_0 is an lc place of (X, Λ).

Since $l(K_X + \Lambda) \sim_C 0$, then $l(K_{X_g} + \Lambda_g) \sim 0$, where (X_g, Λ_g) is a general fiber of $(X, \Lambda) \to C$. Because $(X, \Delta + X_0)$ is lc, then (X_g, Λ_g) is a Calabi–Yau pair and $\mathrm{coeff}\Lambda_g \subset \frac{1}{l}\mathbb{N}$.

Because a general fiber X_g is ϵ-lc, $-(K_{X_g} + \Delta_g)$ is ample, and $\mathrm{coeff}(\Delta_g) \geq c$, by Lemma 6.1, (X_g, Δ_g) is log bounded. Then there exist a natural number m depending only on d, ϵ and an open subset $U \subset C$ such that $-m(K_{X_u} + \Delta_u)$ is very ample without higher cohomology for every $u \in U$. Thus, $-m(K_{X_U} + \Delta_U)$ is relatively very ample over U. Choose a general member $N \in |-m(K_{X_U} + \Delta_U)|$. Because N_g is ample and (X_g, Δ_g) is log bounded, then $\mathrm{vol}(N_g) = N_g^d$ is in a finite set. To prove the result, we may assume $\mathrm{vol}(N_g) = v$ is fixed. Because $N \in |-m(K_{X_U} + \Delta_U)|$ is a general member, then there is a sufficiently small positive rational number t such that $(X_g, \Lambda_g + tN_g)$ is lc. Thus, $(X_g, \Lambda_g), N_g$ is a $(d, \frac{1}{l}, v)$-polarized log Calabi–Yau pair. Then apply Theorem 1.1. □

Acknowledgements. The author would like to thank his advisor Christopher D. Hacon and his postdoctoral advisor Caucher Birkar for their encouragement and constant support. He would also like to thank Jingjun Han, Chen Jiang, Xiaowei Jiang and Jihao Liu for their helpful comments.

Competing interests. The authors have no competing interest to declare.

Financial Support. This work was supported by grants from Tsinghua University, Yau Mathematical Sciences Center.

References

[1] D. Abramovich and K. Karu, ‘Weak semistable reduction in characteristic 0’, *Invent. Math.* **139**(2) (2000), 241–273.
[2] K. Adiprasito, G. Liu and M. Temkin, ‘Semistable reduction in characteristic 0’, Preprint, 2018, arXiv:1810.03131v2.
[3] C. Birkar, ‘Anti-pluricanonical systems on Fano varieties’, *Ann. of Math. (2)* **190** (2019), 345–463.
[4] C. Birkar, ‘Singularities of linear systems and boundedness of Fano varieties’, *Ann. of Math.* **193**(2) (2021), 347–405.
[5] C. Birkar, ‘Moduli of algebraic varieties’, Preprint, 2022, arXiv:2211.11237.
[6] C. Birkar, ‘Geometry of polarised varieties’, *Pub. Math. IHES.* **137** (2023), 47–105.
[7] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, ‘Existence of minimal models for varieties of log general type’, *J. Amer. Math. Soc.* **23**(2) (2010), 405–468.
[8] C. Birkar and S. Qu, ‘Irrationality of degenerations of Fano varieties’, Preprint, 2024, arXiv:2401.07233.
[9] R. Hartshorne, *Algebraic Geometry* (Graduate Texts in Mathematics) (Springer New York, NY, 1977).
[10] Y. Hu, C. H. Liu and S. T. Yau, ‘Toric morphisms and fibrations of toric Calabi–Yau hypersurfaces’, *Adv. Theor. Math. Phys.* **6**(3) (2002), 457–505.
[11] P. Hacking and Y. Prokhorov, ‘Smoothable del Pezzo surfaces with quotient singularities’, *Compos. Math.* **146**(1) (2010), 169–192.
[12] J. Kollár, *Singularities of the Minimal Model Program* (Cambridge Tracts in Math.) vol. 200 (Cambridge, Cambridge Univ. Press, 2013). With a collaboration of Sándor Kovács.
[13] J. Kollár, *Families of Varieties of General Type* (Cambridge, Cambridge Univ. Press, 2023).
[14] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings 1* (Springer Berlin, Heidelberg, 1973).
[15] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties* (Cambridge Tracts in Math.) vol. 134 (Cambridge, Cambridge Univ. Press, 1998).
[16] J. Kollár and C. Xu, ‘Moduli of polarized Calabi–Yau pairs’, *Acta Math. Sinica* **36**(6) (2020), 631–637.
[17] R. Lazarsfeld, *Positivity in Algebraic Geometry. II. Positivity for Vector Bundles and Multiplier Ideals* (Ergebnisse der Mathematik und ihrer Grenzgebiete) 3. Folge.

https://doi.org/10.1017/fms.2024.149 Published online by Cambridge University Press