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A BOUND FOR THE DEGREE OF $H^2(G, Z_p)$

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1. Introduction. Let G be a group and N a trivial G-module. We say an element $\xi \in H^2(G, N)$ is of degree $\leqq n$ if a 2-cocycle representative of ξ is a polynomial 2-cocycle of degree $\leqq n$ [**1**]. Let $P_nH^2(G, N)$ denote the subgroup of $H^2(G, N)$ consisting of elements with degree $\leqq n$. Then we have a filtration

$$0 = P_0H^2(G,N) \leqq P_1H^2(G, N) \leqq P_2H^2(G,N) \leqq \ldots \leqq P_nH^2(G,N) \leqq \ldots$$

of $H^2(G, N)$. We say that the degree of $H^2(G, N)$ is $\leqq n$ if $P_nH^2(G, N) = H^2(G, N)$. Passi and Stammbach [**5**] have studied this filtration for the case when the coefficients are in T, the additive group of rationals mod 1. We are interested in the filtration of $H^2(G, Z_p)$, where Z_p is the additive group of integers mod p and is regarded as a trivial G-module. Our main result is

$$\deg H^2(G, Z_p) \leqq p(M\text{-class of } G) - 1,$$

i.e., $P_{pn-1}H^2(G, Z_p) = H^2(G, Z_p)$ where $n = M$-class of G. (See section 2 for the definition of M-class.) As a consequence we deduce that if π is a group and N is a normal subgroup of π which is an elementary abelian p-group and is contained in the centre of π and π/N is of M-class n, then

$$N \cap (1 + \Delta_{Z_p}{}^{pn+1}(\pi) + \Delta_{Z_p}(\pi)\,\Delta_{Z_p}(N)) = 1,$$

where Z_p denotes the field of p elements and $\Delta_{Z_p}(G)$ denotes the augmentation ideal of the group algebra $Z_p(G)$.

Finally we give an example which shows that $p(M\text{-class of } G) - 1$ is the best possible bound for the degree of $H^2(G, Z_p)$.

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2. Notations and preliminaries. For a group G, $M_i(G)$ denotes the ith term in its Brauer-Jennings-Zassenhaus series which is defined inductively as follows:

$$M_1(G) = G,\ M_i(G) = [G, M_{i-1}(G)]\ M_{(i/p)}(G)^p \quad \text{for } i \geqq 2$$

where (i/p) is the least integer $\geqq i/p$ and $[G, M_{i-1}(G)]$ denotes the subgroup generated by all commutators

$$[x, y] = x^{-1}y^{-1}xy, \quad x \in G, y \in M_{i-1}(G).$$

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G is said to be of M-class n if $M_n(G) \neq 1$, $M_{n+1}(G) = 1$. If k is a field of characteristic $p > 0$ and $\Delta_k(G)$ is the augmentation ideal of the group algebra $k(G)$, then it is shown (see for example [**2**] or [**7**]) that

$$M_i(G) = \{g \in G \mid g - 1 \in \Delta_k{}^i(G)\}.$$

Arguing as in [**3**, Theorem 2.1] we can prove the following:

PROPOSITION 1. *Let π be a group of M-class $n + 1$. Let $\alpha : M_{n+1}(\pi) \to Z_p$ be a homomorphism. Then α is extendable to a Z_p-polynomial map $\phi : \pi \to Z_p$ of degree $\leqq n + 1$ if and only if the central extension induced by α is of degree $\leqq n$.*

LEMMA [**8**, Proposition 3.9, Chapter II]. *Let G be an arbitrary group, N an abelian group regarded as a trivial G-module. Let $f : G \times G \to N$ be an arbitrary 2-cocycle. Let*

$$N \overset{i}{\rightarrowtail} \pi \overset{\theta}{\twoheadrightarrow} G$$

be a central extension of N by G which corresponds to the 2-cocycle f. If H, K are subgroups of G such that $[\theta^{-1}(H), \theta^{-1}(K)] = 1$, then $f(h, k) = f(k, h)$ for all $h \in H$, $k \in K$.

Analogous to [**6**, Proposition 4.6] we establish the following:

PROPOSITION 2. *Let G be a group of M-class n. Let $\xi \in H^2(G, Z_p)$ and*

$$Z_p \overset{i}{\rightarrowtail} \pi \overset{\theta}{\twoheadrightarrow} G$$

be a central extension corresponding to ξ. If M-class of π = M-class of $G = n$, then

$$\xi \in \operatorname{Im}(\inf : H^2(G/M_n(G), Z_p) \to H^2(G, Z_p)).$$

Proof. G being of M-class n,

$$1 = M_{n+1}(G) = [G, M_n(G)]\, M_{((n+1)/p)}(G)^p.$$

Thus

(i) $M_n(G) \leqq$ centre of G, and

(ii) $M_n(G)$ is of exponent p because $n > ((n + 1)/p)$ and therefore $M_n(G) \leqq M_{((n+1)/p)}(G)$ which is of exponent p. Similarly, $M_n(\pi) \leqq$ centre of π and $M_n(\pi)$ is of exponent p.

Therefore the sequence

$$Z_p \overset{i}{\rightarrowtail} M_n(\pi)Z_p \overset{\theta}{\twoheadrightarrow} M_n(G)$$

splits and we have a homomorphism

$$\phi : M_n(G) \to M_n(\pi)\, Z_p$$

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such that $(\theta \circ \phi)(z) = z$ for all $z \in M_n(G)$. We have a central extension

$$M_n(G) \overset{i}{\rightarrowtail} G \overset{\alpha}{\twoheadrightarrow} G/M_n(G)$$

where α is the natural projection. Let $\{w(h)\}$ be a set of representatives in G of elements $h \in G/M_n(G)$; then every element of G is uniquely expressible as $w(h)z$ where $z \in M_n(G)$ and $h \in G/M_n(G)$. Also, $M_n(G)$ being in the centre of G, we have $w(h)z = zw(h)$. We choose representatives $\{\phi(g)\}_{g \in G}$ in π as follows: Choose arbitrarily representative $\phi(w(h))$ in π of the element $w(h)$ and set $\phi(g) = \phi(z)\phi(w(h))$, where $g = w(h)z$, $h \in G/M_n(G)$ and $z \in M_n(G)$. Let $f : G \times G \to Z_p$ be the 2-cocycle corresponding to the above choice of representatives in π of elements $g \in G$. Then

(2.1) $f(z_1, z_2) = 0$ for all $z_1, z_2 \in M_n(G)$

(2.2) $f(z, w(h)) = 0$ for all $z \in M_n(G)$, $h \in G/M_n(G)$.

(2.1) and (2.2) imply that

$$f(w(h_1)z_1, w(h_2)z_2) = f(w(h_1), w(h_2)) + f(w(h_1), z_2),$$
$$h_1, h_2 \in G/M_n(G) \text{ and } z_1, z_2 \in M_n(G).$$

In particular,

$$f(z, g) = 0 \quad \text{for all } z \in M_n(G),\ g \in G.$$

But $f(z, g) = f(g, z)$ for all $z \in M_n(G)$, $g \in G$ (by the Lemma). Therefore

(2.3) $f(g, z) = 0 = f(z, g)$ for all $z \in M_n(G)$, $g \in G$.

It follows that

(2.4) $f(w(h_1)\, z_1, w(h_2)\, z_2) = f(w(h_1), w(h_2))$.

Define $\bar{f} : G/M_n(G) \times G/M_n(G) \to Z_p$ by

$$\bar{f}(h_1, h_2) = f(w(h_1), w(h_2)).$$

$\bar{f}$ is clearly a 2-cocycle and it defines an element η, say, of $H^2(G/M_n(G), Z_p)$ whose image under the inflation is ξ. This completes the proof of the proposition.

3. Main result.

THEOREM. *Let G be any group of M-class n. Then*

$$\deg H^2(G, Z_p) \leqq pn - 1 = p(M\text{-class of } G) - 1.$$

Proof. We proceed by induction on the M-class of G. Let G be a group of M-class 1. Then

$$1 = M_2(G) = [G, G]\, M_{(2/p)}(G)^p = [G, G]\, M_1(G)^p = [G, G]\, G^p.$$

$[G, G]G^p = 1$ implies that G is an elementary abelian p-group. Let $\xi \in H^2(G, Z_p)$ and

(3.1) $Z_p \overset{i}{\rightarrowtail} \pi \overset{\theta}{\twoheadrightarrow} G$

be the central extension corresponding to ξ. Now

$$\theta(M_2(\pi)) = M_2(G) = 1,$$

i.e., $M_2(\pi) \leqq Z_p$ which is cyclic of order p, so either $M_2(\pi) = 1$ or $M_2(\pi) = Z_p$.

Case (i) $M_2(\pi) = 1$: Then π is also an elementary abelian p-group and hence the sequence (3.1) splits. Consequently deg $\xi = 0$.

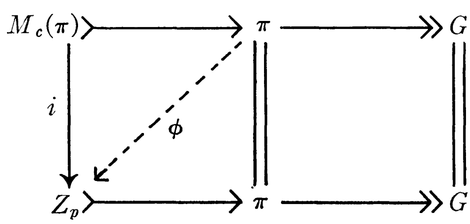
Case (ii). $M_2(\pi) = Z_p$: In this case the M-series of π becomes

$$\pi = M_1(\pi) \geqq M_2(\pi) = Z_p \geqq M_3(\pi)$$
$$\geqq M_4(\pi) \geqq \ldots \geqq M_p(\pi) \geqq M_{p+1}(\pi) = 1.$$

Therefore M-class of π is $\leqq p$. Suppose M-class of $\pi = c$, $1 < c \leqq p$, i.e., $M_c(\pi) = Z_p$, $M_{c+1}(\pi) = 1$. We embed $M_c(\pi)$ into $Z_p(\pi)/\Delta_{Z_p}{}^{c+1}(\pi)$. Therefore the homomorphism $i : M_c(\pi) \rightarrow Z_p$ is extendable to a homomorphism $\beta : Z_p(\pi)/\Delta_{Z_p}{}^{c+1}(\pi) \rightarrow Z_p$. Then $\phi : \pi \rightarrow Z_p$ given by

$$\phi(x) = \beta((x - 1) + \Delta_{Z_p}{}^{c+1}(\pi))$$

is a polynomial map of degree $\leqq c$ and $\phi|M_c(\pi) = i$ and we have a commutative diagram:



The lower row of this diagram is the central extension induced by the embedding i and therefore by Proposition 1, it is of degree $\leqq c - 1 \leqq p - 1$. Hence deg $H^2(G, Z_p) \leqq c - 1 \leqq p - 1$. Thus we have shown that deg $H^2(G, Z_p) \leqq p - 1$ if G is a group of M-class 1.

Suppose now that the result is true for the groups of M-class $< n$. Let G be a group of M-class n. Then

$$1 = M_{n+1}(G) = [G, M_n(G)]\ M_{((n+1)/p)}(G)^p.$$

This implies that $M_n(G) \leqq$ centre of G and $M_i(G)$ is of exponent p where i is the least integer $\geqq (n + 1)/p$. Let $\xi \in H^2(G, Z_p)$ and

$$Z_p \overset{i}{\rightarrowtail} \pi \overset{\theta}{\twoheadrightarrow} G$$

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be the corresponding central extension. Now

$$\theta(M_{n+1}(\pi)) = M_{n+1}(G) = 1,$$

i.e., $M_{n+1}(\pi) \leqq Z_p$. Therefore, either $M_{n+1}(\pi) = 1$ or $M_{n+1}(\pi) = Z_p$.

Case (i)* $M_{n+1}(\pi) = 1$: In this case, M-class of $\pi = n = M$-class of G. Therefore, by Proposition 2, $\xi \in \text{Im}$ (inf: $H^2(G/M_n(G), Z_p) \to H^2(G, Z_p)$). Since M-class of $G/M_n(G) = n - 1$ [**2**, Theorems 4.1 and 5.5], induction gives deg $H^2(G/M_n(G), Z_p) \leqq p(n-1) - 1$. It is not hard to see that if deg $H^2(G/N, Z_p) \leqq k$, where N is a normal subgroup of G, then

$$\deg (\text{Im } (\text{inf} : H^2(G/N, Z_p) \to H^2(G, Z_p))) \leqq k.$$

Hence deg $\xi \leqq p(n-1) - 1 < pn - 1$.

Case (ii)* $M_{n+1}(\pi) = Z_p$: M-series of π in this case is as follows:

$$\pi = M_1(\pi) \geqq M_2(\pi) \geqq \ldots \geqq M_{n+1}(\pi) = Z_p \geqq M_{n+2}(\pi)$$
$$\geqq M_{n+3}(\pi) \geqq \ldots \geqq M_{pn}(\pi) \geqq M_{pn+1}(\pi) = 1$$

for

$$M_{pn+1}(\pi) = [\pi, \ M_{pn}(\pi)] \ \ M_{((pn+1)/p)}(\pi)^p = M_{(n+1/p)}(\pi)^p = M_{n+1}(\pi)^p = 1.$$

Therefore M-class of $\pi \leqq pn$. Suppose that M-class of π is c^*, $n < c^* \leqq pn$, i.e., $M_{c^*}(\pi) = Z_p$, $M_{c^*+1}(\pi) = 1$. Now proceeding analogously as in case (ii) above, we get

$$\deg H^2(G, Z_p) \leqq c^* - 1 \leqq pn - 1.$$

This completes the induction. Hence deg $H^2(G, Z_p) \leqq p$ (M-class of G) $- 1$.

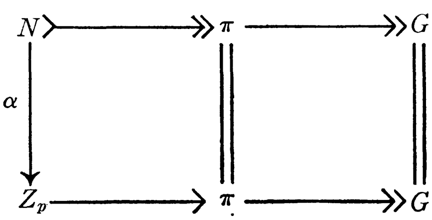
COROLLARY. *Let π be a group and N be a normal subgroup of π, which is elementary abelian p-group and is contained in the centre of π. Let π/N be of M-class n. Then*

$$N \cap (1 + \Delta_{Z_p}{}^{pn+1}(\pi) + \Delta_{Z_p}(\pi) \ \Delta_{Z_p}(N)) = 1.$$

Proof. Let $G = \pi/N$. Consider the central extension

$$N \rightarrowtail \pi \twoheadrightarrow G.$$

Let $1 \neq x \in N$. Then there exists a homomorphism α: $N \to Z_p$ such that $\alpha(x) \neq 0$. Let $Z_p \rightarrowtail \pi \twoheadrightarrow G$ be the central extension induced by α. Then we have the following commutative diagram:



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Now the induced central extension is of degree $\leqq pn - 1$ as deg $H^2(G, Z_p) \leqq pn - 1$. Hence by (the mod p-version of [**4**, Theorem 2.1]), α can be extended to a map $\phi : \pi \rightarrow Z_p$ whose linear extension to $Z_p(\pi)$ vanishes on $\Delta_{Z_p}{}^{pn+1}(\pi) + \Delta_{Z_p}(\pi)\ \Delta_{Z_p}(N)$. Therefore $\alpha(x) = \phi(x) = 0$, a contradiction. Hence

$$N \cap (1 + \Delta_{Z_p}{}^{pn+1}(\pi) + \Delta_{Z_p}(\pi)\ \Delta_{Z_p}(N)) = 1.$$

Remark. "$pn - 1$" is the best possible bound for the "deg $H^2(G, Z_p)$" where G is of M-class n. For example, take $G = Z_p$. Then M-class of G is 1 and it is easily seen that deg $H^2(G, Z_p)$ is exactly $p - 1$.

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