A GROUP OF AUTOMORPHISMS OF THE HOMOTOPY GROUPS

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It is well known that the fundamental group $\pi_1(X)$ of an arcwise connected topological space X operates on the n-th homotopy group $\pi_n(X)$ of X as a group of automorphisms. In this paper I intend to construct geometrically a group $\mathfrak{A}(X)$ of automorphisms of $\pi_n(X)$, for every integer $n \geqq 1$, which includes a normal subgroup isomorphic to $\pi_1(X)$, so that the factor group of $\mathfrak{A}(X)$ by $\pi_1(X)$ is completely determined by some invariant $\Sigma(X)$ of the space X. The complete analysis of the operation of the group on $\pi_n(X)$ is given in §3, §4, and §5.

Throughout the whole paper, X denotes an arcwise connected topological space which has such suitable homotopy extension properties as a polyhedron does, and all mappings are continuous transformations.

§1. Definition of the group $\mathfrak{A}(X)$.

Let x_0 be an arbitrary point of the space X, and Ω a collection $X^X(x_0, x_0)$ of all the mappings that transform X into X and x_0 into x_0. For two maps $a, b \in \Omega$, a is said to be homotopic to b (in notation : $a \sim b$) if there exists a homotopy $h_t \in \Omega$ (for $1 \geqq t \geqq 0$) such that $h_0 = a$ and $h_1 = b$. A mapping $a \in \Omega$ is called to have a (two sided) homotopy inverse, if there is a map $\varphi \in \Omega$ such that $a\varphi \sim 1$ and $\varphi a \sim 1$, where 1 denotes the identity transformation of X onto itself. Let Ω^* be the collection of all the mappings belonging to Ω, each of which has a homotopy inverse.

Now let $X \times I$ be the topological product of X and the line segment I between 0 and 1, and let us consider the totality U of the mappings $\theta : X \times I \to X$ which satisfy the following conditions :

$$(1.1) \qquad \left.\begin{array}{ll} \text{i)} & \theta \mid X \times 0 \in \Omega^* \\ \text{ii)} & \theta(x_0, 1) = x_0 \end{array}\right\}$$

For two maps $\theta, \theta' \in U$, θ is homotopic to θ' (notation : $\theta \sim \theta'$) if there exists a homotopy $h_t : X \times I \to X$ (for $1 \geqq t \geqq 0$) such that

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$$(1.2)\qquad \left.\begin{array}{ll} \text{i)} & h_0 = \theta, \quad h_1 = \theta', \\ \text{ii)} & h_t(x_0, 0) = h_t(x_0, 1) = x_0. \end{array}\right\}$$

It is easily verified that this relation is an equivalent relation, and therefore U is divided into equivalent classes in this sense.

We shall denote by $[\theta]$ the class containing θ. For $\theta \in U$ we construct a mapping $\sigma_\theta \in U$ as follows: a mapping $\bar{\sigma}_\theta$ which is defined continuously on the set $\{(X \times 0) \smile (x_0 \times I)\}$ such that $\bar{\sigma}_\theta(x, 0) \equiv x$ and $\bar{\sigma}_\theta(x_0, t) \equiv \theta(x_0, t)$, can be extended to a mapping $\sigma_\theta \in U$, provided that $\{x_0\}$ has a homotopy extension property in X relative to X. The extended mapping is, of course, not unique but the homotopy class containing σ_θ is uniquely determined if the set $\{(x_0 \times I) \smile (X \times 0) \smile (X \times 1)\}$ has a homotopy extension property in $X \times I$ relative to X; another arbitrarily extended map σ_θ' is homotopic to σ_θ. Now two maps θ_1, $\theta_2 \in U$ are 'multiplied' together by the rule,

$$(1.3)\qquad \theta_1 \times \theta_2(x, t) \equiv \begin{cases} \rho(x, 2t), & \frac{1}{2} \geqq t \geqq 0, \\ \sigma_{\theta_2}(\rho(x, 1), 2t - 1), & 1 \geqq t \geqq \frac{1}{2}, \end{cases}$$

where $\rho(x, t) \equiv \theta_2(\theta_1(x, t), 0)$. Then we have

LEMMA 1.1 $\theta_1 \times \theta_2$ is again a member of the collection U.

Proof. Let $a_1(x) \equiv \theta_1(x, 0)$, $a_2(x) \equiv \theta_2(x, 0)$, then both a_1 and a_2 belong to Ω^*, so that a_1 and a_2 have homotopy inverses φ_1, φ_2 respectively. From the considerations that $\varphi_1\varphi_2$ is a homotopy inverse of a_2a_1 and that $\theta_1 \times \theta_2(x, 0) = \rho(x, 0) = \theta_2(\theta_1(x, 0), 0) = \theta_2(a_1(x), 0) = a_2(a_1(x))$, we have $\theta_1 \times \theta_2 \mid X \times 0 \in \Omega^*$ and therefore the condition (1.1) i) is satisfied. Also we have $\theta_1 \times \theta_2(x_0, 1) = \sigma_{\theta_2}(\rho(x_0, 1), 1) = \sigma_{\theta_2}(x_0, 1) = \theta_2(x_0, 1) = x_0$. This proves the Lemma.

LEMMA 1.2 The class $[\theta_1 \times \theta_2]$ depends only on the classes $[\theta_1]$ and $[\theta_2]$.

Proof. Let $\theta_1' \in [\theta_1]$ and $\theta_2' \in [\theta_2]$, then there exist two homotopies h_s, k_s: $X \times \overset{t}{I} \to X$ $(1 \geqq s \geqq 0)$ such that $h_0 = \theta_1$, $h_1 = \theta_1'$, $k_0 = \theta_2$, and $k_1 = \theta_2'$. Putting $\rho_s(x, t) \equiv k_s(h_s(x, t), 0)$, we have

$$(1.4)\qquad \left.\begin{array}{ll} \text{i)} & \rho_0(x, t) = \theta_2(\theta_1(x, t), 0), \quad \rho_1(x, t) = \theta_2'(\theta_1'(x, t), 0), \\ \text{ii)} & \rho_s(x_0, 0) = k_s(h_s(x_0, 0), 0) = k_s(x_0, 0) = x_0, \\ \text{iii)} & \rho_s(x_0, 1) = k_s(h_s(x_0, 1), 0) = k_s(x_0, 0) = x_0. \end{array}\right\}$$

Since $k_s(x_0, 0) = k_s(x_0, 1) = x_0$, we can construct, in virtue of the homotopy extension properties previously mentioned, $\sigma_{k_s} \in U$ $(1 \geqq s \geqq 0)$, which is also continuous with respect to s, just as in case of σ_θ. Then clearly we have $\sigma_{k_s}(x, 0) = x$ and $\sigma_{k_s}(x_0, t) = k_s(x_0, t)$ by the construction of the function σ_{k_s}.

$$H_s(x, t) \equiv \begin{cases} \rho_s(x, 2t), & \frac{1}{2} \geqq t \geqq 0, \\ \sigma_{k_s}(\rho_s(x, 1), 2t - 1), & 1 \geqq t \geqq \frac{1}{2}, \end{cases}$$

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is obviously continuous and satisfies the conditions (1.2) of the homotopy ; as to the condition ii), we have $H_s(x_0, 0) = \rho_s(x_0, 0) = x_0$ from (1.4) ii) and $H_s(x_0, 1) = \sigma_{k_s}(\rho_s(x_0, 1), 1) = \sigma_{k_s}(x_0, 1) = k_s(x_0, 1) = x_0$ from (1.4) iii).

Since (1.2) i) is evidently satisfied from (1.4) i), the lemma has been proved. Thus the multiplication in U induces a multiplication in the set of the homotopy classes ; $[\theta_1] \times [\theta_2] \equiv [\theta_1 \times \theta_2]$.

THEOREM 1. *By the multiplication defined above, all the homotopy classes of U constitute a group $\mathfrak{A}(X)$ with x_0 as the base point.*

Proof. Let us prove that the multiplicatoin is associative. Let $\theta_1, \theta_2, \theta_3 \in U$, then $([\theta_1] \times [\theta_2]) \times [\theta_3]$ and $[\theta_1] \times ([\theta_2] \times [\theta_3])$ are represented by mappings $(\theta_1 \times \theta_2) \times \theta_3$ and $\theta_1 \times (\theta_2 \times \theta_3)$ respectively. By definition

$$(\theta_1 \times \theta_2) \times \theta_3(x, t) = \begin{cases} \theta_3(\theta_2(\theta_1(x, 4t), 0), 0), & \frac{1}{4} \geqq t \geqq 0,\ x \in X, \\ \theta_3(\sigma_{\theta_2}(\theta_2(\theta_1(x, 1), 0), 4t-1), 0), & \frac{1}{2} \geqq t \geqq \frac{1}{4},\ x \in X, \\ \sigma_{\theta_3}(\theta_3(\sigma_{\theta_2}(\theta_2(\theta_1(x, 1), 0), 1), 0), 2t-1), & 1 \geqq t \geqq \frac{1}{2},\ x \in X, \end{cases}$$

and

$$\theta_1 \times (\theta_2 \times \theta_3)(x, t) = \begin{cases} (\theta_3(\theta_2(\theta_1(x, 2t), 0), 0), & \frac{1}{2} \geqq t \geqq 0,\ x \in X, \\ \sigma_{\theta_2 \times \theta_3}(\theta_3(\theta_2(\theta_1(x, 1), 0), 0), 2t-1), & 1 \geqq t \geqq \frac{1}{2},\ x \in X. \end{cases}$$

As it is rather difficult to show directly the existence of homotopy between $(\theta_1 \times \theta_2) \times \theta_3$ and $\theta_1 \times (\theta_2 \times \theta_3)$, we prove it by making use of the homotopy extension property referred to above. From the relation above we have $(\theta_1 \times \theta_2) \times \theta_3(x, 0) = \theta_3(\theta_2(\theta_1(x, 0), 0), 0) = \theta_1 \times (\theta_2 \times \theta_3)(x, 0)$, and from the property of σ_θ we have

$$(1.6) \qquad (\theta_1 \times \theta_2) \times \theta_3(x_0, t) = \begin{cases} \theta_3(\theta_2(\theta_1(x_0, 4t), 0), 0), & \frac{1}{4} \geqq t \geqq 1, \\ \theta_3(\theta_2(x_0, 4t-1), 0), & \frac{1}{2} \geqq t \geqq \frac{1}{4}, \\ \theta_3(x_0, 2t-1), & 1 \geqq t \geqq \frac{1}{2}. \end{cases}$$

Since $\sigma_{\theta_2 \times \theta_3}(\theta_3(\theta_2(\theta_1(x_0, 1), 0), 0), 2t-1) = \sigma_{\theta_2 \times \theta_3}(x_0, 2t-1) =$

$$\theta_2 \times \theta_3(x_0, 2t-1) = \begin{cases} \theta_3(\theta_2(x_0, 4t-2), 0), & \frac{3}{4} \geqq t \geqq \frac{1}{2}, \\ \sigma_{\theta_3}(\theta_3(\theta_2(x_0, 1), 0,)\ 4t-3) \\ \quad = \sigma_{\theta_3}(x_0, 4t-3) = \theta_3(x_0, 4t-3), & 1 \geqq t \geqq \frac{3}{4}, \end{cases}$$

we have

$$(1.7) \qquad \theta_1 \times (\theta_3 \times \theta_3)(x_0, t) = \begin{cases} \theta_3(\theta_2(\theta_1(x_0, 2t), 0), 0), & \frac{1}{2} \geqq t \geqq 0, \\ \theta_3(\theta_2(x_0, 4t-2), 0), & \frac{3}{4} \geqq t \geqq \frac{1}{2}, \\ \theta_3(x_0, 4t-3), & 1 \geqq t \geqq \frac{3}{4}. \end{cases}$$

From (1.6) and (1.7) there exists a homotopy $h(x, s, t)$ defined on $\{x_0\} \times \overset{s}{\check{I}} \times \overset{t}{\check{I}}$

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such that

$$h(x_0, 0, t) = (\theta_1 \times \theta_2) \times \theta_3(x_0, t), \quad 1 \geqq t \geqq 0,$$
$$h(x_0, 1, t) = \theta_1 \times (\theta_2 \times \theta_3)(x_0, t), \quad 1 \geqq t \geqq 0,$$

and
$$h(x_0, s, 0) = h(x_0, s, 1) = x_0, \quad 1 \geqq s \geqq 0.$$

Moreover putting

$$h(x, 0, t) = (\theta_1 \times \theta_2) \times \theta_3 (x, t), \quad x \in X, \quad 1 \geqq t \geqq 0,$$
$$h(x, 1, t) = \theta_1 \times (\theta_2 \times \theta_3) (x, t), \quad x \in X, \quad 1 \geqq t \geqq 0,$$

and
$$h(x, s, 0) = \theta_3(\theta_2(\theta_1(x, 0), 0), 0), \quad x \in X, \quad 1 \geqq s \geqq 0,$$

h is defined continuously on the set $\{(X \times \overset{s}{I} \times 0) \cup [(x_0 \times \overset{s}{I}) \cup (X \times 0) \cup (X \times 1)] \times \overset{t}{I}\}$. Thus, if $\{(x_0 \times I) \cup (X \times 0) \cup (X \times 1)\}$ has a homotopy extension property in $X \times I$ relative to X, h can be extended to a mapping $X \times \overset{s}{I} \times \overset{t}{I} \to X$, which gives a homotopy between $(\theta_1 \times \theta_2) \times \theta_3$ and $\theta_1 \times (\theta_2 \times \theta_3)$.

Next we must prove the existence of the unity in $\mathfrak{A}(X)$. Let $\theta_0(x, t) \equiv x$, then clearly $\theta_0 \in U$. For any $\theta \in U$ we have from the definition of multiplication

$$(\theta \times \theta_0)(x, t) = \begin{cases} \rho(x, 2t), & x \in X, \quad \frac{1}{2} \geqq t \geqq 0, \\ \sigma_{\theta_0}(\rho(x, 1), 2t - 1), & x \in X, \quad 1 \geqq t \geqq \frac{1}{2}, \end{cases}$$

where $\rho(x, 2t) = \theta_0(\theta(x, 2t), 0) = \theta(x, 2t)$, and $\sigma_{\theta_0}(x, t) = x$ may be assumed. Since $\sigma_{\theta_0}(\rho(x, 1), 2t - 1) = \rho(x, 1) = \theta_0(\theta(x, 1), 0) = \theta(x, 1)$ for $1 \geqq t \geqq \frac{1}{2}$, we have

$$(\theta \times \theta_0)(x, t) = \begin{cases} \theta(x, 2t), & x \in X, \quad \frac{1}{2} \geqq t \geqq 0, \\ \theta(x, 1), & x \in X, \quad 1 \geqq t \geqq \frac{1}{2}. \end{cases}$$

Let us define a homotopy $h_s(x, t)$ for $1 \geqq s \geqq 0$ as follows ;

$$h_s(x, t) \equiv \begin{cases} \theta\left(x, \dfrac{2t}{1+s}\right), & x \in X, \quad \dfrac{s+1}{2} \geqq t \geqq 0, \\ \theta(x, 1), & x \in X, \quad 1 \geqq t \geqq \dfrac{s+1}{2}, \end{cases}$$

then h_s satisfies the conditions of the homotopy (1.2), so that $h_0 = \theta \times \theta_0$ and $h_1 = \theta$. Thus θ_0 represents the right side unity of the group $\mathfrak{A}(X)$.

Lastly we proceed to show the existence of the inverse element of any element $[\theta] \in \mathfrak{A}(X)$. By the assumption on an element θ in U, we have $\theta \mid X \times 0 \in \Omega^*$, so that $\theta \mid X \times 0$ has a homotopy inverse $\varphi \in \Omega^*$. Now we define a mapping $\theta^{-1} \in U$ as follows : if we put

$$\theta^{-1}(x, 0) \equiv \varphi(x), \quad x \in X,$$
$$\theta^{-1}(x_0, t) \equiv \varphi(\theta(x_0, 1 - t)), \quad 1 \geqq t \geqq 0.$$

then θ^{-1} can be extended to a map: $X \times I \to X$ because of the homotopy

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extension property of $\{x_0\}$. This extended map θ^{-1} is shown to represent the inverse of $[\theta]$. Indeed, we have

$$\theta \times \theta^{-1}(x, t) = \begin{cases} \rho(x, 2t), & \frac{1}{2} \geqq t \geqq 0,\ x \in X, \\ \sigma_{\theta^{-1}}(\rho(x, 1), 2t - 1), & 1 \geqq t \geqq \frac{1}{2},\ x \in X, \end{cases}$$

where $\rho(x, t) = \theta^{-1}(\theta(x, t), 0) = \varphi(\theta(x, t))$, $\sigma_{\theta^{-1}}(x, 0) = x$, and $\sigma_{\theta^{-1}}(x_0, t) = \theta^{-1}(x_0, t) = \varphi(\theta(x_0, 1 - t))$. As φ is a homotopy inverse of $\theta \mid X \times 0$, and on the other hand $\sigma_{\theta^{-1}} \mid x_0 \times I$ represents the inverse element of $[\rho \mid x_0 \times I]$, we have a continuous function h defined on $\{(X \times \overset{s}{I} \times 0) \cup [(X \times 0) \cup (X \times 1) \cup (x_0 \times \overset{s}{I})] \times \overset{t}{I}\}$ such that

$$\begin{aligned} &h(x, s, 0) = k(x, s), && x \in X, \quad s \in \overset{s}{I}, \\ &h(x_0, s, t) = l(s, t), && s \in \overset{s}{I}, \quad t \in \overset{t}{I}, \\ &h(x, 0, t) = \theta \times \theta^{-1}(x, t), && x \in X, \quad t \in \overset{t}{I}, \\ &h(x, 1, t) = x, && x \in X, \quad t \in \overset{t}{I}, \end{aligned}$$

where k is a homotopy obtained by the relation $\varphi\theta \sim 1$, and l is also a homotopy whose existence is assured by $\rho(x_0, 1 - t) = \sigma_{\theta^{-1}}(x_0, t)$. Again, by the aid of a homotopy extension property of $\{(x_0 \times I) \cup (X \times 0) \cup (X \times 1)\}$, h can be extended to a map : $X \times I \times I \to X$, which gives a desired homotopy. This completes the proof.

In order to clarify the conditions preassigned to the space X we put down here all the homotopy extension properties assumed in the arguments of the above Theorem ;

(1.8)
- i) $\{x_0\}$ has a homotopy extension property in X relative to X,
- ii) $\{(x_0 \times I) \cup (X \times 0) \cup (X \times 1)\}$ has a homotopy extension property in $X \times I$ relative to X.

These assumptions are, of course, satisfied by a polyhedron.

§ 2. A group of automorphisms $\Sigma(X)$ and the structure of $\mathfrak{A}(X)$.

Now we define a group $\Sigma(X)$, which operates on $\pi_n(X)$, as we shall see later, as a group of automorphisms, and study a homomorphism of $\mathfrak{A}(X)$ onto $\Sigma(X)$, the kernel of which is isomorphic to the fundamental group $\pi_1(X)$ of X.

Let us define a homotopy concept in Ω^* in the following sense : we shall write $a \sim b$ for $a, b \in \Omega^*$ if there exists a homotopy $h_t \in \Omega(1 \geqq t \geqq 0)$ such that $h_0 = a$ and $h_1 = b$. Then Ω^* is divided into homotopy classes. Let us denote by $\Sigma(X)$ the set of all the homotopy classes. For two maps $a, b \in \Omega^*$ we define $(a \times b)(x) \equiv b(a(x))$ for any $x \in X$. Then $a \times b \in \Omega^*$ because $a \times b \in \Omega$ follows immediately from the definition and, if φ and ψ are homotopy inverses of a

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and b respectively, $\psi \times \varphi \in \Omega^*$ is a homotopy inverse of $a \times b$. Furthermore, if $a \sim a'$ and $a \sim b'$, $a \times b \sim a' \times b'$. Thus the multiplication in Ω^* induces a multiplication in $\Sigma(X)$.

THEOREM 2. *$\Sigma(X)$ constitutes a group.*

Proof. It is evident from the definition of multiplication that the associative law holds. As to the existence of unity, let E be a class containing the identity transformation of X, then $E \cdot A = A$ and $A \cdot E = A$ for any $A \in \Sigma(X)$. Lastly for any $A = [a]$ we choose $A^{-1} = [\varphi]$ containing a homotopy inverse φ of a. Then $AA^{-1} = E$ and $A^{-1}A = E$ is clear from the definition of homotopy inverse.

THEOREM 3. *$\Sigma(X)$ operates on the n-th homotopy group $\pi_n(X, x_0)$, for every integer $n \geqq 1$, as a group of automorphisms.*

Proof. Let f be a representative of an element α of $\pi_n(X)$ and let a be a representative of $A \in \Sigma(X)$. Let us take the mapping $af: S^n \to X$ as a representative of $A\alpha$. The correspondence A ; $\alpha \to A\alpha$ is a transformation of $\pi_n(X)$ into itself because, if f' is another representative of α, we have $af \sim af'$, and if a' is another representative of A, we have also $af \sim a'f$. Then it is easily proved that this correspondence is an automorphism of $\pi_n(X)$.

Example of $\Sigma(X)$:

Let X be an n-sphere S^n, then from the concept of Brouwer's degree we have $\Sigma(S^n) = \{E = [1], A = [-1]\}$ where E is a class containing the identity transformation and A is a class containing a mapping of degree -1. Since clearly $A^2 = A \cdot A = E$, the group is a cyclic group of order 2.

Now we intend to define a homomorphism φ of $\mathfrak{A}(X)$ onto $\Sigma(X)$. Let $\theta \in U$ be a representative of an element of $\mathfrak{A}(X)$, then $a_\theta = \theta \mid X \times 0$ represents an element of $\Sigma(X)$. From the homotopy concepts given in § 1 and § 2, it is obvious that if $\theta \sim \theta'$, we have $a_\theta \sim a_{\theta'}$. By the correspondence φ : $[\theta] \to [a_\theta]$ we have the following theorem.

THEOREM 4. *φ is a homomorphism of $\mathfrak{A}(X)$ onto $\Sigma(X)$, the kernel of which is isomorphic to the fundamental group $\pi_1(X)$.*

Proof. For two elements $[\theta_1]$, $[\theta_2] \in \mathfrak{A}(X)$, we have $\varphi([\theta_1]) = [a_{\theta_1}]$ and $\varphi([\theta_2]) = [a_{\theta_2}]$. By definition $\varphi([\theta_1] \times [\theta_2]) = \varphi([\theta_1 \times \theta_2])$ may be represented by a mapping $\theta_1 \times \theta_2 \mid X \times 0 = \rho(x, 0) = \theta_2(\theta_1(x, 0), 0)$, so that $\theta_1 \times \theta_2 \mid X \times 0 = a_{\theta_1} \times a_{\theta_2}$. Thus $\varphi([\theta_1] \times [\theta_2]) = \varphi([\theta_1]) \times \varphi([\theta_2])$ is proved. Clearly φ is an onto-homomorphism from the definition of the group.

Lastly, in order to complete the proof it is sufficient to prove that the kernel of φ is isomorphic to $\pi_1(X)$. If $\varphi([\theta]) = [a_\theta]$ is unity, we may take without loss of generality a representative θ of $[\theta]$ as follows :

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$$\text{(2.1)} \qquad \left.\begin{array}{ll} \text{i)} & \theta : X \times I \to X, \\ \text{ii)} & \theta(x, 0) = x, \\ \text{iii)} & \theta(x_0, 1) = x_0, \end{array}\right\}$$

for (1.8) is assumed. To any element $[\theta]$ belonging to the kernel of φ let there correspond an element $[\xi_0]$ of the fundamental group $\pi_1(X)$ by the rule,

$$\text{(2.2)} \qquad \xi_0(t) \equiv \theta(x_0, t).$$

This correspondence λ has a definite meaning because, if $\theta \sim \theta'$, ξ_0 and $\xi_{\theta'}$ represent the same element of $\pi_1(X)$. Let us prove that λ is an isomorphism. Let $[\theta_1]$, $[\theta_2]$ be two elements belonging to the kernel of φ, then $[\theta_1] \times [\theta_2]$ is represented by a map $\theta_1 \times \theta_2$,

$$\theta_1 \times \theta_2(x, t) = \begin{cases} \theta_2(\theta_1(x, 2t), 0), & \frac{1}{2} \geqq t \geqq 0, \quad x \in X, \\ \sigma_{\theta_2}(\theta_2(\theta_1(x, 1), 0), 2t-1), & 1 \geqq t \geqq \frac{1}{2}, \quad x \in X. \end{cases}$$

Since from (2.1) we have $\theta_2(x, 0) = x$, $\theta_2(\theta_1(x, 2t), 0) = \theta_1(x, 2t)$ and $\sigma_{\theta_2}(\theta_2(\theta_1(x, 1), 0), 2t-1) = \sigma_{\theta_2}(\theta_1(x, 1), 2t-1)$ so that by (2.2)

$$\xi_{\theta_1 \times \theta_2}(t) = \begin{cases} \theta_1(x_0, 2t), & \frac{1}{2} \geqq t \geqq 0, \\ \sigma_{\theta_2}(\theta_1(x_0, 1), 2t-1), & 1 \geqq t \geqq \frac{1}{2}. \end{cases}$$

Since $\theta_1(x_0, 1) = x_0$ and $\sigma_{\theta_2}(x_0, t) = \theta_2(x_0, t)$, we have $\sigma_{\theta_2}(\theta_1(x_0, 1), 2t-1) = \theta_2(x_0, 2t-1)$. Now $\xi_{\theta_1 \times \theta_2}(t)$ may be described as follows :

$$\xi_{\theta_1 \times \theta_2}(t) = \begin{cases} \theta_1(x_0, 2t), & \frac{1}{2} \geqq t \geqq 0, \\ \theta_2(x_0, 2t-1), & 1 \geqq t \geqq \frac{1}{2}. \end{cases}$$

On the other hand, we have, by the definition of the fundamental group,

$$\lambda([\theta_1] \times [\theta_2]) = [\xi_{\theta_1 \times \theta_2}] = [\xi_{\theta_1}] \circ [\xi_{\theta_2}] = \lambda[\theta_1] \circ \lambda[\theta_2],$$

so that the homomorphism is established.

Clearly λ is an onto-homomorphism, because of the homotopy extension property (1.8) i). It remains only to prove that from $\xi_{\theta_1} \sim \xi_{\theta_2}$ follows $\theta_1 \sim \theta_2$. It may be assumed that $\theta_1(x, 0) = x$ and $\theta_2(x, 0) = x$. Since $\xi_{\theta_1} \sim \xi_{\theta_2}$, a homotopy $h_s(t)$ $(1 \geqq s \geqq 0)$ exists such that $h_0(t) = \theta_1(x_0, t)$, $h_1(t) = \theta_2(x_0, t)$ and $h_s(0) = h_s(1) = x_0$. A continuous function h may be defined on the set $\{(X \times \overset{s}{I} \times (0)) \cup [(X \times 0) \cup (X \times 1) \cup (x_0 \times \overset{s}{I})] \times \overset{t}{I}\}$ as follows :

$$\begin{array}{lll} h(x, s, 0) = x, & x \in X, & s \in \overset{s}{I}, \\ h(x, 0, t) = \theta_1(x, t), & x \in X, & t \in \overset{t}{I}, \\ h(x, 1, t) = \theta_2(x, t), & x \in X, & t \in \overset{t}{I}, \\ h(x_0, s, t) = h_s(t), & s \in \overset{s}{I}, & t \in \overset{t}{I}. \end{array}$$

If (1.8) ii) is assumed, it is proved by the aid of the extended map $h : X \times \overset{s}{I} \times \overset{t}{I}$

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$\to X$ that θ_1 is homotopic to θ_2. This completes the proof.

§ 3. Operation of $\mathfrak{A}(X)$ on the homotopy groups.

Let f be a representative of an element $\alpha \in \pi_n(X)$ and θ be a representative of an element $\vartheta \in \mathfrak{A}(X)$. Let us define $\vartheta\alpha = [h] \in \pi_n(X)$ by the rule,

$$h(x) \equiv \theta(f(x), 1). \tag{3.1}$$

This definition has a definite meaning in the sense that $[h]$ depends only on α and ϑ. Then we have,

THEOREM 5. *$\vartheta\alpha = (A\alpha)^{\xi}$ where $A = \varphi(\vartheta) \in \Sigma(X)$ and ξ is an element of $\pi_1(X)$ represented by $\theta(x_0, t)$ $(1 \geqq t \geqq 0)$.*

Proof. From the definition of homomorphism φ, A is represented by $a_\theta(x) = \theta(x, 0)$, and therefore $\theta(f(x), 0) = a_\theta f(x)$. It is an immediate consequence of the operation of A that $a_\theta f$ represents an element $A\alpha$ of $\pi_n(X)$. Moreover if $f(p) = x_0$ for a fixed point $p \in S^n$, $\theta(f(p), t) = \theta(x_0, t)$ represents an element ξ of $\pi_1(X)$, so that according to the operation of π_1 on π_n due to Eilenberg $h(x) = \theta(f(x), 1)$ represents an element $(A\alpha)^{\xi} \in \pi_n$. This completes the proof.

As a direct consequence of Theorem 5 we have,

THEOREM 6. *$\mathfrak{A}(X)$ is a group of automorphisms of $\pi_n(X)$ for every integer $n \geqq 1$.*

Proof. Because of the combination of automorphisms A and ξ, the operation of $\vartheta \in \mathfrak{A}(X)$ on π_n is also an automorphism of $\pi_n(X)$.

§ 4. Algebraic construction of $\mathfrak{A}(X)$.

Now that the operation of $\mathfrak{A}(X)$ on π_n has been clarified by Theorem 5, we can construct the group $\mathfrak{A}(X)$ from a purely algebraic standpoint. Let $\chi(X) = \{)A, \xi) ; A \in \Sigma(X), \xi \in \pi_1(X)\}$; the totality of all the ordered pairs consisting of an arbitrarily chosen element of $\Sigma(X)$ and of an arbitrarily chosen element of $\pi_1(X)$. Defining $(A, \xi)(\alpha) \equiv (A\alpha)^{\xi}$ for any $\alpha \in \pi_n(X)$, (A, ξ) operates on $\pi_n(X)$, for every integer $\pi \geqq 1$, as an automorphism. If we define a multiplication in the set $\chi(X)$ of automorphisms just defined by the rule,

$$(B, \eta)(A, \xi)(\alpha) \equiv (B, \eta)((A, \xi)(\alpha)),$$

then we have $(B, \eta)(A, \xi) \in \chi(X)$. In order to prove this, we need the following lemma.

LEMMA 4.1 $A(\alpha^{\xi}) = (A\alpha)^{A\xi} \equiv (A, A\xi)(\alpha)$ for any $\alpha \in \pi_n$, where $A\xi$ can be interpreted in the sense that $\Sigma(X) \ni A$ operates on the homotopy group of any dimension, especially on the fundamental group too.

Proof. Let α be represented by a mapping $f: S^n \to X$, $S^n \ni p_0 \to x_0$ and let

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$\xi = [e(t), 1 \geqq t \geqq 0]$. We have a mapping $F: \{S^n \times (0) \cup (p_0) \times \overset{t}{I}\} \to X$ such that $F(x, 0) \equiv f(x)$ for any $x \in S^n$, and $F(p_0, t) \equiv e(t)$. From the homotopy extension property of a polyhedron we have an extended map $\overline{F}: S^n \times \overset{t}{I} \to X$ of F. Since $\overline{F}(x, 0) = f(x)$ and $\overline{F}(p_0, t) = e(t)$, $\overline{F}(x, 1)$ represents an element $\alpha^\xi \in \pi_n(X)$. Let a be a representative of A. Putting $a(\overline{F}(x, t)) \equiv G(x, t): S^n \times \overset{t}{I} \to X$ we have $[G(x, 0)] = A\alpha$ from $G(x, 0) = a(f(x))$ and $[G(x, 1)] = A(\alpha^\xi)$ from $G(x, 1) = a(\overline{F}(x, 1))$. Also, from $G(x_0, t) = a(e(t))$ follows $[G(x_0, t)] = A\xi$. Thus we have $A(\alpha^\xi) = (A\alpha)^{A\xi}$. Making use of the lemma, we have

$$\begin{aligned}(B, \eta)(A, \xi)(\alpha) &\equiv (B, \eta)((A, \xi)(\alpha)) = (B, \eta)((A\alpha)^\xi) \\ &= (B((A\alpha)^\xi))^\eta \\ &= ((B(A\alpha))^{B\xi})^\eta \\ &= (B(A\alpha))^{B\xi \cdot \eta} \equiv (A \cdot B, B\xi \cdot \eta)(\alpha).\end{aligned}$$

Thus $$(B, \eta)(A, \xi) = (A \cdot B, B\xi \cdot \eta) \in \chi(X).$$

THEOREM 7. *By this multiplication $\chi(X)$ forms a group.*

Proof. As to the associative law we have

$$\begin{aligned}(C, \zeta)(B, \eta)(A, \xi) &= (C, \zeta)(AB, B\xi \cdot \eta) \\ &= (AB \cdot C, C(B\xi \cdot \eta) \cdot \zeta) \\ &= (ABC, BC\xi \cdot C\eta \cdot \zeta) \\ ((C, \zeta)(B, \eta))(A, \xi) &= (BC, C\eta \cdot \zeta)(A, \xi) \\ &= (A \cdot BC, BC\xi(C\eta \cdot \zeta)) \\ &= (ABC, BC\xi \cdot C\eta \cdot \zeta)\end{aligned}$$

Thus $$(C, \zeta)((B, \eta)(A, \xi)) = ((C, \zeta)(B, \eta))(A, \xi)$$

The existence of the unity is proved as follows :

$(E, e)(A, \xi) = (AE, E\xi \cdot e) = (A, \xi)$ where E, e are the unities of $\Sigma(X)$ and $\pi_1(X)$ respectively.

The existence of an inverse element is proved thus :

$$(A^{-1}, A^{-1}\xi^{-1})(A, \xi) = (AA^{-1}, A^{-1}\xi \cdot A^{-1}\xi^{-1}) = (E, A^{-1}(\xi\xi^{-1})) = (E, e).$$

This completes the proof.

Now the following MAIN THEOREM concerning the relation of two groups $\mathfrak{A}(X)$ and $\chi(X)$ imparts the complete analysis to the structure of $\mathfrak{A}(X)$ and also to the operation of $\mathfrak{A}(X)$ on $\pi_n(X)$ for every integer $n \geqq 1$.

MAIN THEOREM 8. *$\mathfrak{A}(x)$ is isomorphic to the group $\chi(X)$. Moreover, an isomorphism can be established between these groups, preserving the operation on the homotopy groups.*

Proof. The method of proof being analogous as for Theorems 4, 5, we shall

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restrict ourselves to show the correspondence between two groups. Let θ be a representative of $\vartheta \in \mathfrak{A}(X)$ and let $a_\theta = \theta \mid X \times 0$, $\xi_\theta = \theta \mid x_0 \times I$. Then to ϑ let there correspond $([a_\theta], [\xi_\theta]) \in \chi(X)$. It can be shown that this correspondence is an isomorphism and that the operations of ϑ and of the corresponding element $([a_\theta], [\xi_\theta])$ on π_n are the same.

§ 5. Some remarks on the group $\mathfrak{A}(X)$.

By the aid of the main theorem it is advantageous to use $\chi(X)$ in place of $\mathfrak{A}(X)$ in calculating the invariant $\mathfrak{A}(X)$ of the space X. As is easily seen, two distinct elements of $\chi(X)$ do not always operate differently on π_n so that as the group of the operation on π_n, $\chi(X)$ may be reduced to a smaller group. This reduction gives rise to an analogous classification of the space X as the simplicity of a space due to Eilenberg.

Let $\chi^*(X)$ be the totality of all elements in $\chi(X)$ whose operations on any element of $\pi_n(X)$ are trivial; i.e. $\chi^*(X) \equiv \{(A, \xi)\ ;\ (A, \xi)(\alpha) = \alpha$ for any element $\alpha \in \pi_n(X)\}$. Then $\chi^*(X)$ is clearly a normal subgroup of $\chi(X)$. Similary, put $\chi^{**}(X) \equiv \{(A, e)\ ;\ (A, e)(\alpha) = \alpha$ for any $\alpha \in \pi_n(X)\}$ and $\chi^{***}(X) \equiv \{(E, \xi)\ ;\ (E, \xi)(\alpha) = \alpha$ for any $\alpha \in \pi_n(X)\}$, then these two groups are also normal in $\Sigma(X)$ and $\pi_1(X)$ respectively as well as in $\chi(X)$. It is well known that the space is n-simple in the sense of Eilenberg if $\chi^{***}(X) \cong \pi_1(X)$. It may be an interesting problem to consider the spaces satisfying the conditions such as $\chi^*(X) = \chi(X)$ or $\chi^{**}(X) \cong \Sigma(X)$.

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