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Re-nnd SOLUTIONS OF THE MATRIX EQUATION $AXB = C$

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Abstract

In this article we consider Re-nnd solutions of the equation $AXB = C$ with respect to X, where A, B, C are given matrices. We give necessary and sufficient conditions for the existence of Re-nnd solutions and present a general form of such solutions. As a special case when $A = I$ we obtain the results from a paper of Groß ('Explicit solutions to the matrix inverse problem $AX = B$', *Linear Algebra Appl.* **289** (1999), 131–134).

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1. Introduction

Let $\mathbb{C}^{n\times m}$ denote the set of complex $n \times m$ matrices. Here I_n denotes the unit matrix of order n. By A^*, $\mathcal{R}(A)$, $\mathrm{rank}(A)$ and $\mathcal{N}(A)$, we denote the conjugate transpose, the range, the rank and the null space of $A \in \mathbb{C}^{n\times m}$.

The Hermitian part of X is defined as $H(X) = (1/2)(X + X^*)$. We say that X is Re-nnd (Re-nonnegative definite) if $H(X) \geq 0$ and X is Re-pd (Re-positive definite) if $H(X) > 0$.

The symbol A^- stands for an arbitrary generalized inner inverse of A, that is, A^- satisfies $AA^-A = A$. By $A^\dagger$ we denote the Moore–Penrose inverse of $A \in \mathbb{C}^{n\times m}$, that is, the unique matrix $A^\dagger \in \mathbb{C}^{m\times n}$ satisfying

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

For some important properties of generalized inverses see [5, 6, 16] and [15].

Many authors have studied the well-known equation

$$AXB = C, \tag{1.1}$$

with the unknown matrix X, such that X belongs to some special class of matrices. For example, in [18] and [7] the existence of reflexive and anti-reflexive, with respect to a generalized reflection matrix P, solutions of the matrix equation (1.1) was considered, while in [9, 14, 17, 19] necessary and sufficient conditions for the existence of symmetric and antisymmetric solutions of the equation (1.1) were investigated.

The Hermitian nonnegative definite solutions for the equation $AXA^* = B$ were investigated by Khatri and Mitra [14], Baksalary [4], Dai and Lancaster [10], Groß [12], Zhang and Cheng [23] and Zhang [24].

Wu [21] studied Re-pd solutions of the equation $AX = C$, and Wu and Cain [22] found the set of all complex Re-nnd matrices X such that $XB = C$ and presented a criterion for Re-nndness. Groß [11] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [22]. Some results from [22] were extended in the paper of Wang and Yang [20], in which the authors presented criteria for 2×2 and 3×3 partitioned matrices to be Re-nnd, found necessary and sufficient conditions for the existence of Re-nnd solutions of the equation (1.1) and derived an expression for these solutions. In the paper of Dajić and Koliha [3], a general form of Re-nnd solutions of the equation $AX = C$ is given for the first time, where A and C are given operators between Hilbert spaces. In addition to these papers many other papers have dealt with the problem of finding the Re-nnd and Re-pd solutions of some other forms of equations.

In this paper, we first consider the matrix equation

$$AXA^* = C,$$

where $A \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{n\times n}$, and find necessary and sufficient conditions for the existence of Re-nnd solutions. Also, we present a general form of these solutions. Using this result, we obtain necessary and sufficient conditions for the equation

$$AXB = C,$$

where $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times n}$, to have a Re-nnd solution. This way, the results of [22] and [11] follow as a corollary and a general form of those solutions is given in addition. As far as the author is aware, this is the first time necessary and sufficient conditions for the existence of a Re-nnd solution of the equation $AXB = C$ have been given in terms of g-inverses.

Now, we state some well-known results which are used frequently in the next section.

THEOREM 1.1 Ben-Israel and Greville [5]. *Let $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{p\times r}$ and $C \in \mathbb{C}^{n\times r}$. Then the matrix equation*

$$AXB = C,$$

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is consistent if and only if, for some A^-, B^-,

$$AA^-CB^-B = C,$$

in which case the general solution is

$$X = A^-CB^- + Y - A^-AYBB^-,$$

for arbitrary $Y \in \mathbb{C}^{m \times p}$.

The following result was derived by Albert [1] for block matrices, by Cvetković-Ilić *et al.* [8] for C^* algebras, and by Dajić and Koliha [3] for operators between different Hilbert spaces. Here, we give the basic version proved in [1].

THEOREM 1.2. *Let* $M \in \mathbb{C}^{(n+m)\times(n+m)}$ *be a Hermitian block matrix given by*

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

where $A \in \mathbb{C}^{n \times n}$ *and* $D \in \mathbb{C}^{m \times m}$. *Then,* $M \geq 0$ *if and only if*

$$A \geq 0, \quad AA^\dagger B = B, \quad D - B^*A^\dagger B \geq 0.$$

Anderson and Duffin [2] define the parallel sum of matrices for a pair of matrices of the same order as

$$A : B = A(A+B)^-B.$$

It is clear that for this definition to be meaningful, the expression $A(A+B)^-B$ must be independent of the choice of the g-inverse $(A+B)^-$. Hence, a pair of matrices A and B will be said to be parallel summable if $A(A+B)^-B$ is invariant under the choice of the inverse $(A+B)^-$, that is, if

$$\mathcal{R}(A) \subseteq \mathcal{R}(A+B) \wedge \mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*),$$

or, equivalently,

$$\mathcal{R}(B) \subseteq \mathcal{R}(A+B) \wedge \mathcal{R}(B^*) \subseteq \mathcal{R}(A^* + B^*). \tag{1.2}$$

Note that

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow BB^-A = A.$$

By [13, Theorem 2.1],

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow AA^* \leq \lambda^2 BB^* \quad \text{for some } \lambda \geq 0,$$

so, for the nonnegative definite matrices A and B, we have that

$$A \leq A + B \Leftrightarrow \mathcal{R}(A^{1/2}) \subseteq \mathcal{R}((A+B)^{1/2}),$$

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which implies $\mathcal{R}(A) \subseteq \mathcal{R}((A+B)^{1/2})$ or, equivalently,

$$(A+B)^{1/2}((A+B)^{1/2})^{\dagger}A = A.$$

Now,

$$(A+B)\,(A+B)^{\dagger}A = ((A+B)^{1/2}((A+B)^{1/2})^{\dagger})^2 A = A,$$

which is equivalent to $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$.

Hence, nonnegative definite matrices A and B are parallel summable. Furthermore, in [2] it was proved that for a pair of parallel summable matrices the following expression holds:

$$A : B = B : A,$$

that is,

$$A(A+B)^{-}B = B(A+B)^{-}A. \tag{1.3}$$

2. Results

The next result was first proved by Wu and Cain [22] and later derived in a different way by Groß [11]. It gives necessary and sufficient conditions for the matrix equation $AX = C$ to have a Re-nnd solution X, where A, C are given matrices of suitable size and presents a possible explicit expression for X.

THEOREM 2.1. *Let $A \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{n\times m}$. There exists a Re-nnd matrix $X \in \mathbb{C}^{m\times m}$ satisfying $AX = C$ if and only if $AA^{\dagger}C = C$ and AC^* is Re-nnd.*

From the proof of this theorem we can see that

$$X_0 = A^{\dagger}C - (A^{\dagger}C)^* + A^{\dagger}AC^*(A^{\dagger})^*,$$

is one of Re-nnd solutions of $AX = C$. Also, in [11] the author mentions that any matrix of the form

$$X = X_0 + (I - A^{\dagger}A)Y(I - A^{\dagger}A),$$

with $Y \in \mathbb{C}^{m\times m}$ which is Re-nnd is also a Re-nnd solution of $AX = C$, in the case where such solutions exist, but he did not present a general form of such solutions. Our main aim is to generalize these results to the equation $AXB = C$ and to present a general form of Re-nnd solutions of it.

First, we consider the equation

$$AXA^* = C, \tag{2.1}$$

and find necessary and sufficient conditions for the existence of Re-nnd solutions.

The next auxiliary result presents a general form of a solution X of (2.1) which satisfies $H(X) = 0$.

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LEMMA 2.2. *If $A \in \mathbb{C}^{n\times m}$, then $X \in \mathbb{C}^{m\times m}$ is a solution of the equation*

$$AXA^* = 0, \tag{2.2}$$

which satisfies $H(X) = 0$ if and only if

$$X = W(I - A^\dagger A) - (I - A^\dagger A)W^*, \tag{2.3}$$

for some $W \in \mathbb{C}^{m\times m}$.

PROOF. Denote by $r = \operatorname{rank}(A)$. Let us suppose that X is a solution of the equation (2.2) and $H(X) = 0$. Using a singular value decomposition of $A = U^* \operatorname{Diag}(D, 0)V$, where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$ are unitary and $D \in \mathbb{C}^{r\times r}$ is an invertible matrix, we have that

$$A^\dagger = V^* \operatorname{Diag}(D^{-1}, 0)U \quad \text{and} \quad X = V^* \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} V,$$

for some $X_1 \in \mathbb{C}^{r\times r}$ and $X_4 \in \mathbb{C}^{(m-r)\times(m-r)}$.

From $AXA^* = 0$ we obtain that $X_1 = 0$ and, by $H(X) = 0$, that $X_3 = -X_2^*$ and $H(X_4) = 0$. Hence,

$$X = V^* \begin{bmatrix} 0 & X_2 \\ -X_2^* & X_4 \end{bmatrix} V.$$

Taking into account that $H(X_4) = 0$, for

$$W = V^* \begin{bmatrix} I & X_2 \\ 0 & (1/2)X_4 \end{bmatrix} V,$$

we have that

$$X = W(I - A^\dagger A) - (I - A^\dagger A)W^*.$$

In the other direction we can easily check that for arbitrary $W \in \mathbb{C}^{m\times m}$, X defined by (2.3) is a solution of the equation (2.2) which satisfies $H(X) = 0$.

THEOREM 2.3. *Let $A \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{n\times n}$ be given matrices such that the equation (2.1) is consistent and let $r = \operatorname{rank} H(C)$. There exists a Re-nnd solution of the equation (2.1) if and only if C is Re-nnd. In this case the general Re-nnd solution is given by*

$$X = A^{=}C(A^{=})^* + (I - A^-A)UU^*(I - A^-A)^* + W(I - A^\dagger A) - (I - A^\dagger A)W^*, \tag{2.4}$$

with

$$A^{=} = A^- + (I - A^-A)Z(H(C)^{1/2})^-, \tag{2.5}$$

where A^- and $(H(C)^{1/2})^-$ are arbitrary but fixed generalized inverses of A and $H(C)^{1/2}$, respectively, and $Z \in \mathbb{C}^{m\times n}$, $U \in \mathbb{C}^{m\times(m-r)}$, $W \in \mathbb{C}^{m\times m}$ are arbitrary matrices.

PROOF. If X is a Re-nnd solution of the equation (2.1), then

$$AH(X)A^* = H(C) \geq 0.$$

In the other direction, if C is Re-nnd, then $X_0 = A^- C(A^-)^*$ is a Re-nnd solution of the equation (2.1).

Let us prove that a representation of the general Re-nnd solution is given by (2.4). If X is defined by (2.4), then X is Re-nnd and $AXA^* = AA^- C(AA^-)^* = C$.

If X is an arbitrary Re-nnd solution of (2.1), then $H(X)$ is a Hermitian nonnegative-definite solution of the equation

$$AZA^* = H(C),$$

so, by [12, Theorem 1],

$$H(X) = A^= H(C)(A^=)^* + (I - A^- A)UU^*(I - A^- A)^*,$$

where $A^=$ is given by (2.5), for some $Z \in \mathbb{C}^{m \times n}$ and $U \in \mathbb{C}^{m \times (m-r)}$.

Note that,

$$H(X) = H(A^= C(A^=)^* + (I - A^- A)UU^*(I - A^- A)^*),$$

implying

$$X = A^= C(A^=)^* + (I - A^- A)UU^*(I - A^- A)^* + Z,$$

where $H(Z) = 0$ and $AZA^* = 0$. Using Lemma 2.2, we have that

$$Z = W(I - A^\dagger A) - (I - A^\dagger A)W^*,$$

for some $W \in \mathbb{C}^{m \times n}$. Hence, we obtain that X has a representation as in (2.4).

Now, let us consider the equation

$$AXB = C, \tag{2.6}$$

where $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{n \times n}$ are given matrices and find necessary and sufficient conditions for the existence of a Re-nnd solution.

Without loss of generality we may assume that $n = m$ and that matrices A and B are both nonnegative definite. This follows from the fact that whenever the equation (2.6) is consistent then X is a solution of that equation if and only if X is a solution of the equation $A^*AXBB^* = A^*CB^*$. Hence, from now on, we assume that A and B are nonnegative-definite matrices from the space $\mathbb{C}^{n \times n}$.

The next theorem is the main result of this paper which presents necessary and sufficient conditions for the equation (2.6) to have a Re-nnd solution.

THEOREM 2.4. *Let $A, B, C \in \mathbb{C}^{n \times n}$ be given matrices such that A and B are nonnegative definite and the equation (2.6) is consistent. There exists a Re-nnd solution of (2.6) if and only if*

$$T = B(A + B)^- C(A + B)^- A,$$

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is Re-nnd, where $(A+B)^-$ is a g-inverse of $A+B$. In this case a general Re-nnd solution is given by

$$\begin{aligned} X &= (A+B)^{=}(C+Y+Z+W)\,((A+B)^{=})^* \\ &\quad + (I-(A+B)^-(A+B))UU^*(I-(A+B)^-(A+B))^* \\ &\quad + Q(I-(A+B)^\dagger(A+B)) - (I-(A+B)^\dagger(A+B))Q^*, \end{aligned} \tag{2.7}$$

where Y, Z, W are arbitrary solutions of the equations

$$\begin{aligned} Y(A+B)^-B &= C(A+B)^-A, \\ A(A+B)^-Z &= B(A+B)^-C, \\ A(A+B)^-W(A+B)^-B &= T, \end{aligned} \tag{2.8}$$

such that $C+Y+Z+W$ is Re-nnd, $(A+B)^{=}$ is defined by

$$(A+B)^{=} = (A+B)^- + (I-(A+B)^-(A+B))P(H(C+Y+Z+W)^{1/2})^-,$$

where $U \in \mathbb{C}^{n\times(n-r)}$, $Q \in \mathbb{C}^{n\times n}$, $P \in \mathbb{C}^{n\times n}$ are arbitrary, $r = \operatorname{rank}(C+Y+Z+W)$.

PROOF. Denote by

$$\begin{aligned} E &= (A+B)^-B, \quad & F &= C(A+B)^-A, \\ K &= A(A+B)^-, \quad & L &= B(A+B)^-C. \end{aligned}$$

Now, equations (2.8) are equivalent to

$$YE = F, \quad KZ = L, \quad KWE = T. \tag{2.9}$$

Using (1.2), (1.3) and the fact that E is g-invertible and $E^- = B^-(A+B)$, we have that

$$\begin{aligned} FE^-E &= C(A+B)^-AB^-(A+B)(A+B)^-B \\ &= C(A+B)^-AB^-B = CB^-B(A+B)^-AB^-B \\ &= CB^-A(A+B)^-BB^-B = CB^-A(A+B)^-B \\ &= CB^-B(A+B)^-A = C(A+B)^-A = F, \end{aligned}$$

which implies that the equation $YE = F$ is consistent. In a similar way, we can prove that the other two equations from (2.9) are consistent. Furthermore, $T^* = F^*E = KL^*$ is Re-nnd which implies, by Theorem 2.1, that the first two equations from (2.9) have Re-nnd solutions.

Now, suppose that the equation (2.6) has a Re-nnd solution X. Then

$$\begin{aligned} H(T) &= H(B(A+B)^-AXB(A+B)^-A) \\ &= (B(A+B)^-A)H(X)(B(A+B)^-A)^* \geq 0. \end{aligned}$$

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Conversely, let T be Re-nnd. We can check that

$$X_0 = (A+B)^-(C+Y+Z+W)(A+B)^-, \tag{2.10}$$

is a solution of the equation (2.6), where Y, Z, W are arbitrary solutions of the equations (2.9). This follows from

$$\begin{aligned} AX_0B &= (A+B)(A+B)^-C(A+B)^-(A+B) \\ &= (A+B)(A+B)^-AA^-CB^-B(A+B)^-(A+B) \\ &= AA^-CB^-B = C. \end{aligned}$$

Now, we have to prove that for some choice of Y, Z, W, the matrix $C+Y+Z+W$ is Re-nnd which would imply that X_0 is Re-nnd.

Let

$$\begin{aligned} Y &= FE^- - (FE^-)^* + (E^-)^*F^*EE^- + (I-EE^-)^*(I-EE^-), \\ Z &= K^-L - (K^-L)^* + K^-KL^*(K^-)^* + (I-K^-K)Q(I-K^-K)^*, \\ W &= K^-TE^- - (I-K^-K)S - S(I-EE^-), \end{aligned}$$

where $Q = (C^* - K^-T^*E^-)(C^* - K^-T^*E^-)^*$ and $S = K^-KC^* + C^*EE^-$. Obviously, Y, Z, W are solutions of the equations (2.9) and

$$\begin{aligned} H(Y) &= (E^-)^*H(T)E^- + (I-EE^-)^*(I-EE^-), \\ H(Z) &= K^-H(T)(K^-)^* + (I-K^-K)H(Q)(I-K^-K)^*, \\ H(W) &= K^-TE^- + (E^-)^*T^*(K^-)^* - H(C^*EE^- + K^-KC^* - 2K^-T^*E^-). \end{aligned}$$

Using

$$\begin{aligned} K^-KK^-T^*E^- &= K^-KK^-KL^*E^- = K^-KL^*E^- = K^-T^*E^-, \\ K^-T^*E^-EE^- &= K^-F^*EE^-EE^- = K^-F^*EE^- = K^-T^*E^-, \\ KC^*E &= KL^* = T^*, \end{aligned}$$

we compute,

$$\begin{aligned} H(C+Y+Z+W) &= ((E^-)^* + K^-)H(T)((E^-)^* + K^-)^* \\ &\quad + [(I-EE^-)^* \quad (I-K^-K)]D \begin{bmatrix} I-EE^- \\ (I-K^-K)^* \end{bmatrix}, \end{aligned}$$

where

$$D = \begin{bmatrix} I & C - (E^-)^*T(K^-)^* \\ C^* - K^-T^*E^- & H(Q) \end{bmatrix}.$$

By Theorem 1.2, it follows that D is nonnegative definite, so $H(C+Y+Z+W) \geq 0$.

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Hence, with such a choice of Y, Z, W, it can be seen that X_0 defined by (2.10) is Re-nnd solution of (2.6). So, we proved the sufficient part of the theorem.

Let X be an arbitrary Re-nnd solution of (2.6). It is evident that $Y = AXA$, $Z = BXB$ and $W = BXA$ are solutions of (2.9), and that

$$(A + B)X(A + B) = C + Y + Z + W,$$

is Re-nnd. Now, using Theorem 2.3, we obtain that X has the representation (2.7).

Let us mention that, if we apply Theorem 2.4 to the equation

$$AX = C,$$

we obtain [11, Theorem 1] as a corollary.

Note that if the equation $AX = C$ is consistent then X is a solution of it if and only if $A^*AX = A^*C$. By Theorem 2.4, we obtain that there exists a Re-nnd solution of the equation $AX = C$ if and only if

$$T = (A^*A + I)^{-1}A^*C(A^*A + I)^{-1}A^*A,$$

is Re-nnd. Note that in this case $(I + A^*A)$ is invertible matrix.

Let us prove that T is Re-nnd if and only if CA^* is Re-nnd.

By

$$(A^*A + I)^{-1}A^*A = A^*A(A^*A + I)^{-1},$$

we have that

$$T = ((A^*A + I)^{-1}A^*)(CA^*)((A^*A + I)^{-1}A^*)^*,$$

that is,

$$H(T) = ((A^*A + I)^{-1}A^*)H(CA^*)((A^*A + I)^{-1}A^*)^*.$$

From the last equality, $H(CA^*) \geq 0$ implies $H(T) \geq 0$.

Now, suppose that $H(T) \geq 0$. Owing to the consistence of the equation $AX = C$, it follows that $AA^\dagger C = C$ which implies that

$$(A^\dagger)^*(A^*A + I)T((A^\dagger)^*(A^*A + I))^* = (A^\dagger)^*A^*CA^*AA^\dagger = AA^\dagger CA^* = CA^*,$$

that is,

$$H(CA^*) = ((A^\dagger)^*(A^*A + I))H(T)((A^\dagger)^*(A^*A + I))^* \geq 0.$$

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