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Tracially Quasidiagonal Extensions

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Abstract. It is known that a unital simple C^*-algebra A with tracial topological rank zero has real rank zero. We show in this note that, in general, there are unital C^*-algebras with tracial topological rank zero that have real rank other than zero.

Let $0 \to J \to E \to A \to 0$ be a short exact sequence of C^*-algebras. Suppose that J and A have tracial topological rank zero. It is known that E has tracial topological rank zero as a C^*-algebra if and only if E is tracially quasidiagonal as an extension. We present an example of a tracially quasidiagonal extension which is not quasidiagonal.

1 Introduction

The tracial topological rank was introduced as a noncommutative analog of the covering dimension for topological spaces ([Ln2] and [Ln3]). It plays an important role in the classification of amenable C^*-algebras (see [Ln3], [Ln5] and [Ln6]). A unital commutative C^*-algebra $C(X)$ has tracial topological rank k if and only if $\dim X = k$. It was shown in [HLX1] that if $\dim X = k$ and $\mathrm{TR}(A) = m$ then $\mathrm{TR}\big(C(X) \otimes A\big) \leq k + m$. At this moment, the most interesting case is that of a C^*-algebra with tracial topological rank no more than 1.

If A is a unital separable simple C^*-algebra with tracial topological rank zero, it was shown in [Ln4] that A is quasidiagonal and has real rank zero, stable rank one and weakly unperforated ordered K_0-group. We are also interested in the case of C^*-algebras that are not simple. Let

$$0 \to J \to E \to A \to 0$$

be a short exact sequence with $\mathrm{TR}(J) = 0 = \mathrm{TR}(A)$. It is known that $\mathrm{TR}(E) = 0$ if and only if the extension is tracially quasidiagonal [HLX2]. The definition of tracially quasidiagonal extension is stated in Definition 4.1. Quasidiagonal extensions are tracially quasidiagonal. A natural question is whether there are any tracially quasidiagonal extensions which are not quasidiagonal. We will show in this note that there are tracially quasidiagonal extensions which are not quasidiagonal.

In the case that A is simple, as mentioned above, it has been proved that $\mathrm{TR}(A) = 0$ implies that A has real rank zero. The question remained, if, in general, $\mathrm{TR}(A) = 0$ implies that A has real rank zero. In this note, we will construct a tracially quasidiagonal extension of C^*-algebras which is not quasidiagonal. We will show that the C^*-algebra of this extension has tracial topological rank zero but real rank not equal to zero.

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2 The Construction

Definition 2.1 Let A_0 be a unital separable simple C^*-algebra with tracial topological rank zero and with $K_1(A_0) = \mathbb{Z}\oplus\mathbb{Z}$ which satisfies the Universal Coefficient Theorem. For example, A_0 may be chosen to be a unital simple $A\mathbb{T}$-algebra. Let $B = C(S^2)$. As is known, $K_0(B)$ may be written as $\mathbb{Z}\oplus\mathbb{Z}$ and $K_0(B)_+ = \{(n, m) : n > 0\}\cup\{(0, 0)\}$. Consider the extension E:

$$0 \to I \to E \xrightarrow{\pi} B \to 0,$$

where $I = A_0 \otimes \mathcal{K}$ and the boundary map $\mathrm{ind}\colon K_1(B) \to K_0(I)$ is zero and the boundary map $\partial\colon K_0(B) \to K_1(I)$ is nonzero with $\partial(0, 1) \neq 0$. It follows from [BD] that E is a quasidiagonal C^*-algebra. We will use the fact that if E is quasidiagonal as a C^*-algebra, then there is an injective homomorphism which maps E into $\prod_n M_{k(n)}/\bigoplus_n M_{k(n)}$ for some increasing sequence $\{k(n)\}$.

Set $E_1 = E$. Let $\{e_{ij}\}$ denote the matrix units for $\mathcal{K}$. Write $e_n = \sum_{i=1}^n e_{ii}$, $n = 1, 2, \dots$. Here we identify e_{11} with 1_{A_0}.

Definition 2.2 Let A be a C^*-algebra, $\mathcal{G} \subset A$ be a finite subset and $\varepsilon > 0$ be a positive number. Recall that a positive linear map $L\colon A \to B$ (where B is a C^*-algebra) is said to be $\mathcal{G}$-ε-multiplicative if

$$\|L(a)L(b) - L(ab)\| < \varepsilon$$

for all $a, b \in \mathcal{G}$.

Let $A = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$. By a set of standard generators of A, we mean $\{(a_1, a_2, \dots, a_k)\}$, where $a_i = 0$, or a_i is an element in the matrix units of M_{n_i}.

Proposition 2.3 *For any $\varepsilon > 0$, there is $\delta(\varepsilon, n) > 0$ such that if $L\colon A \to B$ is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where A is a C^*-algebra with* $\dim A \le n$*, B is a unital C^*-algebra, and $\mathcal{G}$ contains a set of standard generators of A, then there is a homomorphism $h\colon A \to B$ such that*

$$\|L - h\| < \varepsilon.$$

Proof Since the unit ball of A is compact, there is a finite subset $\mathcal{F}$ of the unit ball such that, for any $x \in A$ with $\|x\| \le 1$, $\mathrm{dist}(x, \mathcal{F}) < \varepsilon/3$. It is well known that there is $\delta > 0$ such that, for any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map L, there is a homomorphism $h\colon A \to B$ such that $\|L(a) - h(a)\| < \varepsilon/3$ for all $a \in \mathcal{F}$. Therefore

$$\|L - h\| < \varepsilon. \qquad \blacksquare$$

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Definition 2.4 In the above proposition, let $\varepsilon = 1/2^n$. We denote by δ_n the corresponding δ. We may assume that $0 < \delta_{n+1} < \delta_n < 1$.

Definition 2.5 For any k, we will use $\pi_k \colon M_k(E) \to M_k(B)$ for the quotient map induced by π. Let $\{\xi_1, \xi_2, \dots\}$ be a dense sequence of S^2, where each point repeats infinitely many times. Let $\{a_1, a_2, \dots,\}$ be a dense sequence of the unit ball of E. Let $\mathcal{G}_n = \{0, 1_E, a_1, \dots, a_n\}$, $n = 1, 2, \dots$, and let $\mathcal{F}_1 = \mathcal{G}_1$. Since E is quasidiagonal, there is a (unital) contractive completely positive linear map $\psi_1 \colon E_1 \to M_{k(1)}$ which is $\mathcal{F}_1$–$1/2 \cdot 1/2 \cdot 1/2^2$-multiplicative with $\|\psi_1(a)\| \geq (1/2)\|a\|$ for all $a \in \mathcal{F}_1$. Let $p_1 = 1_{A_0} \otimes e_{k(1)}$. So $p_1 \subset I$ and ψ_1 is viewed as a map from E_1 to $p_1(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_1$. Put $\phi_1(a) = \pi(a)(\xi_1) \cdot (1_{E_1} - p_1)$. Define $L_1 \colon E_1 \to E_2 = M_2(E_1)$ by

$$L_1(a) = \operatorname{diag}\big(a, \phi_1(a), \psi_1(a)\big)$$

for $a \in E_1$. Set $C_1 = \phi_1(E) \oplus p_1(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_1$ and $C_1' = p_1(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_1$.

Set $I_2 = M_2(I)$. Let $\mathcal{F}_2$ be a finite subset of E_2 containing $L_1(\mathcal{F}_1)$, $\{(a_{ij})_{i,j=1}^2 : a_{ij} = 0, a_1, \text{ or } a_2\}$, a set of standard generators for C_1 and $\{u_{ij}\}_{i,j=1}^2$, a matrix unit, where u_{11} and u_{22} are identified with $\operatorname{diag}(1_{E_1}, 0)$, $\operatorname{diag}(0, 1_{E_1})$. Since E is quasidiagonal, there is a (unital) $\mathcal{F}_2$–$1/3 \cdot 1/2^2 \cdot \delta_{\dim C_1}/2^2$-multiplicative contractive completely positive linear map $\psi_2 \colon E_2 \to M_{k(2)}$ such that $(\psi_2)|_{M_2(\mathbb{C} \cdot 1_E)}$ is a homomorphism and $\|\psi_2(a)\| \geq (1 - 1/4)\|a\|$ for all $a \in \mathcal{F}_2$, and such that there is homomorphism $h_2 \colon C_1 \to M_{k(2)}$ such that

$$\|(\psi_2)|_{C_1} - h_2\| < 1/4$$

(by Proposition 2.3, such h_2 exists). Let $E_3 = M_{2+1}(E_2) = M_{3!}(E)$ and $I_3 = M_{2+1}(I_2)$. Let $p_2' = 1_{A_0} \otimes e_{k(2)}$ and $p_2 = \operatorname{diag}(p_2', p_2') \in I_2$. Define $\phi_1^{(2)}(a) = \pi_2(a)(\xi_1)$ for $a \in E_2$ but the image of $\phi_1^{(2)}$ is identified with $M_2(\mathbb{C} \cdot 1_E)$. Define $\phi_2(a) = \pi_2(a)(\xi_2)$ for $a \in E_2$ but the image of ϕ_2 is identified with $M_2\big(\mathbb{C} \cdot (1_E - p_2')\big)$. Let $\Psi_2(a) = \operatorname{diag}\big(\psi_2(a), \psi_2(a)\big)$ for $a \in E_2$. We now view $\Psi_2 \colon E_2 \to p_2 M_2(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_2 \subset p_2 I_2 p_2$. In particular $\Psi_2(1_{E_2}) = p_2$. Define $L_2 \colon E_2 \to E_3$ by

$$L_2(a) = \operatorname{diag}\big(a, \phi_1^{(2)}(a), \phi_2(a), \Psi_2(a)\big).$$

It should be noted that $\operatorname{diag}\big(\phi_2(a), \Psi_2(a)\big)$ is in E_2 and L_2 is unital. Let

$$C_2 = \phi_1^{(2)}(E_2) \oplus \phi_2(E_2) \oplus p_2(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_2 \quad \text{and} \quad C_2' = p_2(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_2.$$

Let $E_4 = M_4(E_3)$ and $I_4 = M_4(I_3)$. Let $\mathcal{D}_2$ be a finite subset containing 1_{C_2} and the standard generators of C_2. Let $\mathcal{F}_3$ be a finite subset of E_3 containing $L_2(\mathcal{F}_2)$, $\{(a_{ij})_{i,j=1}^{3\times 2} : a_{ij} = 0, a_1, a_2, \text{ or } a_3\}$, $\mathcal{D}_2$ and $\{u_{ij}\}_{i,j=1}^3$, a matrix unit and where u_{ii} is identified with a diagonal element with 1_{E_3} on the i-th place and zero elsewhere.

Since E is quasidiagonal and $E_3 = M_{3!}(E)$, there is a $\mathcal{F}_3$–$1/4 \cdot 1/2^3 \cdot \delta_{\dim C_2}/2^3$-multiplicative contractive completely positive linear map $\psi_3 \colon E_3 \to M_{k(3)}$ such that $(\psi_3)|_{M_{3!}(\mathbb{C} \cdot 1_E)}$ is a homomorphism with $\|\psi_3(a)\| \geq (1 - 1/2^3)\|a\|$ for $a \in \mathcal{F}_3$ and there is a homomorphism $h_3 \colon C_2 \to M_{k(3)}$ such that

$$\|\psi_3|_{C_2} - h_3\| < 1/2^3.$$

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Define $\phi_i^{(3)}(a) = \pi_{3!}(a)(\xi_i)$ for $a \in E_3$ but the image of $\phi_i^{(3)}$ is identified with $M_{3!}(\mathbb{C} \cdot 1_E)$, $i = 1, 2$. Let $p_3' = 1_{A_0} \otimes e_{k(3)}$ and $p_3 = \operatorname{diag}(p_3', \ldots, p_3')$, where p_3' repeats 3! times. So $p_3 \in I_3$. Let $\Psi_3(a) = \operatorname{diag}\big(\psi_3(a), \ldots, \psi_3(a)\big)$ for $a \in E_3$, where $\psi_3(a)$ repeats 3! many times. We view $\Psi_3 \colon E_3 \to p_3 M_3(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_3$. Define $\phi_3(a) = \pi_{3!}(a)(\xi_3)$ for $a \in E_3$ but its image is identified with $M_{3!}\big(\mathbb{C} \cdot (1_E - p_3')\big)$ (so its unit is $1_{E_3} - p_3$). Define $L_3 \colon E_3 \to E_4$ by (for any $a \in E_3$)

$$L_3(a) = \operatorname{diag}\big(a, \phi_1^{(3)}(a), \phi_2^{(3)}(a), \phi_3(a), \Psi_3(a)\big).$$

Note that $\operatorname{diag}\big(\phi_3(a), \Psi_3(a)\big) \in E_3$. Put

$$C_3 = \bigoplus_{i=1}^{2} \phi_i^{(3)}(E_3) \oplus \phi_3(E_3) \oplus p_3 M_{3!}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_3 \quad \text{and} \quad C_3' = p_3 M_{3!}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_3.$$

We continue the construction in this fashion. With $C_n = \bigoplus_{i=1}^{n-1} \phi_i^{(n)}(E_n) \oplus \phi_n(E_n) \oplus p_n\big(M_{n!}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})\big)p_n$, let $E_{n+1} = M_{n+1}(E_n)$ and $I_{n+1} = M_{n+1}(I_n)$. Let $\mathcal{D}_n$ be a finite subset of C_n containing 1_{C_n} and a standard set of generators of C_n and $\mathcal{F}_{n+1}$ be a finite subset of E_{n+1} containing $L_n(\mathcal{F}_n)$, $\{(a_{ij})_{i,j=1}^{n!} : a_{ij} = 0, a_1, \ldots, \text{or } a_n\}$, $\mathcal{D}_n$ and $\{u_{ij}\}_{i,j=1}^{n}$, a matrix unit, where u_{ii} is identified with $\operatorname{diag}(0, \ldots, 0, 1_{E_n}, 0, \ldots, 0)$ (the i-th place is 1_{E_n}). Since E is quasidiagonal and $E_{n+1} = M_{(n+1)!}(E)$, there is a unital $\mathcal{F}_{n+1}$–$1/(n+2) \cdot 1/2^{n+1} \cdot \delta_{\dim C_n}/2^{n+1}$-multiplicative contractive completely positive linear map $\psi_{n+1} \colon E_{n+1} \to M_{k(n+1)}$ such that $(\psi_{n+1})|_{M_{(n+1)!}}(\mathbb{C} \cdot 1_E)$ is a homomorphism, $\|\psi_{n+1}(a)\| \geq (1 - 1/2^{n+1})\|a\|$ for $a \in \mathcal{F}_{n+1}$ and there is a homomorphism $h_{n+1} \colon C_n \to M_{k(n+1)}$ such that

$$\|(\psi_{n+1})|_{C_n} - h_{n+1}\| < 1/2^{n+1}.$$

Define $\phi_i^{(n+1)}(a) = \pi_{(n+1)!}(a)(\xi_i)$ for $a \in E_{n+1}$ and identify the image of $\phi_i^{(n+1)}$ with $M_{(n+1)!}(\mathbb{C} \cdot 1_E)$, $i = 1, 2, \ldots, n$. Let $p_{n+1}' = 1_{A_0} \otimes e_{k(n+1)}$ and $p_{n+1} = \operatorname{diag}(p_{n+1}', \ldots, p_{n+1}')$, where p_{n+1}' repeats $(n+1)!$ times. Put $\Psi_{n+1}(a) = \operatorname{diag}\big(\psi_{n+1}(a), \ldots, \psi_{n+1}(a)\big)$, where $\psi_{n+1}(a)$ repeats $(n+1)!$ many times. Thus the image of Ψ_{n+1} is identified with $p_{n+1} M_{(n+1)!}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_{n+1}$. Note that $\Psi_{n+1}(1_{E_{n+1}}) = p_{n+1}$. Define $\phi_{n+1}(a) = \pi_{(n+1)!}(a)(\xi_{n+1})$ but identify its image with $M_{(n+1)!}\big(\mathbb{C} \cdot (1_E - p_{n+1}')\big)$ (so its unit is $1_{E_{n+1}} - p_{n+1}$). Define

$$L_{n+1}(a) = \operatorname{diag}\big(a, \phi_1^{(n+1)}(a), \phi_2^{(n+1)}(a), \ldots, \phi_n^{(n+1)}(a), \phi_{n+1}(a), \Psi_{n+1}(a)\big),$$

where $a \in E_{n+1}$. Note that $\operatorname{diag}\big(\phi_{n+1}(a), \Psi_{n+1}(a)\big) \in E_{n+1}$. Let

$$C_{n+1} = \bigoplus_{i=1}^{n} \phi_i^{(n+1)}(E_{n+1}) \oplus \phi_{n+2}(E_{n+1}) \oplus p_{n+1}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_{n+1} \quad \text{and}$$

$$C_{n+1}' = p_{n+1} M_{(n+1)!}(\mathbb{C} \cdot 1_{A_0} \otimes \mathcal{K})p_{n+1}.$$

It is easy to verify that (E_n, L_n) forms a generalized inductive limit in the sense of [BE]. Denote by A the C^*-algebra defined by this inductive limit. We will use

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$L_{n,n+k}\colon E_n \to E_{n+k}$ for the decomposition $L_{n+k-1}\circ\cdots\circ L_n$ and $L_{n,\infty}\colon E_n \to A$ for the map induced by the inductive limit. We will also use the fact that $\|L_n(a)\| = \|a\| = \|L_{n,\infty}(a)\|$ for all $a \in E_n$, $n = 1, 2, \dots$.

Let $I_1 = I$, $I_{n+1} = M_{(n+1)!}(I)$. Then $I_n \cong A_0 \otimes \mathcal{K}$ and I_n is an ideal of E_n. Set $J_0 = \bigcup_{n=1}^{\infty} L_{n,\infty}(I_n)$ and $J = \bar{J_0}$.

Proposition 2.6 *J is an ideal of A.*

Proof Let $a \in A$ and $b \in J$. We want to show that $ab, ba \in J$. For any $\varepsilon > 0$, there are $a' \in \bigcup_{n=1}^{\infty} L_{n,\infty}(E_n)$ and $b' \in J_0$ such that $\|a - a'\| < \varepsilon$ and $\|b - b'\| < \varepsilon$. It suffices to show that $a'b', b'a' \in J$. To simplify notation, without loss of generality, we may assume that $a \in \bigcup_{n=1}^{\infty} L_{n,\infty}(E_n)$ and $b \in J_0$. Therefore, there is an integer $n > 0$ such that $a = L_{n,\infty}(a_1)$ and $b = L_{n,\infty}(b_1)$, where $a_1 \in E_n$ and $b_1 \in I_n$. There is an integer $N > n$ such that

$$\left\| L_{N,N+k}\circ L_{n,N}(a_1) L_{N,N+k}\circ L_{n,N}(b_1) - L_{N,N+k}\big(L_{n,N}(a_1)L_{n,N}(b_1)\big)\right\| < \varepsilon$$

for all $k > 0$. By the definition of $L_{n,N}$, $L_{n,N}(b_1) \in I_N$. Therefore

$$L_{N,N+k}\big(L_{n,N}(a_1)L_{n,N}(b_1)\big) \in I_{N+k}.$$

This implies that

$$\operatorname{dist}(ab, J) < \varepsilon$$

for all $\varepsilon > 0$. Hence $ab \in J$. Similarly $ba \in J$. ■

Definition 2.7 Let $B_1 = C(S^2)$ and $B_{n+1} = M_{(1+n)!}\big(C(S^2)\big)$, $n = 1, 2, \dots$. Define $h_n\colon B_n \to B_{n+1}$ by $h_n(b) = \operatorname{diag}\big(b, b(\xi_1), \dots, b(\xi_n)\big)$, $n = 1, 2, \dots$. Let $B_\infty = \lim_n(B_n, h_n)$. Then B_∞ is a unital simple C^*-algebra with $\mathrm{TR}(B_\infty) = 0$ (see Definition 3.2), $K_1(B_\infty) = \{0\}$ and $K_0(B_\infty) = \mathbb{Q} \oplus \mathbb{Z}$ with $(K_0(B_\infty))_+ = \big\{(r, m) : r \in \mathbb{Q}_+ \setminus \{0\}, m \in \mathbb{Z}\big\} \cup \{(0, 0)\}$.

Proposition 2.8 *Let $\pi\colon A \to A/J$ be the quotient map. Then $\pi(A) \cong B_\infty$.*

Proof We first show that, for each n, $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n)$.

Let $a \in E_n \setminus I_n$. Then, by the construction, for all $m > 0$,

$$\operatorname{dist}\big(L_{n,m}(a), I_{n+m}\big) \geq \|\pi_{n!}(a)\|,$$

where $\pi_{n!}\colon E_n \to E_n/I_n$ is the quotient map. This implies that

$$\operatorname{dist}\big(L_{n,\infty}(a), J\big) \geq \|\pi_{n!}(a)\|.$$

Therefore $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n)$.

Now we have

$$L_{n,\infty}(E_n)/J \cong B_n.$$

From the construction there is an isomorphism from $L_n(E_n)/I_{n+1}$ to $L_{n,\infty}(E_n)/J$. Denote by $j_n\colon L_{n,\infty}(E_n)/J \to L_{n+1,\infty}(E_{n+1})/J$ the map induced by L_n and by γ_n the isomorphism from $L_{n,\infty}(E_n)/J$ onto B_n. We obtain the following intertwining:

$$\begin{array}{ccc} L_{n,\infty}(E_n)/J & \xrightarrow{\;j_n\;} & L_{n+1,\infty}(E_n)/J \\ \Big\downarrow{\scriptstyle \pi_{n!}} & & \Big\downarrow{\scriptstyle \pi_{(n+1)!}} \\ B_n & \xrightarrow{\;h_{n,\infty}\;} & B_{n+1}. \end{array}$$

This implies that $B_\infty \cong A/J$. ∎

3 The Tracial Topological Rank of the C^*-Algebra A

Throughout the rest of the paper, we will use $f_{\delta_2}^{\delta_1}$ (where $0 < \delta_2 < \delta_1 < 1$) for the following non-negative continuous function on $[0,\infty)$ defined by

$$f_{\delta_2}^{\delta_1}(t) = \begin{cases} 1 & t \geq \delta_1, \\ \frac{t-\delta_2}{\delta_1-\delta_2} & \delta_2 < t < \delta_1, \\ 0 & t \leq \delta_2. \end{cases}$$

Definition 3.1 Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leq [b]$ if there exists $x \in A$ such that $a = x^*x$ and $xx^* \in \overline{bAb}$, and $[a] = [b]$ if $a = x^*x$ and $b = xx^*$. For more information on this relation, see [Cu1], [Cu2] and [HLX1].

Definition 3.2 **([Ln4] and [HLX1])** Recall that a unital C^*-algebra A is said to have tracial topological rank zero if the following holds: for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there is a projection $p \in A$ and a finite dimensional C^*-subalgebra B of A with $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$, and
(3) $\big[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\big] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$.

If A has tracial topological rank zero, we will write $\mathrm{TR}(A) = 0$. If A is non-unital, we will say that A has tracial topological rank zero if $\mathrm{TR}(\tilde{A}) = 0$.

Lemma 3.3 *Let $0 < \sigma_4 < \sigma_3 < 1$, there is $\delta_1 = \delta(\sigma_3, \sigma_4) > 0$ such that for any C^*-algebra A, any $a, b \in A_+$ and $x \in A$ with $\|x\| \leq 1$, $\|a\| \leq 1$, $\|b\| \leq 1$ and any σ_1, σ_2 with $\sigma_3 < \sigma_2 < \sigma_1 < 1$, then $\|x^*x - a\| < \delta_1$ and $\|xx^* - b\| < \delta_1$ imply*

$$[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\sigma_4}^{\sigma_3}(b)].$$

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Proof Let $\sigma_4' = \sigma_4 + \frac{\sigma_3-\sigma_4}{4}$ and let $\sigma_3' = \sigma_4 + \frac{\sigma_3-\sigma_4}{2}$. It follows from Lemma 1.8 of [HLX1] that there is δ (depending only on $\sigma_4 < \sigma_4' < \sigma_3' < \sigma_3$) such that

$$[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\sigma_4'}^{\sigma_3'}(x^*x)]$$

if $\|x^*x - a\| < \delta$ with $0 < \sigma_3' < \sigma_2 < \sigma_1 < 1$. On the other hand, by Lemma 1.8 of [HLX1] that there is $\delta(\sigma_4, \sigma_3) > 0$, such that if $\|xx^* - b\| < \delta(\sigma_4, \sigma_3)$,

$$[f_{\sigma_4'}^{\sigma_3'}(xx^*)] \leq [f_{\sigma_4}^{\sigma_3}(b)].$$

Since

$$[f_{\sigma_4'}^{\sigma_3'}(x^*x)] = [f_{\sigma_4'}^{\sigma_3'}(xx^*)],$$

we conclude that

$$[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\sigma_4}^{\sigma_3}(b)].$$

Note both δ and $\delta(\sigma_4, \sigma_3)$ depend only on σ_3 and σ_4. ■

Lemma 3.4 $\mathrm{TR}(A) = 0$.

Proof By 1.11 in [HLX1], it suffices to show the following: for any $\varepsilon > 0$, any $0 < \sigma_2 < \sigma_1 < 1$, any finite subset $\mathcal{F}$ of A and a nonzero element $a \in A_+$, there is a projection $p \in A$ and a finite dimensional C^*-subalgebra $C \subset A$ with $1_C = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $\mathrm{dist}(pxp, C) < \varepsilon$ for all $x \in \mathcal{F}$, and
(3) $\left[f_{\sigma_2}^{\sigma_1}\big((1-p)a(1-p)\big)\right] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$ for some $0 < \sigma_4 < \sigma_3 < \sigma_2$.

Without loss of generality, we may assume that $\|a\| = 1$. Fix $0 < d_2 < d_1 < \min\{1/8, \sigma_2\}$. Let $\delta(d_1, d_2) > 0$ be as in Lemma 3.3. There is an integer n such that $1/n < \varepsilon/4$, and a finite subset $S \subset E_n$ such that $\mathcal{F} \cup \{a\} \subset L_{n,\infty}(S)$. Suppose that $L_{n,\infty}(b) = a$, where $0 \leq b \leq 1$ is in E_n and $\|b\| = 1$. We may also assume $L_{n,\infty}(S') \subset L_{l,\infty}(\mathcal{F}_l)$ where $S' = S \cup \{cd : c, d \in S\}$ and where $\mathcal{F}_l$ is as in Definition 2.5. Choose a large integer $l > (n+1)^2$ such that $\max\{1/2^{l-2}, 1/l\} < \delta(d_1, d_2)/2$ and $\left\|\psi_l\big(L_{n,l-1}(b)\big)\right\| \geq (1/2)\|L_{n,l-1}(b)\| = (1/2)\|b\|$. For $s \in S$, we may write (in E_l for some contractive completely positive linear map L)

$$L_{n,l}(s) = \mathrm{diag}\big(s, L(s)\big) \quad \text{with} \quad L_{n,l}(1_{E_n}) = \mathrm{diag}\big(1_{E_n}, L(1_{E_n})\big),$$

where $L(s) \in C_l$. Since $L_{l,\infty}$ is $\mathcal{F}_l$–$1/(l+1)2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative, by Proposition 2.3, there is a homomorphism $h\colon C_l \to A$ such that

$$\|L_{l,\infty}|_{C_l} - h\| < 1/2^{l-1}.$$

Let $p' = \mathrm{diag}\big(0, L(1_{E_n})\big)$. Then $p' \in C_l$. So there is a projection $p \in h(C_l)$ such that $\|L_{l,\infty}(p') - p\| < \min\{1/2^{l-1}, \varepsilon/2\}$. Since $L_{l,\infty}$ is $\mathcal{F}_l$–$1/(l+1)2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative, we have

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(1) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$ and
(2) $pxp \in_\varepsilon h(C_l)$ for $x \in \mathcal{F}$.

To show (3) we consider two cases. The case that $b \in E_n \setminus I_n$ is rather standard. We start with case (i): $b \in (I_n)_+$. We may assume that

$$\|e_l b - b\| < \min\{\delta(d_1, d_2)/4, \varepsilon/4\}.$$

Let $b_1 = e_l b e_l$ and $b_1' = L_{n,l-1}(b_1)$. So $\psi_l(b_1') \neq 0$. In fact $\|\psi_l(b_1')\| > 1/4$. We have

$$L_{n,l}(b_1) = \operatorname{diag}\bigl(b_1, \Phi_n(b_1), \psi_l(b_1'), \dots, \psi_l(b_1')\bigr),$$

where $\Phi_n\colon I_n \to I_l$ is a contractive completely positive linear map such that $\Phi_n(I_n)$ is contained in C_l' and $\psi_l(b_1')$ repeats l times. Note that $\|\psi_l(b_1')\| > 1/4$. So $\operatorname{diag}\bigl(\psi_l'(b_1'), \dots, \psi_l'(b_1')\bigr)$ has an eigenvalue λ with $\lambda \geq 1/4$ and its rank (in C_l') at least l. We have

$$[b_1] \leq [e_l] \quad \text{and} \quad (1/4)[e_l] \leq \bigl[\operatorname{diag}\bigl(\psi_l(b_1'), \dots, \psi_l(b_1')\bigr)\bigr],$$

where $\psi_l(b_1')$ repeats l times. Put $c = \operatorname{diag}\Bigl(\bigl(0, \Phi_n(b_1), \psi_l(b_1'), \dots, \psi_l(b_1')\bigr)\Bigr)$ and $b' = \operatorname{diag}(b_1, 0, \dots, 0)$. Since $\{u_{ij}\}_{i,j=1}^l \subset \mathcal{F}_l$, there is $x \in \mathcal{F}_l$ such that

$$x^*x = b' \quad \text{and} \quad xx^* \in C',$$

where $C' = e_l C_l' e_l$. Moreover, c admits an eigenvalue λ such that $\lambda \geq 1/4$ with corresponding spectral projection e larger than a projection in C_l' with rank l. Therefore there exists $v \in C_l$ such that

$$v^*v = e_l \quad \text{with} \quad e_l \in C_l' \quad \text{and} \quad vv^* \leq e.$$

Note that $f_{1/8}^{1/4}(c) \geq e$. This implies that $z \in C_l$ such that

$$z^*z = xx^* \quad \text{and} \quad zz^* f_{1/8}^{1/4}(c) = zz^*.$$

Let $y = L_{l,\infty}(x)$ and $b'' = (1-p)L_{l,\infty}(b')(1-p)$. Since $L_{l,\infty}$ is $\mathcal{F}_l$–$1/(l+1)2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative and $\|L_{l,\infty}|_{C_l} - h\| < 1/2^{l-1}$, we have

$$\|y^*y - b''\| < 1/2^{l-2} \quad \text{and} \quad \|yy^* - h(xx^*)\| < 1/2^{l-2}.$$

We also estimate that

$$\|b'' - (1-p)a(1-p)\| < 1/2^{l-2} \quad \text{and} \quad \|h(c) - pap\| < 1/2^{l-2}.$$

Moreover,

$$h(z^*z) = h(xx^*) \quad \text{and} \quad h(zz^*)h\bigl(f_{1/8}^{1/4}(c)\bigr) = h(zz^*).$$

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Therefore, by Lemma 3.3,

$$\left[f_{\sigma_2}^{\sigma_1}\left((1-p)a(1-p)\right)\right] \leq \left[f_{d_2}^{d_1}\left(h(xx^*)\right)\right].$$

We also have that $\left[f_{d_2}^{d_1}\left(h(xx^*)\right)\right] = [f_{d_2}^{d_1}\left(h(zz^*)\right)]$. Therefore

$$\left[f_{d_2}^{d_1}\left(h(zz^*)\right)\right] \leq [h(zz^*)] \leq \left[f_{1/8}^{1/4}\left(h(c)\right)\right].$$

It then follows from Lemma 3.3 again that there are $0 < \sigma_4 < \sigma_3 < d_2$ such that

$$\left[f_{1/8}^{1/4}\left(h(c)\right)\right] \leq [f_{\sigma_4}^{\sigma_3}(pap)].$$

Therefore

$$\left[f_{\sigma_2}^{\sigma_1}\left((1-p)a(1-p)\right)\right] \leq [f_{\sigma_4}^{\sigma_3}(pap)].$$

Case (ii): $b \in (E_n)_+ \setminus J_n$. This part of the proof is just a slight modification of that of case (i). We note that (for $0 < i < n$ and $a \in E_n$) $\phi_i^{(n+1)} \circ L_n(a)$ has the form:

$$\operatorname{diag}\left(\pi_{n!}(a)(\xi_i), \phi_1^{(n)}(a), \ldots, \phi_{n-1}^{(n)}(a), \pi_{n!}(a)(\xi_n)\right).$$

Since $\{\xi_n\}$ is dense in S^2, without loss of generality, we may assume that $\pi_{m!}(b) \neq 0$ and $n < m < (m+1)m < l$. By the construction, we may write

$$L_{n,l}(b) = \operatorname{diag}\left(b, L'(b), \phi_m^{(m+1)}(b), \ldots, \phi_m^{(m+1)}(b), L''(b)\right),$$

where $\phi_m(b)$ repeats m many times and $L'(b), L''(b) \in C_l$. Note that

$$\begin{aligned}\operatorname{diag}&\left(0, L'(b), \phi_m^{(m+1)}(b), \ldots, \phi_m^{(m+1)}(b), L''(b)\right)\\ &\geq \operatorname{diag}\left(0, 0, \phi_m^{(m+1)}(b), \ldots, \phi_m^{(m+1)}(b), 0\right).\end{aligned}$$

Since $\{u_{ij}\} \subset \mathcal{F}_l$, there is $z_k \in \mathcal{F}_l$ such that

$$z_k^* z_k = \operatorname{diag}(b, 0, 0, \ldots, 0) \quad \text{and} \quad z_k z_k^* = \operatorname{diag}(0, \ldots, 0, b, 0),$$

where b is on the $k+1$ place. We also have (in $M_{l!/n!}(\mathbb{C} \cdot 1_{E_n})$)

$$[1_{E_n}] \leq \left[\operatorname{diag}\left(0, \phi_m^{(m+1)}(b), \ldots, \phi_m^{(m+1)}(b), 0\right)\right].$$

There is $c \in M_{(l)!/n!}(\mathbb{C} \cdot 1_{E_n})$ such that

$$c^* c = 1_{E_n} \quad \text{and} \quad cc^* \leq \operatorname{diag}\left(0, \phi_m^{(m+1)}(b), \ldots, \phi_m^{(m+1)}(b), 0\right).$$

Note also $\operatorname{diag}(0, b, 0, \ldots, 0) \leq \operatorname{diag}(0, 1_{E_n}, 0, \ldots, 0)$. Since $(L_{l,\infty})|_{M_{l!}(\mathbb{C} \cdot 1_{E_1})}$ is a homomorphism, the same argument in the proof of case (i) shows that this implies that

$$\left[f_{\sigma_2}^{\sigma_1}\left((1-p)a(1-p)\right)\right] \leq [f_{\sigma_4}^{\sigma_3}(pap)].$$

This shows that $\mathrm{TR}(A) = 0$. ∎

Corollary 3.5 $\mathrm{TR}(J) = 0$.

Proof A similar proof shows that $\mathrm{TR}(J) = 0$. ∎

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4 Tracially Quasidiagonal Extensions

Definition 4.1 Let

$$0 \to I \to E \to A \to 0$$

be a short exact sequence of C^*-algebras. Recall that (E, I) is said to be quasidiagonal if there exists an approximate identity $\{e_n\}$ for I consisting of projections such that

$$\|e_n a - a e_n\| \to 0 \quad \text{for all } a \in E.$$

In [HLX2], the extension (E, I) is said to be *tracially quasidiagonal* if, for any $\varepsilon > 0$, any nonzero $a \in E_+$, any finite subset $\mathcal{F} \subset E$ and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exists a C^*-subalgebra $D \subset E$ with $1_D = p$ such that

(1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon D$ for all $x \in \mathcal{F}$,
(3) $D \cap I = pIp$ and $(D, D \cap I)$ is quasidiagonal, and
(4) $\big[f_{\sigma_4}^{\sigma_3}\big((1-p)a(1-p)\big)\big] \le [f_{\sigma_2}^{\sigma_1}(pap)]$.

In [HLX2] we showed that if $\mathrm{TR}(I) = 0 = \mathrm{TR}(A) = 0$ then $\mathrm{TR}(E) = 0$ if and only if (E, I) is tracially quasidiagonal.

It is clear that if (E, I) is quasidiagonal, then (E, I) is tracially quasidiagonal. Theorem 4.4 says that there are tracially quasidiagonal extensions that are not quasidiagonal.

Theorem 4.2 *The extension*

$$0 \to J \to A \to B_\infty \to 0$$

is tracially quasidiagonal.

Proof One can show directly that the extension is tracially quasidiagonal, but the proof will be similar to that of Lemma 3.4. Note, however, we have $\mathrm{TR}(J) = \mathrm{TR}(B_\infty) = 0$. It follows from [HLX2] that $\mathrm{TR}(A) = 0$ if and only if the extension is tracially quasidiagonal. ■

Lemma 4.3 *Let A_n be a sequence of C^*-algebras and $A = \lim_{n\to\infty}(A_n, L_n)$ be a generalized inductive limit (in the sense of [BE]). Suppose that $\|L_n(a)\| = \|a\|$ for all $a \in A_n$, $n = 1, 2, \ldots, n$. Suppose also that $p \in A$ is a projection. Then, for any $\varepsilon > 0$, there is $n > 0$ and a projection $e \in A_n$ such that*

$$\|L_{n,\infty}(e) - p\| < \varepsilon.$$

Proof By the definition, there is a sequence $\{L_{n_k,\infty}(a_k)\}$, where $a_k \in A_{n_k}$ such that it converges to p. By replacing a_k by $(a_k + a_k^*)/2$, we may assume that a_k is self-adjoint. Since $p^2 = p$, we have that $L_{n_k,\infty}(a_k^2) \to p$. Therefore we may assume that

$$\|L_{n_k,\infty}(a_k - a_k^2)\| < 1/2^{k+1} \quad \text{and} \quad \|L_{n_k,\infty}(a_k^2) - p\| < 1/2^{k+1}.$$

Since $\|L_n(a)\| = \|a\|$ for all $a \in A_n$, $n = 1, 2, \dots$, we may assume that

$$\|a_k - a_k^2\| < 1/2^{k+1} \quad k = 1, 2, \dots.$$

Thus for large k, there is a projection $p_k \in A_k$ such that

$$\|a_k - p_k\| < 1/2^k.$$

We have

$$\|p - L_{k,\infty}(p_k)\| < \varepsilon$$

provided that k is large enough. ■

Theorem 4.4 $RR(A) \neq 0$.

Proof Suppose that $RR(A) = 0$. Then $RR(J) = 0$. It follows from [Zh] that the following holds. If $p \in A/J$ is a projection, then there is a projection $q \in A$ such that $\pi(q) = p$. Since B_∞ is a simple unital $A\mathbb{T}$-algebra, there is a projection $p \in B_\infty$ such that $[p] = (1, 1)$. If there were a projection $q \in A$ such that $\pi(q) = p$, then, by Lemma 4.3, there were an integer $n > 0$ and a projection $e \in E_n$ such that

$$\|L_{n,\infty}(e) - q\| < 1/4.$$

Let $\pi_{n!} \colon E_n \to B_n$ be the quotient map. From the commutative diagram

$$\begin{array}{ccc} E_n & \xrightarrow{L_{n,\infty}} & A \\ \big\downarrow \pi_{n!} & & \big\downarrow \pi \\ B_n & \xrightarrow{h_{n,\infty}} & B_\infty \end{array}$$

we conclude that $\pi_{n!}(e) = p_n$ is a projection in B_n and $h_{n,\infty}(p_n) = \pi \circ L_{n,\infty}(e)$. Set $r = \pi \circ L_{n,\infty}(e)$. Then $r \in B_\infty$ is a projection such that

$$\|r - p\| < 1/4.$$

Therefore $[r] = [p]$ in $K_0(B_\infty)$. In other words, $[h_{n,\infty}(p_n)] = [p] = (1, 1)$ in $K_0(B_\infty)$. From the definition, this implies that $[p_n] = (n!, 1)$ in $K_0(B_n)$ and $(\pi_{n!})_*([e]) = (n!, 1)$. However, since $\partial(n!, 1) \neq 0$ in $K_1(I_n)$, such e (in E_n) does not exist, a contradiction. So $RR(A) \neq 0$. ■

Corollary 4.5 *The extension*

$$0 \to J \to A \to B_\infty \to 0$$

is not quasidiagonal.

Proof The proof of Theorem 4.4 shows that $\partial \colon K_0(B_\infty) \to K_1(J)$ is not zero. It follows from [S] (see also [Sch]) that the extension is not quasidiagonal. This also follows from the fact: If J is σ-unital, the extension is quasidiagonal and $RR(J) = RR(B_\infty) = 0$, then $RR(A) = 0$. ■

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