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Complemented hereditary radicals

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The complemented elements of the lattice of hereditary radicals are characterized. A hypernilpotent complemented hereditary radical is the upper radical determined by a finite number of finite matrix rings. As a corollary, Stewart's characterization of radical semisimple classes is obtained. The methods are universal algebraic in nature.

In [*14*], we showed that the natural order on the class of all radicals for associative rings gives rise to a complete lattice in which the hereditary radicals form a complete sublattice, where for hereditary radicals α and β, the meet $(\alpha \wedge \beta)(R) = \alpha(R) \cap \beta(R)$ for any ring R. The semisimple class of the join $\alpha \vee \beta$ is the intersection of the semisimple class of α and the semisimple class of β. We also showed that the lattice of hereditary radicals is Brouwerian and hence distributive. In [*14*], we raised the question of characterizing the complemented elements of this lattice. In this paper, we completely characterize the complemented hereditary radicals by a detailed study of the polynomial identities of certain algebras. As an application, we quickly obtain a recent result of Stewart [*15*] characterizing radical semisimple classes.

Our approach is somewhat universal algebraic in nature. We suggest the reader unfamiliar with this approach see Grätzer [*8*]. For elementary definitions and notions concerning radicals, see [7] or [*11*].

We shall always use lower case Greek letters to denote radicals. If R is a ring, then the $n \times n$ matrix ring over R will be denoted by R_n. $|A|$ will denote the cardinality of the set A.

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For a hereditary radical α, let α^* denote its pseudocomplement where the pseudocomplement is the largest radical λ such that $\alpha \wedge \lambda = 0$. α^* must exist since our lattice is Brouwerian. If α has a complement, its complement must necessarily be α^*. Suppose now that α is complemented. Since $(\alpha \vee \alpha^*)(Z_0) = Z_0$ and $(\alpha \wedge \alpha^*)(Z_0) = 0$ where Z_0 is the integers with zero multiplication, we have either $\alpha(Z_0) = Z$ and $\alpha^*(Z_0) = 0$ or $\alpha(Z_0) = 0$ and $\alpha^*(Z_0) = Z_0$. In the remainder of the paper, we shall always assume $\alpha(Z_0) = Z_0$. This means that α is hypernilpotent. We shall characterize the hypernilpotent complemented radicals. A non-hypernilpotent complemented hereditary radical is then just the complement of a hypernilpotent one.

Let α be a hereditary radical. In [4], Andrunakievič constructed the largest radical α' among all those radicals β with $\alpha(R) \cap \beta(R) = 0$ for every ring R. Clearly if α' is hereditary, $\alpha' = \alpha^*$. If α is hypernilpotent, he showed that α' and α'' were hereditary. Therefore, if α is also complemented, we have $\alpha' = \alpha^*$ and $\alpha'' = \alpha^{**} = \alpha$. It then follows from [4] that

$$\alpha^*(R) = \cap\{I : R/I \text{ is subdirectly irreducible with } \alpha\text{-radical heart}\},$$

and

$$\alpha(R) = \alpha^{**}(R) = \cap\{I : R/I \text{ is subdirectly irreducible with } \alpha\text{-semisimple heart}\}.$$

THEOREM 1. *If S is an α-semisimple simple ring where α is a complemented hypernilpotent radical, then S is finite.*

Proof. Let C be the centroid of S. C is a field since S is simple and S can be regarded as an algebra over C. In [14], we showed that C is finite. We show S satisfies a polynomial identity over C. Suppose not. Let F be the free algebra over C with $\max\{\aleph_0, |S|\}$ generators. Let $f \neq 0$ be in F; then there exists a homomorphism $h : F \to S$ such that $h(f) \neq 0$ since S does not satisfy a polynomial identity. If $x_1, \ldots, x_n$ are the generators of F in the expression of f, we define $h' : F \to S$ by $h'(x_i) = h(x_i)$, $i = 1, \ldots, n$ and defining h' on the other generators making h' onto. This can be done

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since F has at least as many generators as S has elements. Hence $\cap\{\ker h : h : F \to S \text{ is onto}\} = 0$, that is F is a subdirect sum of copies of S . Therefore $\alpha(F) = 0$. Let H be an infinite field containing C and G a simple ring which satisfies no polynomial identities but contains H in its center (for example, a division ring over H infinite dimensional over its center). Let F' be the free ring with $\max\{\aleph_0, |S|, |G|\}$ elements. Repeating the above argument, we obtain $\alpha(F') = 0$. Also F' is a subdirect sum of copies of G . $\alpha(G) = G$ since G has infinite centroid [14]. Therefore $\alpha^*(G) = 0$ and hence $\alpha^*(F') = 0$. We then have $(\alpha \vee \alpha^*)(F') = 0$, contradicting the fact that $\alpha \vee \alpha^* = 1$. (1 is the radical for which all rings are radical.) Since S satisfies a polynomial identity, S is primitive by a theorem of Herstein in [13]. By a theorem of Kaplansky [9], S is finite dimensional over its center which must be C . Therefore S is finite since C is finite.

LEMMA 2. *$\mathrm{GF}(p^q)_n$ satisfies the identity $x^{n+r} - x^n = 0$ where r is the exponent of $\mathrm{GL}(n, p^q)$.*

Proof. Let $\mathrm{GF}(p^q)_n$ act on an n dimensional $\mathrm{GF}(p^q)$ vector space V . If A is in $\mathrm{GF}(p^q)_n$, then we have a descending chain of subspaces, $AV \supseteq A^2V \supseteq \ldots \supseteq A^nV \supseteq \ldots$ which must terminate at A^nV since V has dimension n . A then induces an automorphism ϕ on A^nV which can be extended to an automorphism $\bar{\phi}$ of V . Hence $\bar{\phi}^r = 1$. For any v in V , we have $A^n(v) = \phi^r A^n(v) = \bar{\phi}^r A^n(v) = A^{n+r}(v)$. Therefore $A^n = A^{n+r}$.

LEMMA 3. *Let V be the variety of $\mathrm{GF}(p)$ algebras generated by $\mathrm{GF}(p^q)_n$. If S is in V and A is a subalgebra of S isomorphic to $\mathrm{GF}(p)_n$, then A annihilates every nilpotent ideal of S .*

Proof. The proof is by induction on the index of nilpotence. Let N be an ideal of S and suppose $N^2 = 0$. Let e denote the identity of A . It is sufficient to show that $eN = Ne = 0$. First suppose $eNe \neq 0$. Since the action of $\mathrm{GF}(p)e$ is the same on both sides of eNe ,

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we can regard eNe as a left $A \otimes_{GF(p)} A^{o}$ module where A^{o} denotes the opposite algebra of A. Multiplication is by $(a \otimes b)(ene) = aeneb$. $A \otimes_{GF(p)} A^{o} \cong GF(p)_{n^2}$. All modules over $GF(p)_{n^2}$ are the sum of simple modules; hence eNe contains a simple submodule. All simple modules over $GF(p)_{n^2}$ are isomorphic; hence we may identify one such simple submodule with all

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \otimes_{GF(p)} \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} .$$

Identify A with $GF(p)_n$. Let I_s denote the $s \times s$ identity matrix. Let

$$y = \begin{bmatrix} 0 & 0 \\ & \\ I_{n-1} & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

which is in S. Recalling $N^2 = 0$, we compute

$$y^2 = \begin{bmatrix} 0 & 0 \\ & \\ I_{n-2} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} ,$$

$$y^{n-1} = \begin{bmatrix} 0 & 0 \\ & \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} ,$$

$$y^n = 0 + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

and $y^{n+1} = 0$.

By the previous lemma, we have

$$0 \neq \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = y^n = y^{n+r} = 0 ,$$

a contradiction.

Therefore $eNe = 0$. Suppose now that $eN \neq 0$. eN is a unital A module and hence has a simple submodule which we may identify with all $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ since all simple A modules are isomorphic. Let

$$z = \begin{bmatrix} 0 & 0 \\ & \\ I_{n-1} & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} .$$

As before using $eNe = 0$, we compute

$$z^n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} ,$$

and $z^{n+1} = 0$. We now obtain

$$0 \neq \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = z^n = z^{n+r} = 0$$

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as before. Therefore $eN = 0$. Similarly $Ne = 0$.

Suppose now that A annihilates all nilpotent ideals of index less than k. Suppose N has index k. Clearly $A \cap N = 0$. Passing to S/N^2, we have A isomorphically embedded in S/N^2. Therefore by the above, we have $(e+N^2)N/N^2 = 0$ or $eN \subseteq N^2$. N^2 has index of nilpotence less than k; therefore, our induction hypothesis applies. We obtain $eN = e^2N \subseteq eN^2 = 0$. Similarly $Ne = 0$.

LEMMA 4. *Let F be a finite field of characteristic p. If V is the variety of* GF(p) *algebras generated by F_n, then the free algebra over V with $\aleph_0$ generators can be described as follows: let $A_k = \left[x_{ij}^{(k)}\right]$ be the $n \times n$ matrix with entries commuting indeterminates with $\left(x_{ij}^{(n)}\right)^{|F|} = x_{ij}^{(k)}$ and $px_{ij}^{(k)} = 0$. The free algebra is the algebra generated by the A_k with ordinary matrix multiplication and addition.*

REMARK. Amitsur [2] states this without proof for infinite fields of characteristic not necessarily p.

Proof. Let R be the algebra generated by the A_k's. R is clearly free over V since every substitution by elements of F for the $x_{ij}^{(k)}$'s clearly induces a homomorphism of R into F_n. Therefore we need only show R is in V. To see this, consider the algebra T generated by the $x_{ij}^{(k)}$'s. T is clearly the free algebra of $V(F)$. Hence $\mathrm{Id}(F) = \mathrm{Id}(T)$. From [12, Theorem 3], $\mathrm{Id}(F_n) = \mathrm{Id}(T_n)$. Clearly $R \subseteq T_n$. Therefore R is in V since T_n is.

For a class of rings M, we denote the upper radical of M by UM.

THEOREM 5. *Let $F = \mathrm{GF}(p^q)$. If $\alpha = U\{F_n\}$, then α is a complemented hereditary radical.*

Proof. $\alpha \wedge \alpha^* = 0$. Suppose then that $\alpha \vee \alpha^* < 1$. Since the semisimple class of the join is the intersection of the semisimple classes [14], there exists a ring $R \neq 0$ such that $\alpha(R) = \alpha^*(R) = 0$. $\{F_n\}$ is a special class since every class of simple rings with unity is special [4], hence $\alpha(R) = \cap\{I : R/I \cong F_n\}$; that is, R is a subdirect sum of copies of F_n . It follows that R is in the variety generated by F_n . Also since α is hypernilpotent, we have $\alpha^* = \alpha'$ where α' is the complementary radical of Andrunakievič [4]. Therefore

$\alpha^*(R) = \cap\{I : R/I$ is subdirectly irreducible with α-radical heart$\}$.

By Lemma 2, F_n and hence R satisfies the identity $x^{n+r} - x^n = 0$ where r is the exponent of $GL(n, F)$. Therefore R is a P.I. algebra and also an algebraic algebra and hence R is locally finite [10]. There exists an epimorphism $h : R \to F_n$. Let $\{x_i\}$ be a complete set of liftings of F_n . The subalgebra $\langle x_i\rangle$ generated by $\{x_i\}$ is finite since R is locally finite. $h|\langle x_i\rangle$ is onto. Let $J(\langle x_i\rangle)$ denote the Jacobson radical of $\langle x_i\rangle$. $R/J(\langle x_i\rangle)$ is a separable algebra; hence by a generalization of the Principal Theorem of Wedderburn [1], $\langle x_i\rangle$ and hence R contains a subalgebra A isomorphic to F_n .

R is a subdirect sum of subdirectly irreducible rings $\{S_j\}$ with α-radical hearts. The image of A must be nonzero in some S_i . A is simple; hence the image is an isomorphic copy. We suppose now that $A \subseteq S_i$. Let H be the heart of S_i . Suppose $H^2 = H$. H is then a simple ring. H satisfies a polynomial identity since R does. By a theorem of Herstein in [13], H is primitive and hence by a theorem of Kaplansky [9], H is isomorphic to D_m where D is a division ring finite dimensional over its center C . C must satisfy $x^{n+r} - x^n = 0$. Therefore C is finite and $C = D$. C_m has a unit. It follows that C_m is a direct summand of S . S_i is subdirectly irreducible, hence $C_m = S_i$. Recall $A \subseteq C_m$. Since A satisfies no identities of degree less than $2n$ [3], we have $n \leq m$ since C_m satisfies the standard identity of degree $2m$ [3]. Also C_m satisfies all identities of F_n

since R does. Therefore $m \leq n$. Since $A \subseteq C_n$, $V(A) \subseteq V(C_n)$ and clearly by construction $V(C_n) \subseteq V(A) = V(F_n)$. Let $\langle A_n \rangle$ be the free algebra with $\aleph_0$ generators of $V(F_n)$ described in the previous lemma and $\langle B_k \rangle$. the corresponding free algebra for $V(C_n)$. The mapping $A_k \to B_k$ induces an isomorphism. It is clear then that $C = F$. This is impossible since $C_n = S_i$ was assumed to have α-radical heart, but $\alpha(F_n) = 0$. Therefore $H^2 = 0$.

Let e denote the identity of A. Let X be the collection of all idempotents x of eS_ie such that $ax = xa$ for all a in A. X is not empty since e is in X. Let $I = \sum\{Ax : x$ is in $X\}$. The Jacobson radical $J(S_i)$ of S_i is nilpotent since S_i is an algebraic algebra of bounded index [4]. e annihilates $J(S_i)$ by Lemma 3. It follows that $eS_ie \cap J(S_i) = 0$. Clearly $I \subseteq eS_ie$. It is then sufficient to show that I is an ideal of S_i since this will contradict the subdirect irreducibility of S_i. Let x be in X and y in S_i. Note that $Ax = xA$; hence it sufficies to show $Ay \subseteq I$. Consider the subalgebra $\langle A, y \rangle$ generated by A and y. $\langle A, y \rangle$ is finite dimensional since S_i is an algebraic algebra which satisfies a polynomial identity and hence is locally finite [10]. $\langle A, y \rangle / J(\langle A, y \rangle)$ is separable where $J(\langle A, y \rangle)$ is the Jacobson radical of $\langle A, y \rangle$. By the Principal Theorem of Wedderburn, $\langle A, y \rangle$ contains a subalgebra B which is isomorphic to $\langle A, y \rangle / J(\langle A, y \rangle)$ and $\langle A, y \rangle = B + J(\langle A, y \rangle)$. The previous sum is a vector space direct sum. By Lemma 3, A annihilates the nilpotent ideal $J(\langle A, y \rangle)$ since $A \supseteq \mathrm{GF}(p)_n$. $e = z + j$ with z in B and j in $J(\langle A, y \rangle)$. The projection of A into B is a ring homomorphism since $J(\langle A, y \rangle)$ is an ideal. Therefore $\mathrm{Im}(A)$ is isomorphic to A and hence $zj = jz = 0$ whenever $a = z + k$ for some a in A. We then have $e = e^2 = e(z+j) = ez = (z+j)z = z^2 = z$ for $e = z + j$. Hence $A \subseteq B$. By the Wedderburn-Artin Theorem $B = C_1 \oplus C_2 \oplus \ldots \oplus C_r$ where C_i are matrix rings over division rings.

The projection of A on each factor is either 0 or an embedding. As before, if $F_n = A \subseteq C_i$, we have $A = C_i$. By renumbering if necessary, we have $B = C_1 \oplus C_2 \oplus \ldots \oplus C_t \oplus C_{t+1} \oplus \ldots \oplus C_r$ where $F_n \cong C_i$ and $ec = ce$, $i \leq t$, and $eC_j = 0$, $j > t$. Then $ec_ie = c$ and $c_ia = ac_i$ for each a in A where c_i is the identity of C_i, $i \leq t$. Hence c_i is in X and $Ac_i = C_i$. Now $y = b + j$. $ay = ab + aj = ab$ is in $(C_1 \oplus C_2 \oplus \ldots \oplus C_t)b \subseteq C_1 \oplus C_2 \oplus \ldots \oplus C_t \subseteq I$. Therefore I is an ideal.

THEOREM 6. *Let* α *be a hypernilpotent radical.* α *is complemented if and only if* α *is the upper radical determined by a finite number of matrix rings over finite fields.*

Proof. First suppose $\alpha = U\left\{F_{n_i}^{(i)}\right\}_{i=1}^{r}$. Let $\alpha_i = U\left\{F_{n_i}^{(i)}\right\}$. α_i is complemented by the previous theorem. $\alpha = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_r$ [*14*]. Clearly $\beta = \alpha_1^* \vee \alpha_2^* \vee \ldots \vee \alpha_r^*$ is the complement of α.

Conversely suppose α has a complement. $\alpha = \alpha''$ since α is hypernilpotent. It follows that α is the upper radical determined by the subdirectly irreducible rings with α^*-radical hearts. Let S be such a ring with heart H. Since α is hypernilpotent, S is a semiprime ring and hence $H^2 = H$. Therefore H is a simple ring. Therefore by Theorem 1, H is finite and hence is a matrix ring over a finite field. H has an identity and hence H is a direct summand of S. Therefore $H = S$. We now have that α is the upper radical determined by matrix rings over finite fields. We must show that their number is finite.

Suppose first of all that for some prime p, the number of α-semisimple simple rings is finite. Let $\left\{F_{n_i}^{(i)}\right\}$ be these rings. We first show n is bounded. Suppose not. It follows that no identity over $\mathrm{GF}(p)$ can be satisfied by all the $F_{n_i}^{(i)}$ since all identities satisfied by $F_{n_i}^{(i)}$ have degree at least $2n_i$ [*3*]. Consider the free $\mathrm{GF}(p)$

algebra G with c (cardinality of the continuum) generators. As in the proof of Theorem 1, we see that G is a subdirect sum of the $F^{(i)}_{n_i}$. Therefore $\alpha(G) = 0$. If on the other hand S is a simple algebra satisfying no polynomial identities over $\mathrm{GF}(p)$ and $|S| \le c$ (for example, all linear transformations of finite rank on a vector space of countably infinite dimension over $\mathrm{GF}(p)$), then $\alpha(S) = S$. α is complemented; hence $\alpha^*(S) = 0$. As before we see that G is a subdirect sum of copies of S. Therefore $\alpha(G) = \alpha^*(G) = (\alpha \vee \alpha^*)(G) = 0$, a contradiction since $\alpha \vee \alpha^* = 1$. Therefore n_i is bounded. Hence for some n, there is an infinite number of fields $F^{(j)}$ of characteristic p with $n_j = n$. We distinguish two cases.

Case I. All but finitely many finite fields of characteristic p are in $\left\{F^{(j)} : n_j = n\right\}$.

Case II. Infinitely many finite fields of characteristic p are not $F^{(j)}$'s.

Case I. Let C be the direct limit of the $F^{(j)}$'s where the $F^{(j)}$'s form a direct system with the inclusion maps. C exists since all but finitely many finite fields of characteristic p are represented. C is a field and C_n is the direct limit of the $F^{(j)}_n$. Clearly $F^{(j)}_n \subseteq C_n$, hence $\mathrm{Id}(C_n) \subseteq \cap \mathrm{Id}\left[F^{(j)}_n\right]$. Also if $p(x_1, \ldots, x_n) = 0$ is an identity for each $F^{(j)}_n$ then since C_n is a direct limit of the $F^{(j)}_n$, we have $p(x_1, \ldots, x_n)$ is an identity for C_n. Hence $V(C_n) = V\left\{F^{(j)}_n\right\}$. Therefore the free $\mathrm{GF}(p)$ algebra H over this variety with $\aleph_0$ generators is a subdirect sum of the $F^{(j)}_n$'s as before. Therefore $\alpha(H) = 0$. Now $\alpha^*(C_n) = 0$ and H is a subdirect sum of copies of C_n, hence $\alpha^*(H) = 0 = \alpha(H) = (\alpha \vee \alpha^*)(H)$, a contradiction.

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Case II. Consider the polynomial ring $(\mathrm{GF}(p)[x])_n$. $(\mathrm{GF}(p)[x])_n$ is a subdirect sum of the $F_n^{(j)}$ and hence we have $\alpha(\mathrm{GF}(p)[x])_n = 0$. Also since there are infinitely many F_n with $\alpha^*(F_n) = 0$, we have $\alpha^*(\mathrm{GF}(p)[x])_n = 0$. Again we have $(\alpha \vee \alpha^*)(\mathrm{GF}(p)[x])_n = 0$, a contradiction.

We now have that for each prime p , there are only finitely many α-semisimple simple rings of characteristic p . We now show that only finitely many p are represented. Suppose not. Let $K_{n_i}^{(i)}$ be the α-semisimple simple rings. We first show n_i is bounded above. Suppose not. We show that no polynomial identity over the integers is satisfied by all the $K_{n_i}^{(i)}$. Consider $g(x_1, \ldots, x_n)$. $g(x_1, \ldots, x_n) = 0$ (mod p) for only finitely many primes p . Let s be the degree of $g(x_1, \ldots, x_n)$. We can find some prime p such that $g(x_1, \ldots, x_n) \neq 0$ (mod p) and there is a $K_{n_i}^{(i)}$ of characteristic p with $2n_i \geq s$. Since $K_{n_i}^{(i)}$ satisfies no identity of degree less than $2n_i$, $g(x_1, \ldots, x_n)$ is not identity in K_{n_i} . As before the free algebra over the integers with sufficiently many generators is a subdirect sum of the $K_{n_i}^{(i)}$ and the subdirect sum of copies of some infinite simple ring which satisfies no identities over the integers. It follows that the free algebra is $\alpha \vee \alpha^*$-semisimple. Again this is a contradiction and hence n_i is bounded. As before, we must have infinitely many n_i equal n for some positive integer n . Again we have the polynomial ring $(Z[x])_n$ (Z denotes the integers) is a subdirect sum of the $K_{n_i}^{(i)}$'s . Also for each p , we can pick $G_n^{(p)}$ with $\alpha\left(G_n^{(p)}\right) = G_n^{(p)}$ where G is a finite field of characteristic p . $(Z[x])_n$ is also a subdirect sum of the

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$G_n^{(j)}$. Therefore $\alpha(Z[x]_n) = \alpha^*(Z[x]_n) = 0$, a contradiction.

We are now able to obtain a recent theorem of Stewart [15] characterizing radical semisimple classes. A radical semisimple class M is a radical class which is simultaneously a semisimple class for some other radical.

THEOREM 7 (Stewart). *M is a radical semisimple class if and only if there exists an integer n such that*

$$M = \{R : x^n = x \text{ for every } x \text{ in } R\} .$$

Proof. It is easy to verify that if M is as above, then M is a radical semisimple class. Suppose then that we are given a radical semisimple class M . M is closed under homomorphic images and subdirect sum. Therefore M is a variety. Let β be the radical with M as its radical class and α the radical whose semisimple class is M . We show that $I^2 = I$ for every ring I of M . Suppose not. We then have I/I^2 is in M and I/I^2 is a zero ring. Every ideal of Z_0 , the integers with zero multiplication, can be mapped into I/I^2 with nonzero image. Since M is a semisimple class we have Z_0 is in M . This implies β is larger than the Baer lower radical. Armendariz has shown [6] that this implies that M is all rings, a contradiction. $I^2 = I$ implies β is subidempotent [4], hence the complementary radical β' is hereditary [4] and hence $\beta' = \beta^*$. Clearly $\beta' \geq \alpha$. It is then clear that $\beta^* \vee \beta = 1$ and β is complemented. We then have that β^* is the upper radical determined by a finite number of finite matrix rings $\left\{F_{n_i}^{(i)}\right\}_{i=1}^{n}$. Each $F_{n_i}^{(i)}$ is in M since $\beta\left(F_{n_i}^{(i)}\right) = F_{n_i}^{(i)}$ and hence all subrings are in M since M is a variety. If any $n_i > 1$, then M must contain rings with zero multiplication, but $I^2 = I$ for every ring in M , a contradiction. Therefore $n_i = 1$. Let n be the least common multiple of the $|F^{(i)}|$'s . n clearly is the n demanded in the theorem.

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