A CLASS OF OPERATORS ON THE LORENTZ SPACE M(ϕ)

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In order to deal with certain problems in the theory of interpolation spaces, it is convenient to consider operators of the following form:

Let k be a non-negative measurable function on the half-line R^+, and let f be a measurable function on R^+ with

$$\int_0^\infty k(s)\,|f(st)|\,ds < \infty \qquad \text{for almost all } t \in R^+. \tag{1}$$

Then the operator T is defined by

$$(Tf)(t) = \int_0^\infty k(s)f(st)\,ds \qquad \text{for almost all } t \in R^+, \tag{2}$$

with the domain of T, $D(T)$, consisting of all f which satisfy (1).

An important role is played by the averaging operator P, which is defined for locally integrable functions f by

$$(Pf)(t) = \int_0^1 f(st)\,ds = \int_0^t f(s)\,ds \Big/ t.$$

Note that P commutes with all operators of the form (2) in the sense that if $f \in D(TP)$, then $TPf = PTf$.

It is important to know when T is a bounded operator from a Banach function space X into itself. Some questions of this type are considered in **(1)**. When X is the Lorentz space $M(\phi)$, the situation is very simple, as Theorem 1 shows.

The definition of $M(\phi)$ is as follows: ϕ is a non-negative, non-increasing function on R^+, such that

$$\int_0^t \phi(s)\,ds < \infty \qquad \text{for all } t < \infty.$$

For any measurable function f on R^+, f^* denotes the non-increasing rearrangement of f onto the half-line. That is, if m denotes Lebesgue measure on R^+, f^* is the non-increasing, non-negative, and left-continuous function for which

$$m\{t\colon |f(t)| > y\} = m\{t\colon f^*(t) > y\} \qquad \text{for all } y > 0.$$

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We define

$$(P\phi)(t) = \int_0^1 \phi(st)\,ds = \int_0^t \phi(s)\,ds \Big/ t,$$

and

$$f^{**}(t) = (Pf^*)(t) = \int_0^1 f^*(st)\,ds.$$

Let

$$||f|| = \sup_{t>0} f^{**}(t)/(P\phi)(t)$$

and

$$M(\phi) = \{f\colon ||f|| < \infty\}.$$

Then $M(\phi)$ is a Banach space of equivalence classes of almost everywhere equal measurable functions; see **(3)**.

We shall let $[M(\phi)]$ denote the space of bounded linear operators from $M(\phi)$ into itself, and $||T||$ denote the norm of an operator $T \in [M(\phi)]$. The subset of $M(\phi)$ consisting of non-increasing, non-negative functions will be denoted by $M(\phi)^+$.

THEOREM 1. *Let the operator T be defined by* (2). *Then $D(T) \supset M(\phi)$, and $T \in [M(\phi)]$, with $||T|| = c$, if and only if* (i) *$\phi \in D(T)$ and* (ii) *$T\phi \in M(\phi)$, with $||T\phi|| = c$.*

For the proof, we need two lemmas. In each, T is as in the statement of Theorem 1.

LEMMA 1. *Suppose that f is a measurable function for which $f^* \in D(T)$, and Tf^* is locally integrable. Then $f \in D(T)$ and*

$$(Tf)^{**}(t) \leqslant (Tf^*)^{**}(t).$$

Proof. We recall that if E ranges over measurable subsets of R^+, with $m(E) = t$, then (see **(2)**)

$$tf^{**}(t) = \sup_E \int_E |f(s)|\,ds. \tag{3}$$

Let E be any measurable subset of R^+, with $m(E) = t$. Then

$$\begin{aligned} \int_E dx \int_0^\infty k(s)\,|f(sx)|\,ds &= \int_0^\infty k(s)\,ds \int_E |f(sx)|\,dx \\ &\leqslant \int_0^\infty k(s)\,ds \int_0^t f^*(sx)\,dx \quad \text{(applying (3))}, \\ &= \int_0^t dx \int_0^\infty k(s) f^*(sx)\,ds \\ &= \int_0^t (Tf^*)(x)\,dx < \infty, \end{aligned} \tag{4}$$

since Tf^* was assumed locally integrable.

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From (4) and (1), $f \in D(T)$, so that since Tf^* is non-increasing,

$$\int_E |Tf|(x)\,dx \leqslant \int_0^t (Tf^*)(x)\,dx = t(Tf^*)^{**}(t). \tag{5}$$

The desired result follows upon applying (3) to (5).

LEMMA 2. *Suppose that* $D(T) \supset M(\phi)^+$, *and that*

$$\sup_f \|Tf\| = c \qquad (f \in M(\phi)^+, \qquad \|f\| \leqslant 1). \tag{6}$$

Then $D(T) \supset M(\phi)$ *and* $\|T\| = c$.

Proof. Let $f \in M(\phi)$. Then $f^* \in M(\phi)^+ \subset D(T)$ and, by (6), $Tf^* \in M(\phi)$, so that Tf^* is locally integrable. Hence, by Lemma 1, $f \in D(T)$ and

$$(Tf)^{**}(t) \leqslant (Tf^*)^{**}(t),$$

so that $\|Tf\| \leqslant \|Tf^*\|$.

Proof of Theorem 1. By Lemma 2, we need only consider $f \in M(\phi)^+$, so that $f = f^*$.

First, assume that $\phi \in D(T)$, with $\|T\phi\| = c$. Then, since

$$(T\phi)^{**} = PT\phi = TP\phi,$$

we have

$$(TP\phi)(t) \leqslant \|T\phi\| \cdot (P\phi)(t) = c(P\phi)(t), \tag{7}$$

by definition of the norm in $M(\phi)$.

If $f \in M(\phi)^+$, $f^{**}(t) = (Pf)(t)$, so

$$\begin{aligned} (Pf)(t) &\leqslant [\sup_{s>0} (Pf)(s)/(P\phi)(s)](P\phi)(t) \\ &= \|f\| \cdot (P\phi)(t). \end{aligned} \tag{8}$$

Now, apply T to each member of (8); then, since $k \geqslant 0$,

$$\begin{aligned} (TPf)(t) &\leqslant \|f\| \cdot (TP\phi)(t) \\ &\leqslant c\,\|f\| \cdot (P\phi)(t), \qquad \text{by (7)}. \end{aligned} \tag{9}$$

Since $PTf = TPf$, (9) implies that $\|Tf\| \leqslant c\,\|f\|$, so that $\|T\| \leqslant c$. But, $\phi \in M(\phi)$, with $\|\phi\| = 1$, and $\|T\phi\| = c$, so we must in fact have $\|T\| = c$.

Conversely, if $T \in [M(\phi)]$, with $\|T\| = c$, then $\|T\phi\| = b \leqslant \|T\| \cdot \|\phi\| = c$. By the first part of the proof, $\|T\| = b$, and hence we must have $c = b$.

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