SOME INFINITE INTEGRALS INVOLVING *E*-FUNCTIONS

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1. A function $\phi(p)$ is operationally related to $h(t)$ when they satisfy the integral equation

$$\phi(p) = p\int_0^\infty e^{-pt}h(t)\,dt, \tag{1}$$

provided that the integral is convergent and $R(p) > 0$.

As usual, we shall denote (1) by the symbolic expression

$$\phi(p) \doteqdot h(t).$$

The object of this paper is to evaluate some infinite integrals involving *E*-functions by the methods of the operational calculus. Most of the results obtained are believed to be new.

2. THEOREM. *If* $\phi(p) \doteqdot h(t)$ *and*

$$\psi(p) \doteqdot (t+\alpha)^\nu (t+\beta)^\nu h(t),$$

then

$$\phi(p) = \frac{\pi^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}-\nu}}{\Gamma(\nu)}\,p\int_0^\infty t^{\nu-\frac{1}{2}} \exp\{-\tfrac{1}{2}(\alpha+\beta)t\} I_{\nu-\frac{1}{2}}\{\tfrac{1}{2}(\alpha-\beta)t\}(t+p)^{-1}\psi(t+p)\,dt, \tag{2}$$

provided that the integral is convergent, $R(p) > 0$ *and* $R(\nu) > 0$.

Proof. By hypothesis, we have

$$(t+\alpha)^\nu (t+\beta)^\nu h(t) \doteqdot \psi(p)$$

and hence

$$e^{-at}(t+\alpha)^\nu (t+\beta)^\nu\, h(t) \doteqdot p\,\frac{\psi(p+a)}{p+a}, \tag{3}$$

by virtue of a well-known property.

We also have [**2**, p. 238]

$$p\Gamma(\nu)(p+\alpha)^{-\nu}(p+\beta)^{-\nu} \doteqdot \pi^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}-\nu}t^{\nu-\frac{1}{2}} \exp\{-\tfrac{1}{2}(\alpha+\beta)t\} I_{\nu-\frac{1}{2}}\{\tfrac{1}{2}(\alpha-\beta)t\}, \tag{4}$$

where $R(\nu) > 0$.

Using (3) and (4) in the Parseval-Goldstein theorem [**4**, p. 105] of the operational calculus, which states that if

$$\phi_1(p) \doteqdot g_1(t) \quad \text{and} \quad \phi_2(p) \doteqdot g_2(t),$$

then

$$\int_0^\infty \phi_1(t)g_2(t)t^{-1}dt = \int_0^\infty \phi_2(t)g_1(t)t^{-1}dt, \tag{5}$$

we obtain

$$\int_0^\infty e^{-at}h(t)\,dt = \frac{\pi^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}-\nu}}{\Gamma(\nu)}\int_0^\infty t^{\nu-\frac{1}{2}}\exp\{-\tfrac{1}{2}(\alpha+\beta)t\}I_{\nu-\frac{1}{2}}\{\tfrac{1}{2}(\alpha-\beta)t\}(t+a)^{-1}\psi(t+a)\,dt.$$

On multiplying both sides by a and replacing a by p we arrive at the result.

Example. If we take [**2**, p. 294]

$$h(t) = t^{\lambda-1}(t+\alpha)^{-\nu}$$
$$\doteqdot \frac{p^{1-\lambda}\alpha^{-\nu}}{\Gamma(\nu)}E(\lambda, \nu :: \alpha p) = \phi(p),$$

where $R(\lambda) > 0$, $R(p) > 0$ and $|\arg\alpha| < \pi$, we therefore have

$$(t+\alpha)^{\nu}(t+\beta)^{\nu}h(t) = t^{\lambda-1}(t+\beta)^{\nu}$$
$$\doteqdot \frac{p^{1-\lambda}\beta^{\nu}}{\Gamma(-\nu)}E(\lambda, -\nu :: \beta p) = \psi(p),$$

where $R(\lambda) > 0$, $R(p) > 0$ and $|\arg\beta| < \pi$.

Applying (2) we find that

$$\int_0^\infty t^{\nu-\frac{1}{2}}(p+t)^{-\lambda}\exp\{-\tfrac{1}{2}(\alpha+\beta)t\}I_{\nu-\frac{1}{2}}\{\tfrac{1}{2}(\alpha-\beta)t\}\,E\{\lambda, -\nu :: \beta(p+t)\}dt$$
$$= \Gamma(-\nu)\pi^{-\frac{1}{2}}p^{-\lambda}(\alpha\beta)^{-\nu}(\alpha-\beta)^{\nu-\frac{1}{2}}\,E(\lambda, \nu :: \alpha p), \tag{6}$$

where $R(\nu) > 0$, $R(p) > 0$, $R(\alpha) > 0$ and $R(\beta) > 0$.

3. The following results are to be established here.

$$\int_0^\infty t^{2\lambda-1}(t+z)^{2\sigma-1}\,{}_1F_2(\nu\,;\lambda, \lambda+\tfrac{1}{2}\,; -t^2)\,E[1-\sigma, \tfrac{1}{2}-\sigma, \alpha, \beta : \alpha+\beta+\nu : (t+z)^2]\,dt$$
$$= \frac{\Gamma(2\lambda)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}2^{-2\lambda}z^{2\sigma+2\lambda-2\nu-1}E(1+\nu-\sigma-\lambda, \tfrac{1}{2}+\nu-\sigma-\lambda, \alpha+\nu, \beta+\nu : \alpha+\beta+\nu : z^2), \tag{7}$$

where $R(\lambda) > 0$, $R(2\sigma+\nu-2) < 0$, $R(\lambda+\sigma-\nu-\frac{1}{2}) < 0$ and $|\arg z| < \pi$.

$$\int_0^\infty t^{2\lambda-1}(t+z)^{2\sigma-1}\,{}_1F_2(\beta\,;\lambda, \lambda+\tfrac{1}{2}\,; -t^2)\,E[1-\sigma, \tfrac{1}{2}-\sigma, \alpha :: (t+z)^2]\,dt$$
$$= \frac{\Gamma(2\lambda)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}2^{-2\lambda}z^{2\sigma+2\lambda-2\beta-1}E(1+\beta-\lambda-\sigma, \tfrac{1}{2}+\beta-\lambda-\sigma, \alpha+\beta :: z^2), \tag{8}$$

where $R(\lambda) > 0$, $R(2\sigma+\beta-2) < 0$, $R(\lambda+\sigma-\beta-\frac{1}{2}) < 0$ and $|\arg z| < \pi$.

In the proof we shall require the following results [**5**, p. 70], [**1**, p. 105]:

$$\int_0^\infty e^{-pt}t^{-2\sigma}{}_2F_1(\alpha, \beta ; \gamma ; -t^2)\, dt = \frac{\Gamma(\gamma)2^{-2\sigma}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{1}{2})}\, p^{2\sigma-1}E(1-\sigma, \tfrac{1}{2}-\sigma, \alpha, \beta : \gamma : \tfrac{1}{4}p^2), \tag{9}$$

where $R(\sigma) < \frac{1}{2}$ and $R(p) > 0$;

$${}_2F_1(a, b ; c ; x) = (1-x)^{c-a-b}{}_2F_1(c-a, c-b ; c ; x). \tag{10}$$

(*a*) Starting with (9), we have

$$\begin{aligned}\phi_1(p) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{1}{2})}2^{-2\sigma}p(p+z)^{2\sigma-1}\, E\{\tfrac{1}{2}-\sigma, 1-\sigma, \alpha, \beta : \gamma : \tfrac{1}{4}(p+z)^2\}\\ &\doteqdot e^{-zt}t^{-2\sigma}{}_2F_1(\alpha, \beta ; \gamma ; -t^2)\\ &= g_1(t),\end{aligned}$$

where $R(\sigma) < \frac{1}{2}$ and $R(p) > 0$; also [**2**, p. 238]

$$\begin{aligned}g_2(t) &= \frac{t^{2\lambda+2\gamma-2\alpha-2\beta-1}}{\Gamma(2\lambda+2\gamma-2\alpha-2\beta)}{}_1F_2(\gamma-\alpha-\beta ; \lambda+\gamma-\alpha-\beta, \lambda+\gamma-\alpha-\beta+\tfrac{1}{2} ; -\tfrac{1}{4}t^2)\\ &\doteqdot p^{1-2\lambda}(1+p^2)^{\alpha+\beta-\gamma}\\ &= \phi_2(p),\end{aligned}$$

where $R(\lambda+\gamma-\alpha-\beta) > 0$ and $R(p) > 0$.

Applying (5), using (9) and (10), replacing γ by $\alpha+\beta+\nu$, λ by $\lambda-\nu$, z by $2z$ and t by $2t$, we obtain (7).

(*b*) Now take (9) with $\beta = \gamma$. We have

$$\begin{aligned}\phi_1(p) &= \frac{2^{-2\sigma}}{\Gamma(\alpha)\Gamma(\frac{1}{2})}\, p(p+z)^{2\sigma-1}\, E\{1-\sigma, \tfrac{1}{2}-\sigma, \alpha :: \tfrac{1}{4}(p+z)^2\}\\ &\doteqdot e^{-zt}t^{-2\sigma}(1+t^2)^{-\alpha}\\ &= g_1(t),\end{aligned}$$

where $R(\sigma) < \frac{1}{2}$, $R(z) > 0$; also [**2**, p. 238]

$$\begin{aligned}g_2(t) &= \frac{t^{2\lambda+2\beta-1}}{\Gamma(2\lambda+2\beta)}\, {}_1F_2(\beta ; \lambda+\beta, \lambda+\beta+\tfrac{1}{2} ; -\tfrac{1}{4}t^2)\\ &\doteqdot p^{1-2\lambda}(1+p^2)^{-\beta}\\ &= \phi_2(p),\end{aligned}$$

where $R(\lambda+\beta) > 0$ and $R(p) > 0$.

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Again apply (5), use the formula (9) and replace λ by $\lambda-\beta$; this gives (8). Some interesting particular cases of the results (7) and (8) are given below.

(i) When $\lambda = \nu$, then, by virtue of the relation

$$ {}_0F_1(\nu+1\,;\,-x^2) = \Gamma(\nu+1)x^{-\nu}J_\nu(2x), $$

(7) yields

$$ \int_0^\infty t^{\nu-\frac{1}{2}}(t+z)^{2\sigma-1}\, J_{\nu-\frac{1}{2}}(2t)\, E[1-\sigma, \tfrac{1}{2}-\sigma, \alpha, \beta : \alpha+\beta+\nu : (t+z)^2]\, dt $$
$$ = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\nu)}{2\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}\, \frac{z^{2\sigma-1}}{\pi^{\frac{1}{2}}}\, E(1-\sigma, \tfrac{1}{2}-\sigma, \alpha+\nu, \beta+\nu : \alpha+\beta+\nu : z^2), \tag{11} $$

where $R(\nu) > 0$, $R(2\sigma+\nu-2) < 0$ and $|\arg z| < \pi$.

For $\nu = 1$, (11) reduces to

$$ \int_0^\infty \sin 2t(t+z)^{2\sigma-1}\, E[1-\sigma, \tfrac{1}{2}-\sigma, \alpha, \beta : \alpha+\beta+1 : (t+z)^2]\, dt $$
$$ = \frac{z^{2\sigma-1}}{2\alpha\beta}\, E(1-\sigma, \tfrac{1}{2}-\sigma, \alpha+1, \beta+1 : \alpha+\beta+1 : z^2), \tag{12} $$

where $R(\sigma) < \frac{1}{2}$ and $|\arg z| < \pi$.

(ii) On the other hand, if we take $\sigma = k$, $\alpha = \frac{1}{2}-k+m$, $\beta = \frac{1}{2}-k-m$ and $\nu = 0$, then by virtue of the property of the E-function [3, p. 434]

$$ E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m, \tfrac{1}{2}-k, 1-k : 1-2k : x^2) $$
$$ = \Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-k+m)\Gamma(\tfrac{1}{2}-k-m)x^{-2k}W_{k,m}(2ix)W_{k,m}(-2ix), \tag{13} $$

(7) gives

$$ \int_0^\infty t^{2\lambda-1}(t+z)^{-1}W_{k,m}\{2i(t+z)\}W_{k,m}\{-2i(t+z)\}\, dt $$
$$ = \Gamma(2\lambda)2^{-2\lambda}z^{2\lambda+2k-1}E\,(1-k-\lambda, \tfrac{1}{2}-k-\lambda, \tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m : 1-2k : z^2), \tag{14} $$

where $R(\lambda) > 0$, $R(\frac{1}{2}-k-\lambda) > 0$ and $|\arg z| < \pi$.

(iii) On taking $\lambda = \beta$ in (8), we obtain

$$ \int_0^\infty t^{\beta-\frac{1}{2}}(t+z)^{2\sigma-1}J_{\beta-\frac{1}{2}}(2t)\, E\{1-\sigma, \tfrac{1}{2}-\sigma, \alpha :: (t+z)^2\}\, dt $$
$$ = \frac{B(\alpha, \beta)}{2\pi^{\frac{1}{2}}}\, z^{2\sigma-1}\, E(1-\sigma, \tfrac{1}{2}-\sigma, \alpha+\beta :: z^2), \tag{15} $$

where $R(\beta) > 0$, $R(\beta+2\sigma-2) < 0$ and $|\arg z| < \pi$.

For $\beta = 1$, (15) gives

$$\int_0^\infty \sin 2t(t+z)^{2\sigma-1}\, E\{1-\sigma, \tfrac{1}{2}-\sigma, \alpha :: (t+z)^2\}\, dt = \frac{z^{2\sigma-1}}{2\alpha} E\{1-\sigma, \tfrac{1}{2}-\sigma, \alpha+1 :: z^2\}, \qquad (16)$$

where $R(\sigma) < \frac{1}{2}$ and $|\arg z| < \pi$.

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