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Multiplicity of Resonances in Black Box Scattering

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Abstract. We apply the method of complex scaling to give a natural proof of a formula relating the multiplicity of a resonance to the multiplicity of a pole of the scattering matrix.

1 Introduction

The purpose of this paper is to show the equivalence of two definitions of the multiplicity of the scattering resonances for an operator in the black box formalism. Following Sjöstrand and Zworski [14], this formalism makes it possible to treat at the same time the three traditional perturbations: obstacle, metric, and potential, of the Laplacian $-\Delta$ in $\mathbb{R}^n$. Resonances may be defined either as poles of the meromorphic extension of the resolvent or as poles of the scattering matrix. The equivalence of these two definitions for potential perturbations (as well as some other cases) is well known, *cf.* Jensen [8] and references therein. For the Laplacian on asymptotically hyperbolic manifolds, this equivalence is proved in a recent paper by Borthwick and Perry [3] using a perturbation theorem of Agmon [1] to reduce to the case where all resonances have multiplicity one.

The scattering matrix and resolvent are meromorphic families of operators, and there are (different) natural ways to define multiplicity of poles in the two cases, which we denote $m_S(z)$ and $m_R(z)$, respectively, see [6] and formulas (4) and (3) below. We shall prove

Theorem 1 *For any $z \neq 0$ the multiplicity of the pole of resolvent $m_R(z)$ and of the pole of the scattering matrix $m_S(z)$ are related by*

$$m_S(z) = m_R(z) - m_R(\sigma(z)), \tag{1}$$

where σ is the map on the logarithmic Riemann surface Λ which sends $(\rho, \omega) \to (\rho, -\omega - 2\pi)$ in polar coordinates.

Note that if $m_R(z) \neq 0$ then $m_R(\sigma(z)) = 0$ unless $z \in \mathbb{R}^+$, in which case z is an embedded eigenvalue which does not contribute to the scattering matrix, or else $z \in \mathbb{R}^-$ is a negative eigenvalue of our operator.

The equality of the two multiplicities was shown for compactly supported black box perturbations by Petkov and Zworski [10], using the generic simplicity of resonances from [9]. We give here a natural proof based on complex scaling. Formula (1)

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appeared in [15] in the context of hyperbolic surfaces. As was pointed out to us by the authors the proof there contained an error. A correct proof based on [1] was given in [3].

2 Preliminaries

In this section we review from [13] and [14] the hypotheses of black box scattering and the two definitions of multiplicities.

Denote the open ball in $\mathbb{R}^n$ by $B(x, r_0) = \{y \in \mathbb{R}^n, |x - y| < r_0\}$, $r_0 > 0$. Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{r_0} \oplus L^2(\mathbb{R}^n \setminus B(0, r_0)).$$

We assume that $P\colon \mathcal{D} \to \mathcal{H}$ is a self-adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$ satisfying:

$$1_{\mathbb{R}^n \setminus B(0,r_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, r_0)),$$
$$1_{\mathbb{R}^n \setminus B(0,r_0)} P = -\Delta|_{\mathbb{R}^n \setminus B(0,r_0)}.$$

We assume also that for some $m_0 \in \mathbb{N}$,

$$(2) \qquad 1_{B(0,r_0)}(P + i)^{-m_0} \quad \text{is trace class,}$$

and finally, that $P \geq -C > -\infty$.

Let $r > r_0$ and fix $\varepsilon > 0$, $a > 0$ and $b > r + a$. Then there exists a function f_θ for $\theta > 0$ on $\mathbb{R}^+$ such that

$$f_\theta(t) = \begin{cases} t & \text{if } t \leq r + a, \\ e^{i\theta} t & \text{if } t \geq b, \end{cases}$$

and in addition,

$$\partial_t f_\theta(t) \neq 0,$$
$$0 \leq \operatorname{Arg}(f_\theta(t)) \leq \theta,$$
$$\operatorname{Arg}(f_\theta(t)) \leq \operatorname{Arg}(\partial_t f_\theta(t)) \leq \operatorname{Arg}(f_\theta(t)) + \varepsilon$$

for every $t \in \mathbb{R}^+$. Now, for any $\theta \in \mathbb{R}$, set

$$v_\theta(x) = \begin{cases} f_\theta(|x|)x/|x| & \text{for } \theta > 0 \\ \overline{f}_{-\theta}(|x|)x/|x| & \text{for } \theta < 0; \end{cases}$$

then define $\Gamma_\theta = v_\theta(\mathbb{R}^n) \in \mathbb{C}^n$.

Choose cutoff functions $\chi, \tilde{\chi} \in \mathcal{C}_0^\infty(B(0, r))$ such that $\chi = 1$ on $\operatorname{supp} \tilde{\chi}$ and $\tilde{\chi} = 1$ on $B(0, r_0)$. Writing $P^0 = -\Delta$, we denote by P_θ, resp. P_θ^0, the restriction of the operators P, resp. P^0, on Γ_θ (see [13, 14], for the precise definition). It is known that the resolvents

$$R(z) = (P - z)^{-1} : \mathcal{H} \to \mathcal{D}$$
$$R^0(z) = (P^0 - z)^{-1} : \mathcal{H} \to \mathcal{D}$$

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have meromorphic extensions from the upper half plane $\{\mathrm{Im}(z) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when n is odd, and to the logarithmic plane Λ when n is even. These continuations are operators from $\mathcal{H}_{\mathrm{comp}}$ to $\mathcal{D}_{\mathrm{loc}}$ (see [10]) and all poles have finite rank. The poles of R are called the resonances of P. Write

$$R_\theta = (P_\theta - z)^{-1}, \quad R_\theta^0 = (P_\theta^0 - z)^{-1}.$$

We denote the set of resonances of P by $\mathrm{Res}(P)$; these are listed according to their multiplicities $m_R(z)$, which as in [14] are defined for $z \neq 0$ by

$$m_R(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_\theta(u)\, du\Big) = \mathrm{Tr}\Big(-\frac{1}{2i\pi} \int_{\gamma_\varepsilon(z)} R_\theta(u)\, du\Big)\,; \tag{3}$$

here $\gamma_\varepsilon(z) = \{z + \varepsilon e^{it}, 0 \leq t \leq 2\pi\}$ for ε sufficiently small. We also let

$$m_{R_\theta}(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_\theta(u)\, du\Big)\,.$$

The elements of $\mathrm{Res}(P)$ belonging to $\mathbb{R}$ are eigenvalues of P.

The scattering matrix is defined just as for potential scattering, see [4, 10], and we denote by $S(\lambda)$ the (relative) scattering matrix. This operator also continues meromorphically in λ from $\{\mathrm{Im}(\lambda) > 0\} \cap \{\mathrm{Re}(\lambda) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when n is odd, and to the logarithmic plane Λ when n is even. The poles of its continuation correspond to resonances of P. Following [6] they are of finite rank; we define the multiplicity of a pole of $\det S(z)$ as

$$m_S(z) = -\frac{1}{2i\pi}\, \mathrm{Tr} \int_{\gamma_\varepsilon(z)} \big(S(u)^{-1} \frac{d}{du} S(u)\big)\, du. \tag{4}$$

We recall the following

Proposition 1 *The function*

$$z \longrightarrow (z - z_j) R_\theta(z)$$

is analytic near any $z_j \in \mathbb{R} \cap \mathrm{Res}(P)$.

Proof By the self-adjointness of P, we have

$$\|(P - z)^{-1}\|_{L^2} \leq \frac{1}{\mathrm{Im}(z)},$$

so if $z \in z_j + i\mathbb{R}^+$, then

$$\|(z - z_j)\chi R_0(z)\tilde{\chi}\|_{L^2} \leq 1,$$

which implies that $(z - z_j)\chi R_0(z)\tilde{\chi}$ is analytic near z_j. However,

$$\chi R_\theta(z)\tilde{\chi} = \chi R_0(z)\tilde{\chi},$$

and so $(z - z_j)\chi R_\theta(z)\tilde{\chi}$ is also analytic near z_j. Since the poles of $(z - z_j)R_\theta(z)$ are the same as those of $(z - z_j)\chi R_\theta(z)\tilde{\chi}$, this proves the claim. ∎

3 Analysis of Various Traces

We now review some results concerning traces of functions of P, P_θ and P^0.

Recall the Birman-Krein formula [4, 2, 11]. Fix $f \in \mathcal{C}_0^\infty(\mathbb{R})$ and assume that $\mathrm{Im}\lambda_0 > 0$, $\mathrm{Re}\lambda_0 > 0$. Then with $g(\lambda) = (\lambda - \lambda_0)^{m_0} f(\lambda)$ (where m_0 is defined in (2)),

(5)
$$\begin{aligned}
&\mathrm{Tr}[f(P) - (1-\chi) f(P^0)(1-\tilde{\chi})] - \mathrm{Tr}[\chi f(P^0)] \\
&\qquad = \frac{1}{2i\pi}\int_{\mathbb{R}} \Big(\frac{df}{d\lambda}\log(\det S(\lambda))\Big)\, d\lambda + \sum_{\lambda \in \mathrm{Res}(P)\cap\mathbb{R}} g(\lambda)(\lambda-\lambda_0)^{-m_0} \\
&\qquad = -\frac{1}{2i\pi}\int_{\mathbb{R}} g(\lambda)(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}\big(S'(\lambda)S^{-1}(\lambda)\big)\, d\lambda \\
&\qquad\quad + \sum_{\lambda \in \mathrm{Res}(P)\cap\mathbb{R}} g(\lambda)(\lambda-\lambda_0)^{-m_0}.
\end{aligned}$$

Let $\tilde{g}$ be an almost analytic extension of g with compact support. (This means that $\tilde{g}|_{\mathbb{R}} = g$ and $\overline{\partial}\tilde{g}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$, cf. [7].) Then as in [5, §8]

$$\begin{aligned}
f(P^0) &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\mathrm{Im}(z)|>\varepsilon} \overline{\partial}\tilde{g}(z)(P^0-\lambda_0)^{-m_0}R^0(z)\, dxdy \\
f(P) &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\mathrm{Im}(z)|>\varepsilon} \overline{\partial}\tilde{g}(z)(P-\lambda_0)^{-m_0}R(z)\, dxdy.
\end{aligned}$$

Define the families of operators

(6)
$$\begin{aligned}
G(z) &= (P-\lambda_0)^{-m_0}R(z) \;\; - (1-\chi)(P^0-\lambda_0)^{-m_0}R^0(z)(1-\tilde{\chi}), \\
G_\theta(z) &= (P_\theta-\lambda_0)^{-m_0}R_\theta(z) - (1-\chi)(P_\theta^0-\lambda_0)^{-m_0}R_\theta^0(z)(1-\tilde{\chi}),
\end{aligned}$$

and also let

$$F_\varepsilon = \frac{1}{\pi}\int_{|\mathrm{Im}(z)|>\varepsilon} \overline{\partial}\tilde{g}(z)G(z)\, dxdy.$$

The estimates

$$\|G(z)\|_{\mathcal{S}_1} \le \frac{C\langle z\rangle}{|\mathrm{Im}(z)|^2}$$

(where $\|\cdot\|_{\mathcal{S}_1}$ is the trace class norm), from [5, §9] imply

$$\|F_\varepsilon - F_{\varepsilon'}\|_{\mathcal{S}_1} \le C\sup(\varepsilon, \varepsilon').$$

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Using the completeness of the space of trace class operators, we see that

$$\begin{aligned}\operatorname{Tr}[f(P)-&(1-\chi)f(P^0)(1-\tilde{\chi})]\\&=\frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Tr}\Big[\int_{|\operatorname{Im}(z)|>\varepsilon}\overline{\partial}\tilde{g}(z)G(z)\,dxdy\Big],\\&=-\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\operatorname{Tr}\Big[\int_{\mathbb{R}}\tilde{g}(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde{g}(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\,d\lambda\Big]\\&=-\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{\mathbb{R}}\operatorname{Tr}\big[\tilde{g}(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde{g}(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\big]\,d\lambda.\end{aligned}$$

From Sjöstrand [12, Prop. 5.3], for $\operatorname{Im} z>0$ the function $\theta\to\operatorname{Tr}\big(G_\theta(z)\big)$ is independent of $\theta\geq 0$, which implies that

$$\operatorname{Tr}[f(P)-(1-\chi)f(P^0)(1-\tilde{\chi})]=-\sum_{\pm}\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\operatorname{Im}(z)=\pm\varepsilon}\operatorname{Tr}\big(\tilde{g}(z)G_\theta(z)\big)\,dz.$$

Note that $z\to\operatorname{Tr}\big(\tilde{g}(z)G_\theta(z)\big)$ is smooth in the domain where $R_\theta(z)$ is defined, and is locally independent of θ.

Define

$$d_x^+(\varepsilon)=\partial B(x,\varepsilon)\cap\{\operatorname{Im}(z)\geq 0\},\quad d_x^-(\varepsilon)=\partial B(x,\varepsilon)\cap\{\operatorname{Im}(z)\leq 0\}.$$

Also write

$$Q_\varepsilon=\{x\in\mathbb{R}:|x-z_j|<\varepsilon\ \text{ for some } z_j\in\operatorname{Res}(P)\cap\mathbb{R}\},\quad T_\varepsilon=\mathbb{R}\setminus Q_\varepsilon.$$

For ε sufficiently small,

$$\begin{aligned}-\sum_{\pm}\frac{\pm 1}{2i\pi}\int_{\operatorname{Im}(z)=\pm\varepsilon}\operatorname{Tr}\big(\tilde{g}(z)G_{\pm\theta}(z)\big)\,dz&=-\sum_{\pm}\frac{\pm 1}{2i\pi}\int_{T_\varepsilon}\tilde{g}(z)\operatorname{Tr}\big(G_{\pm\theta}(z)\big)\,dz\\&+\sum_{\pm}\frac{\pm 1}{\pi}\int_{\{0\leq\pm\operatorname{Im}(z)\leq\varepsilon\}\setminus Q_\varepsilon}\overline{\partial}\tilde{g}(z)\operatorname{Tr}\big(G_{\pm\theta}(z)\big)\,dxdy\\&+\sum_{\pm}\sum_{z_j\in\operatorname{Res}(P)\cap\mathbb{R}}\frac{1}{2i\pi}\int_{d_{z_j}^{\pm}(\varepsilon)}\tilde{g}(z)\operatorname{Tr}\big(G_{\pm\theta}(z)\big)\,dz.\end{aligned}$$

Recalling that $\operatorname{supp}(\tilde{g})$ is compact, we have

$$\begin{aligned}\Big|\sum_{\pm}\frac{\pm 1}{\pi}&\int_{\{0\leq\pm\operatorname{Im}(z)\leq\varepsilon\}\setminus Q_\varepsilon}\overline{\partial}\tilde{g}(z)\operatorname{Tr}\big(G_{\pm\theta}(z)\big)\,dxdy\Big|\\&\leq 2\frac{1}{\pi}\int_{\{0\leq|\operatorname{Im}(z)|\leq\varepsilon\}\setminus Q_\varepsilon}|\overline{\partial}\tilde{g}(z)|\,\big|\operatorname{Tr}\big(G(z)\big)\big|\,dxdy\leq C|\varepsilon|.\end{aligned}$$

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Also,

$$\sum_{z_j\in \mathrm{Res}(P)\cap\mathbb{R}} \sum_{\pm} \frac{1}{2i\pi}\int_{d^{\pm}_{z_j}(\varepsilon)} \tilde g(z)\mathrm{Tr}\big(G_{\pm\theta}(z)\big)\, dz = \sum_{z_j\in \mathrm{Res}(P)\cap\mathbb{R}} \frac{1}{2i\pi}\int_{\partial B(z_j,\varepsilon)} \tilde g(z)\mathrm{Tr}\big(G_{\theta}(z)\big)\, dz.$$

By Proposition 1, $z \to (z-z_j)G_\theta(z)$ is analytic near $z_j \in \mathbb{R}\cap \mathrm{Res}(P_0)$; since we have control of the trace norm, $z\to (z-z_j)\mathrm{Tr}\big(G_\theta(z)\big)$ is also analytic. Hence there exist $H_j \in \mathbb{C}$ such that

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big(\sum_{z_j\in \mathrm{Res}(P_0)} \sum_{\pm} \frac{1}{2i\pi}\int_{d^{\pm}_{z_j}(\varepsilon)} \tilde g(z)\mathrm{Tr}\big(G_{\pm\theta}(z)\big)\, dz\Big) &= \lim_{\varepsilon\to 0}\Big(\sum_{z_j\in \mathrm{Res}(P_0)} \sum_{\pm} \frac{1}{2i\pi} g(z_j)\int_{d^{\pm}_{z_j}(\varepsilon)} \mathrm{Tr}\big(G_{\pm\theta}(z)\big)\, dz\Big)\\ &=: \sum_{z_j\in \mathrm{Res}(P_0)} g(z_j)H_j(z_j-\lambda_0)^{-m_0}.\end{aligned}$$

We now take the limit as $\varepsilon\to 0$. When n is odd, we obtain directly

$$\begin{aligned}\mathrm{Tr}\big(f(P)-(1-\chi)f(P^0)(1-\tilde\chi)\big) - \mathrm{Tr}[\chi f(P^0)] = &-\sum_{\pm}\frac{\pm 1}{2i\pi}\int_{\mathbb{R}} g(z)\mathrm{Tr}\big(G_{\pm\theta}(z)\big)\, dz + \sum_{z_j\in \mathrm{Res}(P)\cap\mathbb{R}} g(z_j)H_j(z_j-\lambda_0)^{-m_0}\\ &+\sum_{\pm}\frac{\pm 1}{2i\pi}\int_{\mathbb{R}} g(z)\mathrm{Tr}\big[\chi(P^0_{\pm\theta}-\lambda_0)^{-m_0}R^0_{\pm\theta}(z)\big]\, dz.\end{aligned}$$

When n is even, however, the limits of the terms $G_{-\theta}(z)$, $R^0_{-\theta}(z)$ must be written differently since z lies on the logarithmic Riemann surface Λ, and hence the integral of these terms is over a different preimage of $\mathbb{R}^+$ on Λ. In terms of polar coordinates (ρ,ω) on Λ, we can write the limits of these terms as $\varepsilon\to 0$ as $G_{-\theta}(\varphi(z))$ and $R^0_{-\theta}(\varphi(z))$, respectively, where $\varphi(\rho,\omega)=(\rho,\omega+2\pi)$. For simplicity, we assume that this notation is implied, but do not write it out explicitly in the next few formulas. In any case, together with the Birman-Krein formula, as applied in (5), this implies the equality of measures

$$\begin{aligned}-\frac{1}{2i\pi}(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}\big(S'S^{-1}(\lambda)\big)\, d\lambda + \sum_{\lambda\in \mathrm{Res}(P)\cap\mathbb{R}} \delta_\lambda(\lambda-\lambda_0)^{-m_0} = &-\sum_{\pm}\frac{\pm 1}{2i\pi}\mathrm{Tr}\big(G_{\pm\theta}(\lambda)\big)\, d\lambda + \sum_{\pm}\frac{\pm 1}{2i\pi}\mathrm{Tr}\big(\chi(P^0_{\pm\theta}-\lambda_0)^{-m_0}R^0_{\pm\theta}(\lambda)\big)\, d\lambda\\ &+\sum_{\lambda\in \mathrm{Res}(P)\cap\mathbb{R}} \delta_\lambda H_\lambda(\lambda-\lambda_0)^{-m_0}.\end{aligned}$$

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Taking the continuous part of these measures now gives

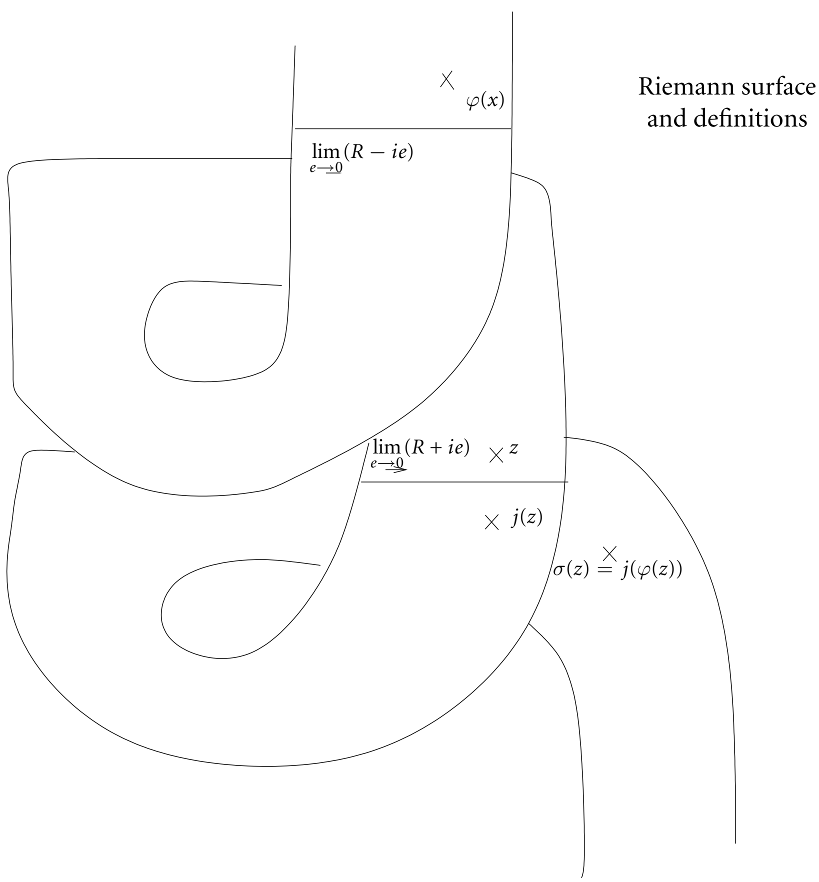
$$(7)\quad (\lambda-\lambda_0)^{-m_0}\operatorname{Tr}\big(S'S^{-1}(\lambda)\big)\,d\lambda = \operatorname{Tr}\big(G_\theta(\lambda)\big)\,d\lambda - \operatorname{Tr}\big(G_{-\theta}(\lambda)\big)\,d\lambda \\ - \sum \pm \pm \operatorname{Tr}\big[\chi(P^0_{\pm\theta}-\lambda_0)^{-m_0}R^0_{\pm\theta}(\lambda)\big]\,d\lambda.$$

From the analyticity of these functions we conclude

Proposition 2 *If S is the scattering matrix and G is as defined in* (6)*, then for all z with* $\operatorname{Re} z > 0$,

$$\operatorname{Tr}(S'S^{-1}(z)) = (z-\lambda_0)^{m_0}\operatorname{Tr}(G_\theta(z)) - (z-\lambda_0)^{m_0}\operatorname{Tr}(G_{-\theta}(z)) \\ + \frac{1}{2i\pi}\operatorname{Tr}\big[\chi(P^0_\theta-\lambda_0)^{-m_0}R^0_\theta(z)\big] - \frac{1}{2i\pi}\operatorname{Tr}\big(\chi(P^0_{-\theta}-\lambda_0)^{-m_0}R^0_{-\theta}(z)\big).$$

When $z \in \mathbb{R}^+$ and n is even, we must replace $G_{-\theta}(z)$ and $R^0_{-\theta}(z)$ by $G_{-\theta}(\varphi(z))$ and $R^0_{-\theta}(\varphi(z))$, respectively.



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4 Proof of the Main Theorem

We now compare the two expressions for the multiplicity of resonances using Proposition 2. First we note that $m_{R^0}(z) = 0$. But this multiplicity is the same as

$$
\begin{aligned}
(8) \qquad m_{R^0_\theta}(z) &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R^0_\theta(u)\,du \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P^0_\theta-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(P^0_\theta-\lambda_0)^{m_0}R^0_\theta(u)\,du\Big) \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P^0_\theta-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}R^0_\theta(u)\,du\Big) \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P^0_\theta-\lambda_0)^{-m_0}R^0_\theta(u)\,du.
\end{aligned}
$$

Writing this formula as $\mathrm{Tr}B = 0$ (*i.e.*, with B denoting the operator given by the integral),

$$\mathrm{Tr}(1-\chi)B(1-\tilde\chi) = \mathrm{Tr}(1-\chi)B - \mathrm{Tr}(1-\chi)B\tilde\chi = -\mathrm{Tr}\chi B$$

since $\mathrm{Tr}(1-\chi)B\tilde\chi = \mathrm{Tr}(1-\chi)\tilde\chi B = 0$. We use this to express

$$
\begin{aligned}
(9) \qquad m_R(z) = m_{R_\theta}(z) &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_\theta(u)\,du \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P_\theta-\lambda_0)^{-m_0}R_\theta(u)\,du \\
&= -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(G_\theta(u)\big)\,du \\
&\quad + \frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(\chi(P^0_\theta-\lambda_0)^{-m_0}R^0_\theta(u)\big)\,du.
\end{aligned}
$$

Integrating (7) around $\gamma_\varepsilon(z)$ and employing (9) gives

$$
\begin{aligned}
(10) \qquad m_S(z) &= -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}\mathrm{Tr}S'S^{-1}(u)\,du \\
&= -\sum_{\pm}\frac{\pm1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}(G_{\pm\theta})(u)\,du \\
&\quad + \sum_{\pm}\frac{\pm1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\times\mathrm{Tr}\big(\chi(P^0_{\pm\theta}-\lambda_0)^{-m_0}R^0_{\pm\theta}(u)\big)\,du \\
&= m_{R_\theta}(z) - m_{R_{-\theta}}(\varphi(z)).
\end{aligned}
$$

In order to use this last formula we must relate the two multiplicities $m_{R_{\pm\theta}}(z)$. Thus let $\mathcal{C}$ be the operator $u \to \overline{u}$. For any trace class operator A we have $\mathrm{Tr}(\mathcal{C}A\mathcal{C}) =$

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$\overline{\mathrm{Tr} A}$. Certainly $\mathcal{C}P\mathcal{C} = P$, and hence $\mathcal{C}(R(z))\mathcal{C} = R(\bar{z})$ for $z \in \mathbb{C}$. This relationship for z in the riemann surface is

$$\mathcal{C}R_{-\theta}(z)\mathcal{C} = R_{\theta}(j(z))$$

where (again in terms of polar coordinates on Λ), $j(\rho,\omega) = (\rho, -\omega)$. Therefore

$$\begin{aligned} m_{R_{-\theta}}(z) &= \frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_{-\theta}(u)\,du \\ &= \overline{\Big(-\frac{1}{2i\pi}\mathrm{Tr}\big(\mathcal{C}(\int_{\gamma_\varepsilon(z)} R_{-\theta}(u)\,du)\mathcal{C}\big)\Big)} \\ &= \overline{\Big(\frac{1}{2i\pi}\mathrm{Tr}\big(\int_{\gamma_\varepsilon(j(z))} R_{\theta}(u)\,du\big)\Big)} \\ &= \overline{m_{R_\theta}(j(z))} = m_{R_\theta}(j(z)). \end{aligned}$$

Equation (10) now implies

$$m_S(z) = m_{R_\theta}(z) - m_{R_\theta}(j(\varphi(z))),$$

which proves Theorem 1 since $j \circ \varphi = \sigma$.

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