Canad. Math. Bull. Vol. 59 (2), 2016 pp. 392–402
http://dx.doi.org/10.4153/CMB-2015-074-5

Total Character of a Group G with $(G, Z(G))$ as a Generalized Camina Pair

S. K. Prajapati and R. Sarma

Abstract. We investigate whether the total character of a finite group G is a polynomial in a suitable irreducible character of G. When $(G, Z(G))$ is a generalized Camina pair, we show that the total character is a polynomial in a faithful irreducible character of G if and only if $Z(G)$ is cyclic.

1 Introduction

Throughout this article, G denotes a finite group. Let $\mathrm{Irr}(G)$, $\mathrm{nl}(G)$ and $\mathrm{lin}(G)$ be the set of all irreducible characters of G, the set of all nonlinear irreducible characters of G and the set of linear characters of G, respectively. Suppose that ρ is the direct sum of all the non-isomorphic irreducible complex representations of G. The character τ_G afforded by ρ is called the *total character* of G, that is, $\tau_G = \sum_{\chi\in\mathrm{Irr}(G)} \chi$. Since τ_G is stable under the action of the Galois group of the splitting field of G, $\tau_G(g) \in \mathbb{Z}$ for all $g \in G$. The dimension of the total character of a group seems to have remarkable connection with the geometry of the group. For instance, in the case of the symmetric group $G = S_n$, $\tau_G(1)$ is the number of involutions of S_n [9], whereas in the case of $G = \mathrm{GL}(n, q)$, $\tau_G(1)$ is the number of symmetric matrices in $\mathrm{GL}(n, q)$ [3]. Degree of total character is discussed by many authors [4, 6, 13, 15, 16].

A consequence of a well-known theorem due to Burnside and Brauer [5, Theorem 4.3] is that the total character of the group G is a constituent of $1+\chi+\cdots+\chi^{m-1}$ if χ is a faithful character which takes exactly m distinct values on G. M. L. Lewis and S. M. Gagola [2] classified all the solvable groups for which $\chi^2 = \tau_G$ for some $\chi \in \mathrm{Irr}(G)$. Motivated by this, K. W. Johnson raised the following question (see [14]).

> Do there exist an irreducible character χ of G and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$?

The aim of the article is to solve a weaker version of the same, *i.e.*, to examine the existence of $f(x) \in \mathbb{Q}[x]$ and $\chi \in \mathrm{Irr}(G)$ such that $f(\chi) = \tau_G$. We call such a polynomial $f(x) \in \mathbb{Q}[x]$, if it exists, *a Johnson polynomial* of G. This problem has been studied for dihedral groups D_{2n} in [14]. In fact, the authors have proved that D_{2n} has a Johnson polynomial if and only if $8 \nmid n$.

To describe the classes of groups to which our results apply, we recall some definitions. A pair (G, N) is said to be a generalized Camina pair (abbreviated GCP) if

Received by the editors April 4, 2014; revised November 5, 2015.
Published electronically January 28, 2016.
Author S.K.P. was supported by the Council of Scientific and Industrial Research (CSIR), India.
AMS subject classification: **20C15**.
Keywords: finite groups, group characters, total characters.

N is normal in G and, all nonlinear irreducible characters of G vanish outside N [12]. There are a number of equivalent conditions for $(G, Z(G))$ to be a GCP. An equivalent condition we will refer to is that a pair $(G, Z(G))$ is a GCP if and only if, for all $g \in G \setminus Z(G)$, the conjugacy class of g in G is gG'.

In this article, we compute the total character τ_G of a group G for which $(G, Z(G))$ is a generalized Camina pair and prove a necessary and sufficient condition for the existence of a Johnson polynomial. Our main results can be stated as follows.

Theorem 1.1 *Let $(G, Z(G))$ be a GCP. Then G has a Johnson polynomial if and only if $Z(G)$ is cyclic. In fact, if $Z(G)$ is cyclic, then a Johnson polynomial of G is given by*

$$f(x) = d^2 \sum_{j=1}^{r} (x/d)^{lj} + d \sum_{\substack{j=1 \\ l \nmid j}}^{m} (x/d)^j,$$

where $d = |G/Z(G)|^{1/2}$, $r = |Z(G)/G'|$, $m = |Z(G)|$, and $l = |G'|$. In particular, $f(x) = d^2(x/d)^m + d\sum_{j=1}^{m-1}(x/d)^j$ when $Z(G)$ is cyclic and $Z(G) = G'$.

In the last section, we apply the above theorem to prove the following.

Theorem 1.2 *If G is a non-abelian p-group of order p^3 or p^4, then G has a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \subseteq Z(G)$.*

The character afforded by the regular representation shares certain properties with the total character of a group, and so the same question may be asked of it. We say that a group G has a *regular-Johnson polynomial* $f(x) \in \mathbb{Q}[x]$ if there is some $\chi \in \mathrm{Irr}(G)$ such that $\rho_G(g) = f(\chi(g))$, where ρ_G is the character of the regular representation of G. In the following theorem a group having a regular-Johnson polynomial is characterized.

Theorem 1.3 *Let G be a finite group. Then G has a regular-Johnson polynomial if and only if G has a faithful irreducible character.*

Proof Let G has a regular-Johnson polynomial $f \in \mathbb{Q}[x]$ with $\chi \in \mathrm{Irr}(G)$ such that $\rho_G(g) = f(\chi(g))$ for all $g \in G$, where ρ_G is the regular character. Now we will show that χ is faithful. On the contrary, let $g \neq 1 \in \ker(\chi)$. Then $0 = \rho_G(g) = f(\chi(g)) = f(\chi(1)) = \rho_G(1)$, which is a contradiction. Conversely, let $\chi \in \mathrm{Irr}(G)$ be a faithful character of G. Suppose that $f(x) = \prod_{g \neq 1 \in G} \frac{x - \chi(g)}{\chi(1) - \chi(g)}$. Then $f(\chi(g)) = \rho_G(g)$ for all $g \in G$. The coefficients of $f(x)$ manifestly lie in the cyclotomic field $\mathbb{Q}[\xi]$, where $\xi = e^{2\pi i/n}$ and $n = |G|$. Next we show that $f(x) \in \mathbb{Q}[x]$. Consider the Galois group $\mathbb{G} := \mathrm{Gal}(\mathbb{Q}[\xi]/\mathbb{Q})$. Then $\mathbb{Z}_n^\times \cong \mathbb{G}$ by $r \mapsto \sigma_r(\xi) := \xi^r$, where $\mathbb{Z}_n^\times$ consisting of all congruence classes mod n of integers coprime to n. The Galois group $\mathbb{G}$ acts on $\mathrm{Irr}(G)$ by $\sigma.\phi(g) = \mathrm{tr}(\sigma\rho(g))$, where $\phi \in \mathrm{Irr}(G)$ and ϕ is afforded by the representation ρ, and $\sigma\rho$ is defined by first realising ρ as matrices over $\mathbb{Q}[\xi]$, and then evaluating $(\sigma\rho)(g) = \sigma(\rho(g))$ entry-wise. Therefore we have $\sigma.\phi(g) = \phi(g^r)$ if $\sigma = \sigma_r$ (as described above), where r is coprime to $n = |G|$. Since $g \mapsto g^r$ is a permutation of G fixing 1, the coefficients of $f(x)$ are rational. ■

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

2 Further Notation and Preliminaries

Throughout this article, C_n denotes the cyclic group of order n. Suppose G is a finite group. Then $Z(G)$, G' and $\mathrm{Cl}(G)$ denote respectively the center, the commutator subgroup and the set of conjugacy classes of G. If $a, b \in G$, then ${}^b a = b^{-1}ab$, $[a, b] = a^{-1}b^{-1}ab$. Here $cd(G)$, $d(G)$, and $\Phi(G)$ denote the set of irreducible character degrees, the minimal number of generators of G, and the Frattini subgroup of G, respectively. Suppose N is a normal subgroup of G. Then we denote by $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) \setminus \mathrm{Irr}(G/N)$. Here we start by recalling some basic results.

Lemma 2.1 ([5, Theorem 2.32(a)]) *If G has a faithful irreducible character, then $Z(G)$ is cyclic.*

Lemma 2.2 *Let G be a non-abelian group. Then $\sum_{\chi\in\mathrm{lin}(G)} \chi(g) = 0$ for each $g \in G \setminus G'$.*

Proposition 2.1 exhibits the relationship between faithful characters and groups having Johnson polynomial.

Proposition 2.1 *Let G be a finite group. Suppose $f(x) \in \mathbb{C}[x]$ and χ is a character of G such that $f(\chi) = \tau_G$. Then χ is a faithful character. In particular, an abelian group has a Johnson polynomial if and only if it is cyclic.*

Proof Suppose $f(x) \in \mathbb{C}[x]$ and χ is a character of G such that $f(\chi) = \tau_G$ with $\ker(\chi) \neq \{1\}$. Since $\cap_{\phi\in\mathrm{Irr}(G)} \ker(\phi) = \{1\}$, $\tau_G(1) \neq \tau_G(g)$ for all $g \neq 1 \in G$. Take $g \neq 1 \in \ker(\chi)$. Then $\tau_G(1) = f(\chi(1)) = f(\chi(g)) = \tau_G(g)$, which is a contradiction. This shows that χ must be a faithful character. Hence an abelian group G having a Johnson polynomial implies that G is a cyclic group. For the converse, consider the polynomial $f(x) = \sum_{i=0}^{|G|-1} x^i$. ■

To show that G provides a negative answer to Johnson's question, we will later introduce a specific character and then attain a contradiction. For this, we need the following proposition, which is a simple observation.

Proposition 2.2 *Let χ be an irreducible character of G. If $g_1, g_2 \in G$ are such that $\chi(g_1) = \chi(g_2)$ but $\tau_G(g_1) \neq \tau_G(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau_G$.*

3 Groups with $(G, Z(G))$ a Generalized Camina Pair

In this section, we study the total character of a group G for which $(G, Z(G))$ is a generalized Camina pair (abbreviated as GCP). The notion of generalized Camina pair was first introduced by Lewis [12]. The groups with $(G, Z(G))$ a GCP were studied under the name VZ-groups by Lewis [11]. First, we record a couple of lemmata that will be useful.

Lemma 3.1 ([12, Lemma 2.1]) *Let $g \in G$. Then the following statements are equivalent.*

(i) *The conjugacy class of g is the coset gG'.*
(ii) *$\chi(g) = 0$ for all nonlinear $\chi \in \mathrm{Irr}(G)$.*

Lemma 3.2 ([12, Lemma 2.4]) *Let H be a normal subgroup of a group G such that (G, H) is a GCP. Then G' is a subgroup of H.*

3.1 Remarks on a Group G with $(G, Z(G))$ a Generalized Camina Pair

Let $(G, Z(G))$ be a GCP. Suppose χ is a nonlinear irreducible character of G. Then

$$\chi\downarrow_{Z(G)} = \chi(1)\lambda$$

for some $\lambda \in \mathrm{Irr}(Z(G))$. Thus

$$\begin{aligned} |G| = \sum_{g \in G} |\chi(g)|^2 &= \sum_{g \in Z(G)} |\chi(g)|^2 \qquad \text{(since } (G, Z(G)) \text{ is a GCP)} \\ &= \sum_{g \in Z(G)} |\chi(1)\lambda(g)|^2 \\ &= \chi(1)^2 |Z(G)|. \end{aligned}$$

Therefore the degree of any nonlinear irreducible character of G is $|G/Z(G)|^{1/2}$. Suppose n is the number of nonlinear irreducible characters of G. Then

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G/G'| + n.\chi(1)^2.$$

Therefore the total number of nonlinear irreducible characters of G is

$$|Z(G)| - |Z(G)/G'|.$$

Let $\eta\colon G \to G/G'$ be the natural homomorphism and let $\phi\colon \mathrm{Irr}(G/G') \to \mathrm{Irr}(Z(G))$ be defined by $\phi(\lambda) := \lambda \circ \eta$. Suppose $X := \{\lambda \in \mathrm{Irr}(Z(G)) \mid \lambda \notin \mathrm{Image}(\phi)\}$ and $\widehat{\Phi}\colon X \to \mathrm{nl}(G)$ defined by

$$\widehat{\Phi}(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G), \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Theorem 3.1 *Suppose $(G, Z(G))$ is a GCP. With the notation in the preceding paragraph, the map $\widehat{\Phi}$ is a bijection. In other words,*

$$\mathrm{nl}(G) = \{\widehat{\Phi}(\lambda) \mid \lambda \in \mathrm{Irr}(Z(G)) \textit{ and } G' \nsubseteq \ker(\lambda)\}.$$

Proof Clearly $\widehat{\Phi}$ is one-to-one. Let $\chi \in \mathrm{nl}(G)$. Then $\chi\downarrow_{Z(G)} = |G/Z(G)|^{1/2}\lambda$, where $\lambda \in \mathrm{Irr}(Z(G))$. We must show that $\lambda \in X$. Suppose $\lambda \notin X$. Then $G' \subseteq \ker(\lambda)$. Hence $\chi\downarrow_{Z(G)}(G') = |G/Z(G)|^{1/2} = \chi(1)$. Thus $\chi \in \mathrm{lin}(G)$, which is a contradiction. Hence $\widehat{\Phi}$ is a bijection. ∎

In the following proposition we discuss the total character τ_G of G.

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

Proposition 3.1 *Let $(G, Z(G))$ be a GCP. Then the total character τ_G is given by*

$$(3.2)\qquad \tau_G(g) = \begin{cases} |G/G'| + (|Z(G)| - |Z(G)/G'|)|G/Z(G)|^{1/2} & \text{if } g = 1,\\ |G/G'| - |Z(G)/G'|.|G/Z(G)|^{1/2} & \text{if } g \in G' \setminus \{1\},\\ 0 & \text{otherwise.}\end{cases}$$

Proof Set $\mathcal{A} := \operatorname{Irr}(Z(G)) \setminus \operatorname{Irr}(Z(G)/G')$. By Theorem 3.1, $\operatorname{nl}(G) = \{\widehat{\Phi}(\lambda) \mid \lambda \in \mathcal{A}\}$ and $\chi(1) = |G/Z(G)|^{1/2}$ for all $\chi \in \operatorname{nl}(G)$.

If $g = 1$, then $\tau_G(1) = |G/G'| + (|Z(G)| - |Z(G)/G'|)|G/Z(G)|^{1/2}$. If $g \in G \setminus Z(G)$, then by the hypothesis of the proposition and Lemma 2.2, we get

$$\tau_G(g) = \sum_{\chi \in \operatorname{Irr}(G)} \chi(g) = \sum_{\chi \in \operatorname{lin}(G)} \chi(g) = 0.$$

For $g \neq 1 \in Z(G)$, we have

$$(3.3)\qquad 0 = \sum_{\psi \in \operatorname{Irr}(Z(G))} \psi(g) = \sum_{\phi \in \operatorname{Irr}(Z(G)/G')} \phi(g) + \sum_{\lambda \in \mathcal{A}} \lambda(g).$$

If $g \neq 1 \in G' \subseteq Z(G)$, then

$$\begin{aligned}\tau_G(g) &= \sum_{\chi \in \operatorname{lin}(G)} \chi(g) + \sum_{\chi \in \operatorname{nl}(G)} \chi(g)\\ &= |G/G'| + |G/Z(G)|^{1/2} \sum_{\lambda \in \mathcal{A}} \lambda(g)\\ &= |G/G'| - |Z(G)/G'|.|G/Z(G)|^{1/2} \qquad \text{(by (3.3))}.\end{aligned}$$

Finally, if $g \in Z(G) \setminus G'$, then by Lemma 2.2 and (3.3), we get $\tau_G(g) = 0$. This completes the proof. ∎

With these technical results we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Suppose that $Z(G) = \langle g \rangle$ is a cyclic group of order m. Since $(G, Z(G))$ is a GCP, by Lemma 3.2, $G' \subseteq Z(G)$. Let $G' = \langle g^k \rangle$, $|G'| = l$, and $|Z(G)/G'| = r$. Set $\zeta_m = e^{\frac{2\pi i}{m}}$. The homomorphism $\lambda_{\zeta_m}: Z(G) \to \mathbb{C}^*$ given by $g \mapsto \zeta_m$ defines a faithful linear character. Hence $Z(G) \cong \operatorname{Irr}(Z(G)) = \langle \lambda_{\zeta_m} \rangle$. The set of irreducible characters of $Z(G)$ whose kernel contains G' is $\{\lambda_{\zeta_m}^{l}, \lambda_{\zeta_m}^{2l}, \dots, \lambda_{\zeta_m}^{rl}\}$. Hence $\operatorname{nl}(G) := \{\widehat{\Phi}(\lambda_{\zeta_m}^{i}) \mid i = 1, \dots, m \text{ and } l \nmid i\}$, where $\widehat{\Phi}$ is the map defined in (3.1). Obviously $|\operatorname{nl}(G)| = |Z(G)| - |Z(G)/G'|$. Let

$$f(x) = d^2 \sum_{j=1}^{r} (x/d)^{lj} + d \sum_{\substack{j=1\\ l \nmid j}}^{m} (x/d)^{j},$$

where $d = |G/Z(G)|^{1/2}$.

Assertion *If $\chi = \widehat{\Phi}(\lambda_{\zeta_m})$, then $f(\chi) = \tau_G$.*

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

Proof of the Assertion If $g = 1$, then

$$\begin{aligned} f(\chi(1)) &= d^2 \sum_{j=1}^{r} (\chi(1)/d)^{lj} + d \sum_{\substack{j=1 \\ l \nmid j}}^{m} (\chi(1)/d)^j \\ &= d^2 r + d(m-r) \\ &= |G/G'| + |G/Z(G)|^{1/2}(|Z(G)| - |Z(G)/G'|) \\ &= \tau_G(1) \qquad \text{(by (3.2))}. \end{aligned}$$

Let $a \neq 1 \in G'$. Then $a = g^{kq}$ where $1 \leq q \leq (l-1)$. So

$$\begin{aligned} f(\chi(g^{kq})) &= d^2 \sum_{j=1}^{r} (\chi(g^{kq})/d)^{lj} + d \sum_{\substack{j=1 \\ l \nmid j}}^{m} (\chi(g^{kq})/d)^j \\ &= d^2 \sum_{j=1}^{r} (e^{\frac{2\pi i k q}{r}})^j + d \sum_{\substack{j=1 \\ l \nmid j}}^{m} (e^{\frac{2\pi i k q}{m}})^j \\ &= d^2 . r + d(-|Z(G)/G'|) \\ &= |G/G'| - |Z(G)/G'| . |G/Z(G)|^{1/2} \\ &= \tau_G(g^{kq}) \qquad \text{(by (3.2))}. \end{aligned}$$

Finally, let $g^s \in Z(G) \setminus G'$. Then s is not a integer multiple of k. Now by using the similar arguments as in the above case we get $f(\chi(g^s)) = 0 = \tau_G(g^s)$. This completes the assertion. ■

On the other hand, if $Z(G)$ is non-cyclic, then G has no faithful irreducible character. Therefore, from Proposition 2.1, G has no Johnson polynomial. This completes the proof. ■

Remark 3.1 Since the set of character values of $\widehat{\Phi}(\lambda^i_{\zeta_m})$ does not depend on i when $(i, m) = 1$, we have $f(\widehat{\Phi}(\lambda^i_{\zeta_m})) = \tau_G$.

As a consequence of Theorem 1.1, we get the following:

Corollary 3.1 *Every extra-special p-group has a Johnson polynomial.*

Proof Suppose G is an extra-special p-group. Then $Z(G) = G'$ and $|Z(G)| = p$ and by [8, Theorem 2.18], $(G, Z(G))$ is a GCP. Therefore by Theorem 1.1, the polynomial

$$f(x) = p^n \sum_{j=1}^{p-1} (x/p^n)^j + p^{2n} (x/p^n)^p$$

is a Johnson polynomial of G and $f(\chi) = \tau_G$ for every $\chi \in \mathrm{nl}(G)$. ■

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

Table 1

Group	Order	Presentation	Polynomial $f(x)$
G_1	p^3	$\langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p \rangle$	$f_1(x) = p \sum_{j=1}^{p-1} (x/p)^j + p^2 (x/p)^p$
G_2	2^3	$\langle a, b \mid a^4 = b^4 = 1, a^2 = b^2 = [a, b] \rangle$	$f_2(x) = x^2 + x$
G_3	p^3 odd	$\langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$	$f_3(x) = p \sum_{j=1}^{p-1} (x/p)^j + p^2 (x/p)^p$
G_4	p^4	$\langle a, b \mid a^{p^3} = b^p = 1, [a, b] = a^{p^2} \rangle$	$f_4(x) = p^2 \sum_{j=1}^{p} (x/d)^{pj} + p \sum_{j=1, j \neq t.p}^{p^2} (x/p)^j$
G_5	p^4	$\langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = [a, c] = 1, [b, c] = a^p, [a, b] = [a, c] = 1, [b, c] = a^p \rangle$	$f_5(x) = p^2 \sum_{j=1}^{p} (x/d)^{pj} + p \sum_{j=1, j \neq t.p}^{p^2} (x/p)^j$
G_6	p^4	$\langle a, b \mid a^{p^2} = b^{p^2} = 1, [a, b] = a^p \rangle$	Does not exist
G_7	p^4	$\langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = a^p, [a, c] = [b, c] = 1 \rangle$	Does not exist
G_8	p^4	$\langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$	Does not exist
G_9	2^4	$\langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = a^2, a^2 = b^2, [a, c] = 1, [b, c] = 1 \rangle$	Does not exist
G_{10}	p^4 odd	$\langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = 1 \rangle$	Does not exist
$G_{11} = D_{16}$	2^4	$\langle a, b \mid a^8 = b^2 = 1, [a, b] = a^6 \rangle$	Does not exist
G_{12}	2^4	$\langle a, b \mid a^8 = b^2 = 1, [a, b] = a^2 \rangle$	Does not exist
G_{13}	2^4	$\langle a, b \mid a^8 = b^4 = 1, [a, b] = a^6, a^4 = b^2 \rangle$	Does not exist
G_{14}	p^4 odd	$\langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = a^p, [a, c] = b, [b, c] = 1 \rangle$	Does not exist
G_{15}	p^4 odd	$\langle a, b, c \mid a^{p^2} = b^p = 1, [a, b] = a^p, a^p = c^p, [a, c] = b, [b, c] = 1 \rangle$	Does not exist
G_{16}	p^4 odd	$\langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = a^p, c^p = a^{\alpha p}, [a, c] = b, [b, c] = 1 \rangle$ α denotes a quadratic non-residue mod p	Does not exist
G_{17}	$p^4, p > 3$	$\langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [b, c] = d, [a, c] = 1, [a, d] = [b, d] = [c, d] = 1 \rangle$	Does not exist
G_{18}	3^4	$\langle a, b, c \mid a^9 = b^3 = c^3 = 1, [a, b] = c, [a, c] = 1, [b, c] = a^6 \rangle$	Does not exist

4 An Application

4.1 p-Groups of Order $\leq p^4$

We quote some known results that we use in the sequel.

Lemma 4.1 ([5, Lemma 2.9]) *Let H be a subgroup of G. Suppose χ is a character of G. Then $\langle \chi{\downarrow_H}, \chi{\downarrow_H} \rangle \leq |G/H| \langle \chi, \chi \rangle$ with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.*

Lemma 4.2 ([1, Theorem 20]) *If G is a p-group, then for each $\chi \in \mathrm{Irr}(G)$, $\chi(1)^2$ divides $|G : Z(G)|$.*

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

Lemma 4.3 *Let G be a non-abelian group of order p^4. Then* $\operatorname{cd}(G) = \{1, p\}$.

Proof Since $Z(G) \neq 1$, $|Z(G)| = p$ or p^2. Therefore $|G/Z(G)| = p^3$ or p^2. So by Lemma 4.2, the result follows. ∎

The list of all non-abelian p-groups of order p^3 and p^4 [10, Table 1] is displayed in Table 1 along with a Johnson polynomial (if exists). Now we prove Theorem 1.2. To prove the theorem, we use the classification of non-abelian p-groups of order p^3 and p^4, and follow the notation in Table 1.

Proof of Theorem 1.2 Suppose $G' \subseteq Z(G)$ then $G = G_i$ $(1 \leq i \leq 10)$. By Lemmata 4.3 and 4.1, for these groups $(G, Z(G))$ is a GCP. Therefore, for G_i $(1 \leq i \leq 10)$ use Theorem 1.1 to determine a Johnson polynomial ($Z(G_i)$ is cyclic if $1 \leq i \leq 5$ and non-cyclic otherwise).

Next suppose $G' \nsubseteq Z(G)$. Then $G = G_i$ $(11 \leq i \leq 18)$. We must show that for these groups there is no Johnson polynomial. For the groups $G = G_i$ $(11 \leq i \leq 13)$, one can easily check that G has no Johnson polynomial.

Next for $G = G_i$ $(14 \leq i \leq 18)$, the nilpotency class of G is 3. Therefore $G/Z(G)$ is non-abelian and $Z(G) \subset G'$. Hence $|Z(G)| = p$. As $|G/G'| \geq p^2$, we deduce that $|G'| = p^2$. Since there is a normal abelian subgroup N (say) of index p, every nonlinear irreducible characters of G must be induced from N. Therefore, $\chi(G \setminus N) = 0$ for all $\chi \in \operatorname{nl}(G)$ and $\operatorname{cd}(G) = \{1, p\}$. Since $G/Z(G)$ is an extra-special group of order p^3, $G/Z(G)$ has $p-1$ nonlinear irreducible characters of degree p which vanish outside $Z(G/Z(G)) = G'/Z(G)$ in $G/Z(G)$. For $\chi \in \operatorname{nl}(G/Z(G))$ we have

$$\chi{\downarrow}_{Z(G/Z(G))} = p\lambda \tag{4.1}$$

for some $\lambda \in \operatorname{Irr}(Z(G/Z(G))) \setminus 1_{Z(G/Z(G))}$, where $1_{Z(G/Z(G))}$ is the trivial character of $Z(G/Z(G))$. In particular, we have all the nonlinear irreducible characters of G having $Z(G)$ in their kernel. Now, let $\psi \in \operatorname{Irr}(G|Z(G))$. Since $|Z(G)| = p$, ψ is faithful and hence ϕ is not G-invariant, where ϕ is an irreducible constituent of $\psi{\downarrow}^G_{G'}$. Therefore, by Clifford's theorem $\psi{\downarrow}^G_{G'} = \sum_1^p \phi_i$, where $\phi_1 = \phi$ and p is the index of the inertia group N of ϕ in G. Now $\phi_i{\downarrow}^{G'}_{Z(G)} = \lambda$, where $\lambda \in \operatorname{Irr}(Z(G)) \setminus 1_{Z(G)}$ for each $1 \leq i \leq p$. Therefore, by using the fact $\psi(1) = p$, we have

$$\psi{\downarrow}^G_{G'} = \sum_{\beta \in \operatorname{Irr}(G'/Z(G))} \beta\phi_1 = \rho_{G'/Z(G)}\phi_1,$$

where $\rho_{G'/Z(G)}$ is the regular character of $G'/Z(G)$. Hence for each $\psi \in \operatorname{Irr}(G|Z(G))$, we have $\psi(G' \setminus Z(G)) = 0$.

Now if $g \in G' \setminus Z(G)$, then

$$\begin{aligned} \tau_G(g) &= \sum_{\chi \in \operatorname{lin}(G)} \chi(g) + \sum_{\chi \in \operatorname{nl}(G/Z(G))} \chi(g) + \sum_{\chi \in \operatorname{Irr}(G|Z(G))} \chi(g) \\ &= |G/G'| + \sum_{\lambda \in \operatorname{Irr}(Z(G/Z(G))) \setminus 1_{Z(G/Z(G))}} p\lambda(g) + \sum_{\chi \in \operatorname{Irr}(G|Z(G))} \chi(g) \quad \text{(by (4.1))} \\ &= p^2 - p + 0 = p^2 - p. \end{aligned} \tag{4.2}$$

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

Now suppose G has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \mathrm{nl}(G)$. Therefore χ is faithful and $\chi \in \mathrm{Irr}(G|Z(G))$. By (4.2), we have

$$f(0) = f(\chi(g)) = \tau_G(g) = p^2 - p$$

for all $g \in G' \setminus Z(G)$. Again for $h \in G \setminus N$ we have, $f(0) = f(\chi(h)) = \tau_G(h) = 0$. Therefore, from Proposition 2.2, G has no Johnson polynomial for $G = G_i$ ($14 \le i \le 18$). This completes the proof. ∎

4.2 Minimal Non-abelian Groups and p-$\mathcal{JFC}$-groups

A non-abelian group G is called a *minimal non-abelian* group if every proper subgroup of G is abelian. For a prime p and $n \ge 2$, $m \ge 3$ define

$$G(n,m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, [a,b] = a^{p^{n-1}} \rangle.$$

Then $G(n,m)$ is a metacyclic group and its order is p^{n+m}. Again for a prime p and $n, m \in \mathbb{N}$ define

$$G(n,m,1) = \langle a, b \mid a^{p^m} = b^{p^n} = 1, [a,b] = c, [a,c] = [b,c] = 1 \rangle.$$

Then $G(n,m,1)$ is not a metacyclic group and its order is p^{n+m+1}. First we recall a result on minimal non-abelian p-groups.

Theorem 4.1 ([17, Lemma 2.1]) *Let G be a minimal non-abelian p-group. Then G is isomorphic to Q_8, $G(n,m)$ or $G(n,m,1)$.*

Proposition 4.1 *Suppose G is a minimal non-abelian p-group. Then G has a Johnson polynomial if and only if G is isomorphic to Q_8.*

Proof Total character of $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ is given by $\tau_{Q_8}(1) = 6$, $\tau_{Q_8}(a^2) = 2$, and $\tau_{Q_8}(a) = \tau_{Q_8}(b) = \tau_{Q_8}(ab) = 0$. If χ is the faithful irreducible character of Q_8, then one can verify that $\chi^2 + \chi = \tau_{Q_8}$ so that $x^2 + x$ is a Johnson polynomial of Q_8. Now observe that $Z(G(n,m)) = \langle a^p, b^p \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}}$ and $Z(G(n,m,1)) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p$ are non-cyclic. Therefore, they do not have any faithful irreducible character. Hence by Proposition 2.1, $G(n,m)$ and $G(n,m,1)$ have no Johnson polynomials. ∎

A p-group G is said to be a p-$\mathcal{JFC}$-group if the Frattini subgroup of every proper subgroup of G is cyclic.

Theorem 4.2 ([17, Theorem 3.1]) *Suppose that G is a p-$\mathcal{JFC}$-group with $|G'| \le p$ and p odd.*

(i) *If $|G'| = 1$, then G is abelian, and one of the following holds.*
  (a) *$G \cong C_{p^n} \times E_p^m$, where n, m are non-negative and $\Phi(G)$ is a cyclic group of order p^{n-1}.*
  (b) *$G \cong C_{p^2} \times C_{p^2}$ and $\Phi(G) = E_p^2$.*

(ii) *If $|G'| = p$ and $d(G) = 2$, then one of the following holds.*
  (a) *$G \cong Mod_{p^{n+1}} = \langle a, b \mid a^{p^n} = b^p = 1, [a,b] = a^{p^{n-1}} \rangle$, where $n \ge 2$ is a positive integer.*

https://doi.org/10.4153/CMB-2015-074-5 Published online by Cambridge University Press

(b) $G = \langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1\rangle$, *where $n \geq 1$ is a positive integer.*

(c) $G = \langle a, b \mid a^{p^2} = b^{p^2} = 1, [a,b] = a^p\rangle$.

(iii) *If $|G'| = p$ and $d(G) \neq 2$, then $\Phi(G)$ is cyclic.*

Proposition 4.2 *Let p be an odd prime. Suppose G is p-$\mathcal{IFC}$-group with $|G'| \leq p$.*

(i) *If $|G'| = 1$, then G has no Johnson polynomials.*

(ii) *If $|G'| = p$ and $d(G) = 2$, then G is a Johnson polynomial if and only if $G \cong Mod_{p^{n+1}}$.*

(iii) *If $|G'| = p$ and $d(G) \neq 2$, then G need not have a Johnson polynomial.*

Proof If $|G'| = 1$, then by Theorem 4.2, G is a non-cyclic abelian group and hence by Proposition 2.1 G has no Johnson polynomials. Now suppose $G \cong \mathrm{Mod}_{p^{n+1}}$ so that $|G'| = p$ and $d(G) = 2$. Here $Z(G) = \langle a^p\rangle$, $|G/Z(G)| = p^2$ and $G' = \langle a^{p^{n-1}}\rangle$. By Lemma 4.2, the degree of every nonlinear irreducible character is p and so by Lemma 4.1 $(G, Z(G))$ is GCP. Hence by Theorem 1.1 the following polynomial

$$f(x) = p^2 \sum_{j=1}^{p^{n-2}} (x/p)^{lj} + p \sum_{\substack{j=1 \\ p \nmid j}}^{p^{n-1}} (x/p)^j$$

is a Johnson polynomial. Next suppose $|G'| = p$ and $d(G) = 2$ and $G \ncong \mathrm{Mod}_{p^{n+1}}$. Then by Theorem 4.2, either $G = \langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1\rangle$ or $G = \langle a, b \mid a^{p^2} = b^{p^2} = 1, [a,b] = a^p\rangle$. In the former case, $Z(G) = \langle a^p, b^p, c\rangle$, and in the latter, $Z(G) = \langle a^p, b^p\rangle$. Hence, in either case the center is non cyclic. Therefore, it does not have faithful irreducible character and hence by Proposition 2.1, none of these groups has a Johnson polynomial.

Finally to justify the third statement of the theorem we will produce examples. Suppose p is an odd prime. Let $G_1 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle$ and $G_2 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a,b] = a^p, [a,c] = [b,c] = 1\rangle$. The groups G_1 and G_2 are both p-$\mathcal{IFC}$-groups. Observe that $\Phi(G_1) = G_1' = Z(G_1) = \langle c\rangle$, $\Phi(G_2) = G_2' = \langle a^p\rangle$, $Z(G_2) = \langle a^p, c\rangle$, $d(G_i) \neq 2$, and $(G_i, Z(G_i))$ is a GCP for $i = 1, 2$. Hence by Theorem 1.1, G_1 has a Johnson polynomial but G_2 has no Johnson polynomials. ■

Acknowledgement The authors thank the anonymous referee for his/her useful comments and suggestions, including the statement of Theorem 1.3.

References

[1] Y. Berkovich, *Groups of prime power order*. Vol. 1, de Gruyter Expositions in Mathematics 46, Walter de Gruyter, Berlin, 2008.

[2] S. M. Gagola, Jr. and M. L. Lewis, *Squares of characters that are the sum of all irreducible characters*. Illinois J. Math. **42**(1998), no. 4, 655–672.

[3] R. Gow, *Properties of the finite linear group related to the transpose-inverse involution*. Proc. London Math. Soc. **47**(1983), no. 3, 493–506. http://dx.doi.org/10.1112/plms/s3-47.3.493

[4] R. Heffernan and D. MacHale, *On the sum of the character degrees of a finite group*. Math. Proc. R. Ir. Acad. **108**(2008), no. 1, 57–63. http://dx.doi.org/10.3318/PRIA.2008.108.1.57

[5] I. M. Isaacs, *Character theory of finite groups*. Pure and Applied Mathematics 69. Academic Press, New York, 2000.
[6] I. M. Isaacs, M. Loukaki, and A. Moreto, *The average degree of an irreducible character of a finite group*. Israel J. Math. **197**(2013), no. 1, 55–67. http://dx.doi.org/10.1007/s11856-013-0013-z
[7] G. James and M. Liebeck, *Representations and characters of groups*. Second edition. Cambridge University Press, New York, 2001.
[8] G. Karpilovsky, *Group representations*. Vol. 1. Part B, In: Introduction to group representations and characters. North-Holland Mathematics Studies, 175. North-Holland, Amsterdam, 1992, pp. i–xiv, 621–1274.
[9] V. Kodiyalam and D. N. Verma, *A natural representation model for symmetric groups*. arxiv:math/0402216.
[10] S. Lemieux, *Finite exceptional p-groups of small order*. Comm. Algebra **35**(2007), no. 6, 1890–1894. http://dx.doi.org/10.1080/00927870701246924
[11] M. L. Lewis, *Character tables of groups where all nonlinear irreducible characters vanish off the center*. In: Ischia group theory 2008. World Sci. Publ. Hackensack, NJ, 2009, pp. 174–182. http://dx.doi.org/10.1142/9789814277808_0013
[12] ———, *The vanishing-off subgroup*. J. Algebra **321**(2009), no. 4, 1313–1325. http://dx.doi.org/10.1016/j.jalgebra.2008.11.024
[13] K. Magaard, and H. P. Tong-Viet, *Character degree sums in finite nonsolvable groups*. J. Group Theory **14**(2011), no. 1, 53–57.
[14] E. Poimenidou and H. Wolfe, *Total characters and Chebyshev polynomials*. Int. J. Math. Math. Sci.(2003) , no. 38, 2447–2453.
[15] J. Soto-Andrade, *Geometrical Gelfand models, tensor quotients and Weil epresentations*. Proc. Symp. Pure Math. 47, Smer. Math. Soc., Providence, RI, 1987, pp. 306–316.
[16] H. P. Tong-Viet, *On groups with large character degree sums*. Arch. Math. (Basel) **99**(2012), no. 5, 401–405. http://dx.doi.org/10.1007/s00013-012-0446-3
[17] J. Zhang and X. Li, *Finite p-groups all of whose proper subgroups have cyclic Frattini subgroups*. J. Group Theory **15**(2012), no. 2, 245–259.

Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, INDIA
e-mail: skprajapati.iitd@gmail.com ritumoni@maths.iitd.ac.in