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OBTAINING PRESCRIBED RATES OF CONVERGENCE FOR THE ERGODIC THEOREM

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1. Introduction. Let $\{Y_n, n \in \mathbf{Z}\}$ be an ergodic strictly stationary sequence of random variables with mean zero, where $\mathbf{Z}$ denotes the set of integers. For $n \in \mathbf{N} = \{1, 2, \dots\}$, let $S_n = Y_1 + Y_2 + \dots + Y_n$. The ergodic theorem, alias the strong law of large numbers, says that $n^{-1} S_n \rightarrow 0$ as $n \rightarrow \infty$ a.s. If the Y_n's are independent and have variance one, the law of the iterated logarithm tells us that this convergence takes place at the rate $(2n^{-1} \log \log n)^{\frac{1}{2}}$ in the sense that

$$(1) \quad \limsup_{n\rightarrow\infty} n^{-1}S_n(2n^{-1} \log \log n)^{-\frac{1}{2}} = 1 \text{ a.s.}$$

It is our purpose here to investigate what other rates of convergence are possible for the ergodic theorem, that is to say, what sequences $\{b_n, n \geqq 1\}$ have the property that

$$(2) \quad \limsup_{n\rightarrow\infty} b_n^{-1}S_n = 1 \text{ a.s.}$$

for some ergodic stationary sequence $\{Y_n, n \in \mathbf{Z}\}$. Theorem 1 shows that the class of such sequences is quite large. We will say $\{Y_n, n \in \mathbf{Z}\}$ is *symmetric* if it and the sequence $\{-Y_n, n \in \mathbf{Z}\}$ have the same distribution. For symmetric sequences, (2) obviously implies that

$$\liminf_{n\rightarrow\infty} b_n^{-1}S_n = -1 \text{ a.s.}$$

THEOREM 1. *Let $\{b_n = b(n), n \in \mathbf{N}\}$ be a sequence of positive real numbers such that*

$$(3) \quad \lim_{n\rightarrow\infty} b_n = \infty$$

and

$$(4) \quad \liminf_{n\rightarrow\infty} n^{-1}b_n = 0.$$

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Then there exists a $\{-1, 1\}$*-valued symmetric ergodic stationary sequence* $\{Y_n, n \in \mathbf{Z}\}$ *such that* (2) *holds.*

The hypothesis (4) is necessary for (2) to hold, by the ergodic theorem itself. The condition (3) is not necessary but it cannot be dropped altogether since the left side of (2) is not measurable with respect to the σ-field of shift-invariant sets for $\{Y_n\}$ if $\{b_n\}$ converges. Some further discussion of (3) will be given at the end of this paper.

In Section 2, we describe a class of ergodic stationary sequences which are used in Section 3 to prove Theorem 1. This class, which may have other applications, first evolved in connection with the work of O'Brien and Vervaat (1983) on self-similar processes. H. Kesten has suggested privately that it might also be possible to prove Theorem 1 using renewal processes. We have not pursued this idea. In Section 4, we consider the analogue of Theorem 1 for two-dimensional arrays.

Several authors have obtained results related to Theorem 1. Halász (1976) has proved (among other things) that convergence in the ergodic theorem can be arbitrarily fast in the sense that, if $\{b_n, n \geqq 1\}$ is non-decreasing and diverges to infinity, then there exists an ergodic stationary sequence $\{Y_n, n \geqq 1\}$ for which

$$\limsup_{n\to\infty} b_n^{-1}|S_n| \leqq 1 \text{ a.s.}$$

On the other hand, Krengel (1978) has proved that if $n^{-1}b_n \to 0$ as $n \to \infty$, then there is an ergodic stationary sequence $\{Y_n, n \geqq 1\}$ for which

$$\limsup_{n\to\infty} b_n^{-1}S_n = \infty \text{ a.s.}$$

These two results obviously follow from Theorem 1. Kakutani and Petersen (1981) have shown that for any sequence $\{c_n, n \geqq 1\}$ of non-negative reals for which

$$\sum_{n=1}^{\infty} c_n = \infty,$$

there exists an ergodic stationary sequence $\{Y_n, n \geqq 1\}$ for which

$$\left|\sum_{n=1}^{k} c_n n^{-1} S_n\right|$$

is almost surely unbounded as a function of k. Our theorem does not directly imply their result, but the construction in our Section 2 can be used to obtain their result if the parameters appearing in our construction are chosen appropriately. We omit the details since they do not add much new insight. The reader should note that the authors cited above all start with an arbitrary probability space $(\Omega, \mathscr{F}, P)$ and an arbitrary invertible non-atomic ergodic measure preserving transformation $T : \Omega \to \Omega$ and then show that for some real-valued function f on Ω the sequence $\{f(T^n)\}$ has the required property. We have not attempted to strengthen Theorem 1 in this direction. These authors all use a lemma due to Rohlin (cf. Halmos (1956), p. 71 or Friedman (1970), p. 108), which states that for any $\epsilon > 0$ and any positive integer n, there exists a set $E \in \mathscr{F}$ such that E, $TE, \dots, T^{n-1}E$ are disjoint and

$$P(E \cup TE \cup \dots \cup T^{n-1} E) \geqq 1 - \epsilon.$$

In our construction, $(\Omega, \mathscr{F}, P)$ and T are chosen in advance in such a way that this holds for certain values of n even with $\epsilon = 0$, so that in addition we get $T^n E = E$, modulo a null set, for these values of n.

Aaronson (1981) has considered the sequence $b_n^{-1} S_n$ for the case in which $n^{-1} b_n$ is non-decreasing and diverges to infinity. Then

$$\limsup_{n\to\infty} b_n^{-1}|S_n| = \infty \text{ a.s.} \quad \text{or} \lim_{n\to\infty} b_n^{-1} S_n = 0 \text{ a.s.}$$

The latter case is automatic if EY_1 exists; he makes no such restriction. This result is in marked contrast to ours. A related result is that of Kesten (1975), who showed that

$$\liminf_{n\to\infty} n^{-1} S_n > 0 \text{ a.s. if } S_n \to \infty \text{ a.s.}$$

2. A class of ergodic stationary random sequences. In this section, we define a class of sequences $\{Y_n, n \in \mathbf{Z}\}$ which will be used in the next section to prove Theorem 1. The construction depends on a sequence $n_1, n_2, \dots$ of even integers, all at least 4, and a sequence $m_1, m_2, \dots$ of integers such that

$$(5) \qquad 1 \leqq m_k \leqq \tfrac{1}{4} n_k$$

for all k. These parameters will be specified later. Let

$$N(k) = N_k = n_1 n_2 \dots n_k, \; M(k) = M_k = m_1 m_2 \dots m_k \quad \text{and}$$
$$A_k = \{0, 1, \dots, n_k - 1\}, \quad k = 1, 2, \dots .$$

Also, let $N_0 = M_0 = 1$ and $A_0 = \{1, -1\}$. Henceforth, $[\cdot]$ will denote the greatest integer function.

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We take as our probability space $(\Omega, \mathscr{F}, P)$ where $\Omega = A_0 \times A_1 \times A_2 \times \ldots$, $\mathscr{F}$ is the σ-field generated by the cylinder sets in Ω, and P is the product of the uniform probability measures on $A_0, A_1, A_2, \ldots$. We will define an invertible ergodic measure-preserving transformation T on $(\Omega, \mathscr{F}, P)$ and will define $\{Y_n, n \in \mathbf{Z}\}$ in terms of T. The above choice of $(\Omega, \mathscr{F}, P)$ is not essential, but it facilitates our definition of T and $\{Y_n\}$.

For $k \geqq 1$, define numbers $x_k(i)$, $i \in A_k$ by

$$(6) \quad x_k(i) = \begin{cases} +1 \text{ if } i \text{ is even or if } i = 2[\tfrac{1}{2}rm_k^{-1}n_k] - 1 \\ \qquad\qquad\qquad\qquad \text{for } r = 1, 2, \ldots, m_k, \\ -1 \text{ otherwise.} \end{cases}$$

In particular

$$(7) \quad x_k(0) = x_k(n_k - 1) = 1$$

for all k. Note that $x_k(i) = 1$ for $\frac{1}{2}n_k + m_k$ values of i and $x_k(i) = -1$ for the remaining $\frac{1}{2}n_k - m_k$ values of i, so that

$$(8) \quad \sum_{i=0}^{n_k-1} x_k(i) = 2m_k.$$

We will refer to the $+1$'s which occur at odd values of i as "reversed" $+1$'s.

Define a transformation $T:\Omega \to \Omega$ as follows. Let $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$. If $\omega_k = n_k - 1$ for all $k \geqq 1$, let $T\omega = (\omega_0, 0, 0, 0, \ldots)$. Otherwise, there is a least $k > 0$ such that $\omega_k < n_k - 1$. Then define $T\omega$ by

$$(9) \quad (T\omega)_j = \begin{cases} \omega_0 x_k(\omega_k) x_k(\omega_k + 1) \text{ if } j = 0, \\ 0 \text{ if } 0 < j < k, \\ \omega_k + 1 \text{ if } j = k, \text{ and} \\ \omega_j \text{ if } j > k. \end{cases}$$

The transformation T can be interpreted as follows. It adds one to the first component (i.e., ω_1) if possible. Otherwise, it still adds one modulo n_1 but then one is "carried", in the sense of decimal system arithmetic, to the second component. The carrying is continued further if necessary (i.e., if $\omega_2 = n_2 - 1$ as well). The role of ω_0 is to provide a parity check for this arithmetical operation. Suppose k is the least positive integer such that $\omega_k < n_k - 1$. By (6),

$$x_j((T\omega)_j) = x_j(\omega_j)$$

for all positive $j \neq k$. Thus $(T\omega)_0 = \omega_0$ or $-\omega_0$, according to whether $x_j((T\omega)_j)$ differs from $x_j(\omega_j)$ for an even or odd number of positive coordinates j, respectively. This interpretation extends in an obvious way to the two special points ω for which $\omega_k = n_k - 1$ for all $k \geqq 1$.

We have the following result.

THEOREM 2. *The transformation T is measurable, measure-preserving, invertible and ergodic.*

Proof. The measurability and invertibility are obvious and the measure-preservation follows immediately from a consideration of cylinder sets in Ω. For $\omega = (\omega_0, \omega_1, \dots) \in \Omega$, let $\omega^* = (-\omega_0, \omega_1, \dots)$. For any event $E \in \mathcal{F}$, let

$$E^* = \{\omega^* : \omega \in E\} = \{\omega : \omega^* \in E\}.$$

Now let E be an event such that $P(E) > 0$ and $TE = E$. We will prove T is ergodic by showing that $P(E) = 1$. Let $E_1 = E \cap E^*$. Since $T(\omega^*) = (T\omega)^*$ for all ω, we have

$$(10) \quad TE_1 = TE \cap T(E^*) = TE \cap (TE)^* = E \cap E^* = E_1.$$

Since $E_1 \subset E$, it is sufficient to prove that $P(E_1) = 1$.

We first show that $P(E_1) > 0$. Fix ϵ such that

$$(11) \quad 0 < \epsilon < (13)^{-1} P(E).$$

Let E' be a cylinder set in Ω, depending on coordinates $0, 1, \dots, k-1$ for some k, such that

$$(12) \quad P(E \,\Delta\, E') < \epsilon,$$

where Δ denotes the symmetric difference. Since $TE = E$,

$$(13) \quad P(E \,\Delta\, T^nE') = P(T^nE \,\Delta\, T^nE') = P(T^n(E \,\Delta\, E')) < \epsilon$$

for any $n \in \mathbf{Z}$. By (12),

$$(14) \quad P(E^*) = P(E) > P(E') - \epsilon = P((E')^*) - \epsilon.$$

Let

$$E'' = \{\omega \in E' : x_k(\omega_k) = -1\}.$$

By (5), (6) and (12),

$$(15) \quad P(E'') \geqq \tfrac{1}{4} P(E') \geqq \tfrac{1}{4} (P(E) - \epsilon).$$

If $\omega = (\omega_0, \omega_1, \dots) \in E''$, then $x_k(\omega_k + 1) = 1$ so that

$$T^{N(k-1)}(\omega) = (-\omega_0, \omega_1, \dots, \omega_{k-1}, \omega_k + 1, \omega_{k+1}, \dots).$$

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Combining this with the fact that E' does not depend on the k'th coordinate, we see that for $\omega \in E''$, $\omega^* \in E'$ if and only if

$$T^{N(k-1)}\omega \in E'.$$

Applying (14), this last fact, (13), (12), (15) and (11), we obtain

$$\begin{aligned} P(E_1) &= P(E \cap E^*) \\ &> P(E \cap E'' \cap (E')^*) - \epsilon \\ &= P(E \cap E'' \cap T^{-N(k-1)} E') - \epsilon \\ &\geqq P(E'') - 3\epsilon \\ &\geqq \tfrac{1}{4}(P(E) - \epsilon) - 3\epsilon > 0. \end{aligned} \tag{16}$$

Now let $\omega \in E_1$ and let $\omega' \in \Omega$ be such that $\omega_k = \omega'_k$ for all but finitely many k's. Then $\omega' = T^n\omega$ or $\omega' = T^n\omega^*$ for some $n \in \mathbf{Z}$. By (10), $\omega' \in E_1$. It follows that E_1 is a tail event for the random sequence $(\omega_0, \omega_1, \dots)$ of independent random variables. By (16) and Kolmogorov's $0 - 1$ law, $P(E_1) = 1$. This proves Theorem 2.

We immediately obtain the next corollary.

COROLLARY 1. *Let $(\Omega, \mathscr{F}, P)$ and T be as defined above. Then the sequence of random variables $\{Y_n, n \in \mathbf{Z}\}$ defined by*

$$Y_n(\omega) = (T^n\omega)_0 \tag{17}$$

is $\{-1, 1\}$-valued, symmetric, ergodic and stationary.

Remarks. The requirement in Theorem 2 that $m_k \leqq \frac{1}{4}n_k$ is only used to prove the ergodicity of T. Some such restriction is necessary, but we do not know a necessary and sufficient condition. A weaker sufficient condition for ergodicity is

$$\liminf_{k\to\infty} m_k n_k^{-1} < \tfrac{1}{2},$$

while a necessary condition is

$$2^k M_k N_k^{-1} \to 0 \text{ as } k \to \infty.$$

It is much easier to show the invariant set E of the foregoing proof must have probability 0, $\frac{1}{2}$, or 1. The ergodicity in Theorem 1 can then be obtained by conditioning on an invariant set E for which $P(E) = \frac{1}{2}$, if such a set exists. The advantages of actually proving ergodicity in Theorom 2 are that we can then represent the sequence required in Theorem 1 explicitly, the proof of (2) is easier, and the symmetry condition of Theorem 1 can also be obtained.

If a stationary sequence $\{Y_n\}$ satisfies some suitable mixing condition, then (1) still holds, except for some constant factor. This was first shown by Ibragimov (1962). Thus, a sequence satisfying (2) for

$$b_n \neq (cn \log \log n)^{\frac{1}{2}}$$

can be expected to exhibit long range dependence. The sequence $\{Y_n\}$ of Corollary 1 has the property that, for any k, the sample paths consist of successive blocks of length N_k, any two of which are either identical or negatives of each other. Thus, the long range dependence is very strong.

3. Proof of theorem 1.

Step 1. *Reduction to the monotonic case*. We first show that the sequence $\{b_n\}$ may be assumed to have certain monotonicity properties, but that in exchange for this simplification we must prove a slightly stronger result. Let $\{\mu_i, i \in \mathbf{N}\}$ be an increasing sequence of positive integers defined as follows. First, choose μ_1 so as to minimize $b(\mu_1)$. This is possible by (3). Now assume $\mu_1, \mu_2, \ldots, \mu_i$ have been chosen and choose μ_{i+1} so as to minimize $b(\mu_{i+1})$ subject to the constraints that

$$\mu_{i+1} > \mu_i \quad \text{and} \quad \mu_{i+1}^{-1} b(\mu_{i+1}) < \mu_i^{-1} b(\mu_i).$$

This is possible by (3) and (4). Now define a sequence $\{a_n = a(n), n \in \mathbf{N}\}$ by linearly interpolating between the points $(\mu_i, b(\mu_i))$, $i \in \mathbf{N}$. Specifically, let

$$a_n = \begin{cases} b(\mu_1) \text{ if } n \leqq \mu_1, \\ b(\mu_i) + (n - \mu_i)(b(\mu_{i+1}) - b(\mu_i))(\mu_{i+1} - \mu_i)^{-1} \\ \qquad\qquad\qquad\qquad \text{if } \mu_i \leqq n \leqq \mu_{i+1}. \end{cases}$$

The sequence $\{a_n, n \geqq 1\}$ has the following properties:

(18) $\quad a(\mu_i) = b(\mu_i)$ for all $i \in \mathbf{N}$,

(19) $\quad a_n \leqq b_n$ for all $n \in \mathbf{N}$,

(20) $\quad a_n \leqq a_{n+1}$ for all $n \in \mathbf{N}$,

(21) $\quad \lim_{n\to\infty} a_n = \infty$,

(22) $\quad n^{-1} a_n \geqq (n+1)^{-1} a_{n+1}$ for all $n \in \mathbf{N}$, and

(23) $\quad \lim_{n\to\infty} n^{-1} a_n = 0.$

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It follows from (18) and (19) that in order to prove Theorem 1 it suffices to prove the following result.

THEOREM 3. *Let* $\{a_n = a(n), n \in \mathbf{N}\}$ *be a sequence of real numbers satisfying* (20), (21), (22) *and* (23). *Let* $\{\mu_i, i \in \mathbf{N}\}$ *be an increasing sequence of positive integers. Then there exists a* $\{-1, 1\}$*-valued symmetric ergodic stationary sequence* $\{Y_n, n \in \mathbf{Z}\}$ *such that*

$$(24) \quad \limsup_{n\to\infty} a_n^{-1} S_n \leqq 1 \text{ a.s.}$$

and

$$(25) \quad \limsup_{i\to\infty} (a(\mu_i))^{-1} S(\mu_i) \geqq 1 \text{ a.s.}$$

The rest of this section is devoted to the proof of Theorem 3. We will actually prove a slightly stronger result than Theorem 3; we will show that, for any $\delta \in (0, 1)$, Theorem 3 holds with (25) replaced by

$$(26) \quad \limsup_{i\to\infty} \min \{n^{-1}a_n^{-1}\mu_i S_n : \delta\mu_i \leqq n \leqq \mu_i\} \geqq 1 \text{ a.s.}$$

The purpose of this extension is to facilitate a certain application of this result to self-similar processes (cf. Section 1). The proof of Theorem 3 is not significantly altered by the extension.

Step 2. *Specification of the parameters.* Let $\{Y_n\}$ be a sequence of the type defined in Corollary 1. We will show that $\{Y_n\}$ meets the requirements of Theorem 3 if the parameters $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ are chosen so as to satisfy the following condition for all $k \in \mathbf{N}$:

$$(27) \quad m_k = [a(N_k)2^{-k}M_{k-1}^{-1}],$$

$$(28) \quad m_k \geqq k,$$

$$(29) \quad n_k \geqq 4k^3 m_k,$$

$$(30) \quad a([k^{-2}m_k^{-1}N_k]) \geqq (2k + 7)a(N_{k-1}),$$

and

$$(31) \quad (1 - k^{-1})N_k \leqq \nu_k \leqq (1 - (2k)^{-1})N_k,$$

where $\nu_k = \mu_i$ for some $i \geqq 1$.

Let us now verify that such choices are possible. For some $k \geqq 1$, assume $n_1, n_2, \ldots, n_{k-1}$ and $m_1, m_2, \ldots, m_{k-1}$ have been chosen. Treat n_k as a variable and define m_k in terms of n_k by (27). By (21), (28) holds for n_k sufficiently large. By (27) and (23),

$$m_k n_k^{-1} \leqq a(N_k) N_k^{-1} 2^{-k} M_{k-1}^{-1} N_{k-1} \to 0 \text{ as } n_k \to \infty.$$

This implies that (29) and, again using (21), (30) hold for sufficiently large n_k. As n_k varies, the possible values of N_k form an arithmetic sequence. Thus, every sufficiently large integer μ lies between $(1 - k^{-1})N_k$ and $(1 - (2k)^{-1})N_k$ for some choice of n_k, so that (31) can be achieved for infinitely many n_k. Thus n_k and m_k can be chosen so as to satisfy (27), . . . , (31) for all k.

By (28), $m_k \to \infty$ as $k \to \infty$ so that (27) implies

(32) $$2^{-k} M_k^{-1} a(N_k) \to 1 \quad \text{as } k \to \infty.$$

It is clear from Corollary 1 that $\{Y_n\}$ is $\{-1, 1\}$-valued, symmetric, ergodic and stationary. It only remains to prove (24) and (26), which we do in Steps 3, 4 and 5.

Step 3. *Some preliminary bounds.* A *block* is defined to be a finite possibly random set of successive integers. More specifically, a *k-block* for the sequence $\{Y_n, n \in \mathbf{Z}\}$ is a random set

$$B = \{R, R + 1, \dots, R + N_k - 1\}$$

of successive integers such that $(T^R\omega)_j = 0$, $j = 1, 2, \dots, k$. For each $j > k$, $(T^n \omega)_j$ is constant over all $n \in B$. We see that the set of integers is composed of juxtaposed k-blocks. We say a k-block $B = \{R, R + 1, \dots, R + N_k - 1\}$ is *positive* if $Y_R = +1$ and *negative* otherwise, and write sign $(B) = +1$ or -1 respectively to indicate the two cases. Note that sign (B) is also random. A *partial k-block C* is a block which is contained in a k-block. We will study S_n by examining the contributions to S_n from the complete and partial k-blocks which intersect $\{1, 2, \dots, n\}$.

Any k-block B is composed of n_k juxtaposed $(k-1)$-blocks C_0, $C_1, \dots, C_{n_k-1}$, which are assumed to be ordered in the obvious way. It is clear that sign (C_0) = sign (B) and, in general, the signs of the C_i's alternate except for the m_k "reversed" $(k-1)$-blocks whose locations match those of the reversed $+1$'s among the numbers $x_k(i)$, $i \in A_k$; in other words,

(33) $$\text{sign } (C_i) = x_k(i) \text{ sign } (C_0).$$

Consider any collection of λ successive $(k-1)$-blocks, not necessarily contained in a single k-block. Suppose that l of these are reversed $(k-1)$-blocks. These l are spaced approximately $m_k^{-1} n_k$ apart by (6) so that l is roughly $\lambda m_k n_k^{-1}$. More precisely, for some r,

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$$\lambda \leqq (2[\tfrac{1}{2}(r + l + 1)\, m_k^{-1} n_k] - 1) - (2[\tfrac{1}{2} r m_k^{-1} n_k] - 1) - 1$$
$$\leqq (l + 1)\, m_k^{-1} n_k + 1.$$

Combining this with a similar lower bound for λ, we obtain

(34) $|l - \lambda\, m_k n_k^{-1}| \leqq 1 + m_k n_k^{-1} \leqq 2.$

Let B be any k-block. If $k = 1$, then

$$|\sum_{n \in B} Y_n| = 2m_1 = 2M_1$$

by (8). By (7), it follows that

$$\sum_{n \in B} Y_n = \text{sign}\,(B) 2M_1.$$

For $k > 1$, B can be decomposed into n_k $(k - 1)$-blocks of which, by (33), $2m_k$ more have the same sign as B than the opposite sign. By induction,

(35) $$\sum_{n \in B} Y_n = \text{sign}\,(B) 2^k M_k$$

for all $k \geqq 1$. Recalling (32), we note that (35) suggests (24) and (26) are at least feasible.

Now let C be any partial k-block which is contained in a k-block B. Assume initially that C contains the left end-point of B. In the notation used above (33),

$$C = C_0 \cup C_1 \cup \ldots \cup C_{r-1} \cup D \quad \text{for some } r$$

where D is a (possibly empty) subset of C_r. If $k = 1$ (so that each C_i is a singleton) and B is positive, then each negative C_i is preceded by a positive one. A similar statement holds if B is negative. Thus,

$$0 \leqq \text{sign}\,(B) \sum_{n \in C} Y_n \leqq 2M_1.$$

Now take $k > 1$. Assume the contribution to S_n from D satisfies

$$0 \leqq \text{sign}\,(C_r) \sum_{n \in D} Y_n \leqq 2^{k-1} M_{k-1}.$$

The roughly alternating pattern of signs among the C_i's then gives

(36) $$0 \leqq \text{sign}\,(B) \sum_{n \in C} Y_n \leqq 2^k M_k,$$

by (35). Next suppose C contains the right end-point of B. Let D be the complement of C in B. Since

$$\sum_{n\in B} Y_n = \sum_{n\in C} Y_n + \sum_{n\in D} Y_n$$

and D contains the left end-point of B (unless $C = B$), we see from (35) and (36) applied to D that (36) holds for C in this case also. Finally, suppose C contains neither end-point of B. Let D be the part of B which precedes C. An argument like that of the previous case shows that

$$(37)\quad \Big|\sum_{n\in C} Y_n\Big| \leqq 2^k M_k$$

in this case.

Step 4. *The proof of* (24). The idea of the proof is as follows. Suppose the block $B = \{1, 2, \ldots, n\}$ intersects λ k-blocks. If λ is small, then it is unlikely that any of these are reversed k-blocks, so that S_n can be bounded by cancelling the contributions of adjacent k-blocks in B, which are opposite in sign. This cancellation allows a little leeway for those n for which only a small number of reversed k-blocks intersect B. On the other hand, if many reversed k-blocks intersect B, then the error allowed by (34) is negligible, which enables us to get an accurate bound on S_n.

First, then, we consider n such that

$$(38)\quad N_k \leqq n \leqq (k + 1)^{-2} m_{k+1}^{-1} N_{k+1}.$$

Such n exist by (29). The block

$$C_k = \{1, 2, \ldots, [(k+1)^{-2} m_{k+1}^{-1} N_{k+1}]\}$$

is composed of at most $(k+1)^{-2} m_{k+1}^{-1} n_{k+1}$ complete k-blocks and at most two partial k-blocks. The contributions to the sum of the complete k-blocks alternate in sign except for the reversed ones and the latter are separated by at least $m_{k+1}^{-1} n_{k+1} - 3$ other k-blocks. It follows that the probability that C_k intersects any of the reversed k-blocks is at most

$$(39)\quad ((k+1)^{-2} m_{k+1}^{-1} n_{k+1} + 2)(m_{k+1}^{-1} n_{k+1} - 3)^{-1} \leqq 2(k+1)^{-2}$$

for large k, where the final inequality follows from (29). By the Borel-Cantelli Lemma, C_k intersects a reversed k-block for at most finitely many k, with probability one. Let $B = \{1, 2, \ldots, n\}$ where n satisfies (38) for some k and suppose C_k and hence B do not intersect any reversed k-blocks. Each pair of adjacent k-blocks in B contributes a total of zero to

S_n since the k-blocks have opposite signs. If the contributions of k-blocks are cancelled in pairs in this way, we are left only with the contributions of one of the following:

(a) a single complete or partial k-block,

(b) two partial k-blocks of opposite signs, or

(c) a complete k-block of one sign and one or two partial k-blocks of the other sign.

Since $n \geqq N_k$, the partial k-blocks in all cases include an end-point of the k-blocks which contain them. Applying (35), (36), (27) and (20), we then obtain

$$(40) \quad |S_n| \leqq 2^k M_k \leqq a(N_k) \leqq a(n)$$

for n sufficiently large and satisfying (38) for some k, almost surely.

Next, consider n satisfying

$$(41) \quad (k+1)^{-2} m_{k+1}^{-1} N_{k+1} \leqq n \leqq (k+1) m_{k+1}^{-1} N_{k+1},$$

for large k. For such n, $B = \{1, 2, \ldots, n\}$ intersects at most $(k+1)m_{k+1}^{-1} n_{k+1}$ complete k-blocks. By (34), there are at most

$$(42) \quad ((k+1)m_{k+1}^{-1} n_{k+1}) m_{k+1} n_{k+1}^{-1} + 2 = k + 3$$

reversed k-blocks among these. Also, B intersects at most two additional partial k-blocks. By (42), (35), (37), (27), (30) and (20), we have

$$\begin{aligned} |S_n| &\leqq (2(k + 3) + 1)2^k M_k + 2(2^k M_k) \\ &= (2k + 9)2^k M_k \\ &\leqq (2k + 9)a(N_k) \\ &\leqq a([(k+1)^{-2} m_{k+1}^{-1} N_{k+1}]) \\ &\leqq a(n), \end{aligned}$$

for n satisfying (41).

Finally, consider n such that

$$(44) \quad (k+1)m_{k+1}^{-1} N_{k+1} \leqq n \leqq N_{k+1}.$$

The block $B = \{1, 2, \ldots, n\}$ intersects at most nN_k^{-1} complete k-blocks and at most two additional partial k-blocks. Of the complete ones, there are at most $nm_{k+1}N_{k+1}^{-1} + 2$ reversed ones, by (34). By (35), (37), (27), (22), (30) and (20),

$$
\begin{aligned}
|S_n| &\leqq (2(nm_{k+1}N_{k+1}^{-1} + 2) + 1)2^kM_k + 2(2^kM_k) \\
&= n2^{k+1}M_{k+1}N_{k+1}^{-1} + 7(2^kM_k) \\
&\leqq na(N_{k+1})N_{k+1}^{-1} + 7a(N_k) \qquad (45) \\
&\leqq a(n) + 7k^{-1}a(\,[\,(k+1)^{-2}m_{k+1}^{-1}N_{k+1}]\,) \\
&\leqq a(n)(1 + 7k^{-1}),
\end{aligned}
$$

for n satisfying (44). Combining (40), (43) and (45) yields (24).

Step 5. *Proof of* (26). By (31), we may choose an increasing subsequence $\{\nu_k, k \geqq 1\}$ from the sequence $\{\mu_i, i \geqq 1\}$ such that

(46) $\quad (1 - k^{-1})N_k \leqq \nu_k \leqq (1 - (2k)^{-1})N_k$

for $k \geqq 1$. For each k, let E_k be the event that $B = \{1, 2, \dots, \nu_k\}$ is contained in a single k-block. The second inequality in (46) assures us that B intersects at most $(1 - (2k)^{-1})n_k + 2$ $(k - 1)$-blocks. Thus E_k occurs if 1 lies in any of the first $(2k)^{-1}n_k - 1$ $(k - 1)$-blocks of some k-block, i.e., if $(T\omega)_k$ takes any of the values $0, 1, \dots, (2k)^{-1}n_k - 2$. This latter event has probability

$$((2k)^{-1}n_k - 1)n_k^{-1} > (4k)^{-1}$$

by (29). Since $(T\omega)_1, (T\omega)_2, \dots$ are independent, we deduce from the Borel-Cantelli Lemma that

(47) $\quad P(E_k \text{ for infinitely many } k) = 1.$

Now suppose E_k holds for some large k. Let n be an integer such that

$$\delta\nu_k \leqq n \leqq \nu_k,$$

and let $B = \{1, 2, \dots, n\}$. Then B contains at least $nN_{k-1}^{-1} - 2$ $(k - 1)$-blocks and at most two additional partial $(k - 1)$-blocks. By (34), B contains at least

$$(nN_{k-1}^{-1} - 3)m_kn_k^{-1} - 1 \geqq nm_kN_k^{-1} - 2$$

reversed $(k - 1)$-blocks. By (35) and (37),

$$
\begin{aligned}
|S_n| &\geqq (2(nm_kN_k^{-1} - 2) - 1)2^{k-1}M_{k-1} - 2\,(2^{k-1}M_{k-1}) \\
&\geqq 2^knM_kN_k^{-1} - 7(2^{k-1}M_{k-1}).
\end{aligned}
$$

Applying (28), (32) and the fact that $n \geqq \frac{1}{2}\delta N_k$, we have

$$|S_n| \geqq na(N_k)N_k^{-1}(1 + o(1)\,),$$

where $o(1) \to 0$ as $k \to \infty$. By (20) and (31), we then deduce that

$$\begin{aligned} n^{-1}a_n^{-1}\nu_k|S_n| &\geqq n^{-1}(a(N_k))^{-1}(1 - k^{-1})N_k|S_n| \\ &\geqq (1 - k^{-1})(1 + o(1)) \\ &= 1 + o(1). \end{aligned}$$

By (47), we thus have

(48) $$\limsup_{k\to\infty} \min \{n^{-1}a_n^{-1}\nu_k|S_n|:\delta\nu_k \leqq n \leqq \nu_k\} \geqq 1 \text{ a.s.}$$

Since $\{Y_n, n \in \mathbf{Z}\}$ is symmetric, the inequality in (26) must hold with probability at least one half. Since $a_n \to \infty$, the left side of (26) is in the σ-field of shift-invariant events. Since $\{Y_n\}$ is ergodic by Theorem 2, (26) must hold. This completes the proofs of Theorem 3 and Theorem 1.

4. Two-dimensional arrays. A two-dimensional array is defined to be a collection $\{Y_{r,n}, r \in \mathbf{N}, n \in \mathbf{Z}\}$ of random variables such that the sequences $\{Y_{r,n}, n \in \mathbf{Z}\}$, $r \in \mathbf{N}$, are independent and identically distributed. Given such an array, let

$$S_{r,n} = S(r, n) = \sum_{i=1}^{n} Y_{r,i}.$$

It is our purpose to examine the following question: given a sequence $\{b_n = b(n), n \in \mathbf{N}\}$ of positive real numbers such that (3) and (4) hold and given an increasing sequence $\{\mu_r, r \in \mathbf{N}\}$ of positive integers, does there exist a two-dimensional array $\{Y_{r,n}, r \in \mathbf{N}, n \in \mathbf{Z}\}$ such that for each r the sequence $\{Y_{r,n}\}$ is $\{-1, 1\}$-valued, symmetric, stationary and ergodic and such that

(49) $$\limsup_{r\to\infty} (b(\mu_r))^{-1}S(r, \mu_r) = 1 \text{ a.s. ?}$$

Our purpose is primarily that we need a positive answer to this question for our aforementioned application to self-similar processes, but we note that two-dimensional arrays (and more particularly triangular arrays with $\mu_r = r$) have received much attention in connection with the law of the iterated logarithm. Our methods do not seem to provide an answer to the question as asked, so we impose some additional conditions. First, we simplify matters by assuming that $\{b_n, n \in \mathbf{N}\}$ satisfies the monotonicity requirements

(50) $$b_n \uparrow \infty \quad \text{and} \quad n^{-1}b_n \downarrow 0.$$

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This means that we can go directly to the analogue of Theorem 3, bypassing Step 1 of the proof of Theorem 1. Second, we impose a roughly geometric minimum rate-of-growth condition on $\{\mu_r, r \geqq \mathbf{N}\}$. Assume that for some constant K,

$$(51) \quad \sum_{i=1}^{r} \mu_i \leqq K\mu_r, \quad r \geqq 1.$$

Under these assumptions we can obtain a result which is sufficient for our needs and which can be proved by making minor modifications to the proof of Theorem 1.

THEOREM 4. *Let* $\{b_n, n \in \mathbf{N}\}$ *satisfy* (50) *and let* $\{\mu_r, r \in \mathbf{N}\}$ *satisfy* (51) *for some* K. *Let* $\delta \in (0, 1)$. *Then there exists a two-dimensional array* $\{Y_{r,n}\}$ *such that for each* r *the sequence* $\{Y_{r,n}, n \in \mathbf{N}\}$ *is* $\{-1, 1\}$*-valued, symmetric, ergodic and stationary and such that*

$$(52) \quad \limsup_{r\to\infty} \max \{ (b(\mu_r))^{-1} S_{r,n} : \delta\mu_r \leqq n \leqq \mu_r\} \leqq 1 \text{ a.s., } \textit{and}$$

$$(53) \quad \limsup_{r\to\infty} \min \{ (nb_n)^{-1}\mu_r S_{r,n} : \delta\mu_r \leqq n \leqq \mu_r\} \geqq 1 \text{ a.s.}$$

Proof. Construct the array $\{Y_{r,n}, r \in \mathbf{N}, n \in \mathbf{Z}\}$ by constructing $\{Y_{1,n}, n \in \mathbf{N}\}$ as in the proof of Theorem 3. Make the obvious notational changes. Most of the bounds in Steps 3-5 of that proof are non-random and hence carry over directly. The only difficulties occur at the two points involving the Borel-Cantelli Lemma. The second of these points presents no problem: the assumed independence of $\{Y_{r,n}, n \in \mathbf{N}\}$ for different values of r actually simplifies the proof somewhat. We will provide the details only for the modification required when the Borel-Cantelli Lemma is applied in Step 4.

Fix k and suppose r is such that (38) holds for some n between $\delta\mu_r$ and μ_r. The block $D_r = \{1, 2, \dots, \mu_r\}$ is composed of at most $\mu_r N_k^{-1}$ k-blocks and at most two partial k-blocks. Thus, the probability that D_r intersects any of the reversed k-blocks is at most

$$(\mu_r N_k^{-1} + 2)(n_{k+1}m_{k+1}^{-1} - 3)^{-1} \leqq 4\mu_r N_{k+1}^{-1} m_{k+1}$$

for large k. Summing over all r as described above, for which all μ_r are at most

$$\delta^{-1}(k+1)^{-2} m_{k+1}^{-1} N_{k+1},$$

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we see that the probability that D_r intersects any reversed k-blocks for any r is at most

$$(K\delta^{-1}(k + 1)^{-2}m_{k+1}^{-1}N_{k+1})(4m_{k+1}N_{k+1}^{-1}) = 4K\delta^{-1}(k+1)^{-2}$$

for large k. With probability 1, therefore, there are only finitely many k's for which any D_r intersects a reversed k-block.Thus, there are only finitely many r's for which D_r intersects a reversed k-block, where k is determined by (38) and $\delta\mu_r \leqq n \leqq \mu_r$.

Remark. If we drop the requirement that (51) holds, we can obtain a similar result to Theorem 4 but with a weaker conclusion. If n satisfies (38) for some k, then $B = \{1, 2, \dots, n\}$ intersects at most one reversed k-block. It follows by an argument like that leading to (40) that then

$$|S_{r,n}| \leqq 3(2^k M_k) \leqq 3b_n.$$

We see that the conclusions of Theorem 4 hold with the right side of the inequality in (52) replaced by 3.

5. Complements. One of the properties of the random sequences $\{Y_n, n \in \mathbf{Z}\}$ constructed in Section 3 is that

(54) $\quad P(S_{2N(k)} \neq 0 \text{ for infinitely many } k) = 0 \text{ a.s.}$

This is easily deduced from the arguments leading up to (40).

One consequence of (54) is that we may modify the sequence $\{b_n\}$ in such a way that lim inf $b_n = 0$; specifically, let $b_n = k^{-1}$ if $n = 2N_k$ and leave b_n unchanged otherwise. The same sequence $\{Y_n\}$ still satisfies the requirements of Theorem 1. Thus (3) is not necessary.

Let us suppose now that $\{b_n\}$ is a sequence of positive real numbers such that

(55) $\quad \limsup_{n\to\infty} b_n < \infty$

and that $\{Y_n\}$ is integer-valued. Since the sequence $\{b_n\}$ is bounded, it is clear that, if (2) holds, then $\{S_n, n \geqq 1\}$ is bounded above and $\{S_n, n \geqq 0\}$ achieves its maximum a.s. By applying the stationarity of $\{Y_n\}$, it is clear that

$$P(S_n \leqq 0, n \geqq 0)$$
$$= P(\{S_n, n \geqq 0\} \text{ achieves its maximum at } n = 0) > 0.$$

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Thus (2) is impossible. If (55) holds and $\{Y_n\}$ is real-valued, then

$$P(\limsup_{n\to\infty} S_n < \epsilon) > 0 \quad \text{for every } \epsilon > 0,$$

so that a necessary condition for (2) is that

$$\limsup_{n\to\infty} b_n = \infty \quad \text{or} \quad \liminf_{n\to\infty} b_n = 0.$$

The following example shows that (55) does not preclude (2). Let $b_n = 2^{-i}$ if $n = 2^i k$ where k is an odd integer and i is a non-negative integer. Obviously,

$$\limsup b_n = 1 < \infty.$$

Let $U_1, U_2, \dots$ be independent random variables with

$$P(U_n = 0) = P(U_n = 1) = \tfrac{1}{2} \quad \text{for all } n.$$

Divide $\mathbf{Z}$ into juxtaposed blocks, each composed of two successive integers, so that each block has the form $\{2n - 1, 2n\}$ if $U_1 = 0$ and each has the form $\{2n, 2n + 1\}$ if $U_1 = 1$. In each block $\{n, n + 1\}$, $Y_n > 0$ and $Y_{n+1} = -Y_n$. Every second block, using U_2 to decide which ones, has the values $Y_n = \frac{1}{2}$ and $Y_n = +1$ and $Y_{n+1} = -\frac{1}{2}$. Of the remaining blocks, every second one has the values $\frac{3}{4}$ and $-\frac{3}{4}$. Of those still remaining, every second one has the values $\frac{7}{8}$ and $-\frac{7}{8}$, and so on. If $U_1 = 0$, then $S_{2n} = 0$ for all n and S_{2n-1} takes each value $1 - 2^{-k}$ infinitely often, $k = 1, 2, \dots$. Since $b_{2n-1} = 1$ for all n, we have (2) when $U_1 = 0$. Now suppose $U_1 = 1$. Then $Y_1 = -1 + 2^{-i}$ for some $i \geqq 1$. It can be seen that $S_n \leqq 0$ except for n of the form $k2^i$ for k odd. If $n = k2^i$ for odd k, then S_n takes each value $2^{-i} - 2^{-j}$, $j > i$, infinitely often, so that

$$\limsup_{n\to\infty} b_n^{-1} S_n = \limsup_{\substack{k\to\infty \\ k \text{ odd}}} (b_{k2^i})^{-1} S_{k2^i} = \sup_j (2^{-i})^{-1}(2^{-i} - 2^{-j}) = 1.$$

Thus (55) is not incompatible with (2). We note however that this example does not have the symmetry property of Theorem 1.

It follows from the construction in Section 3 that the sequence $\{b_n^{-1} S_n\}$ is tight; in fact

$$P(|b_n^{-1} S_n| > 1 + \epsilon) \to 0 \quad \text{for every } \epsilon > 0.$$

Thus, there exists at least one weakly convergent subsequence, whose limit distribution is concentrated on $[-1, 1]$. By (54), there is a subsequence which converges to 0 (even almost surely). It can be seen from the arguments in Step 5 of the proof of Theorem 3 that

$$P(b_{\frac{1}{2}N(k)}^{-1} S_{\frac{1}{2}N(k)} \geqq \tfrac{1}{2}) \geqq \tfrac{1}{4}$$

for all k, so that there is a subsequence which converges to a distribution Q with $Q([\frac{1}{2}, 1]) \geqq \frac{1}{4}$. Thus there are subsequences of $\{b_n^{-1} S_n\}$ with a non-degenerate weak limit. Note the contrast with the i.i.d. case, where the law of the interated logarithm has a larger denominator than the central limit theorem.

It was pointed out to me by W. Vervaat that the dynamical system (i.e., $(\Omega, \mathscr{F}, P)$ and T) introduced in Section 2 can be viewed in the following way. The set $\Omega' = A_1 \times A_2 \times \ldots$ is a group under component-wise addition, with carrying in the sense described in Section 2. Then $T': \Omega' \to \Omega'$, obtained from T by ignoring A_0, simply adds $(1, 0, 0, \ldots)$. The ergodicity of T' can then be obtained from the general theory of topological dynamics on groups. See for example Walters (1975). This provides an alternative to our use of Kolmogorov's $0 - 1$ law.

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