Canad. Math. Bull. Vol. **31** (4), 1988

# APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. The paper gives a good approximation of a $C^k$ function on $\mathbf{R}^n$ and its derivatives by the restriction of an entire function on $\mathbf{C}^n$ and its derivatives respectively.

In this paper we deal with approximation of a $C^k$ function on $\mathbf{R}^n$ and its derivatives by the restriction of an entire function and its derivatives respectively in a way which is better than uniform approximation. As a particular case we obtain, for $k = 0$, Carleman's theorem [3]. Similar results were given in the case $n = 1$ by Hoischen [1].

This work was inspired by the proof of Whitney's theorem by Narasimhan, (see [2]).

We shall use the following notations. $H(\mathbf{C}^n)$ will denote the space of entire functions. If

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n,\ |\alpha| \text{ denotes } \sum_{i=1}^{n} \alpha_i \text{ and } D^\alpha = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}.$$

For $S \subset \mathbf{R}^n$, $k$ a non negative integer and $f \in C^k(\mathbf{R}^n)$, we set

$$\|f\|_k^s = \sum_{|\alpha| \leqq k} \frac{1}{\alpha!} \sup_{x \in S} |D^\alpha f(x)|.$$

Note that if $f, g \in C^k(\mathbf{R}^n)$ we have

$$\|fg\|_k^s \leqq \|f\|_k^s \|g\|_k^s.$$

THEOREM. For some $0 \leqq k \leqq \infty$, let $f$ and $\epsilon$ be such that, $f \in C^k(\mathbf{R}^n)$, $\epsilon \in C(\mathbf{R}^n)$ and $\epsilon$ is positive. Let $(K_p)$ be a sequence of compact sets in $\mathbf{R}^n$ with $K_0 = \phi$, $K_p \subset K_{p+1}^0$ and $\cup K_p = \mathbf{R}^n$. Let $(n_p)$ be a sequence of non negative integers and set $k_p = \min(k, n_p)$. Then, there exists $g \in H(\mathbf{C}^n)$ such that

$$|D^\alpha f(x) - D^\alpha g(x)| < \epsilon(x), \text{ for } x \in \mathbf{R}^n \backslash K_p^0 \text{ and } |\alpha| \leqq k_p,$$

Received by the editors July 30, 1987, and, in revised form, October 27, 1987.
Research supported by NSERC of Canada and FCAR of Québec.
AMS Subject Classification (1980): 32A15, 41A30.




https://doi.org/10.4153/CMB-1988-071-1 Published online by Cambridge University Press

for every $p \geqq 0$.

For the proof we shall use the following lemma which can be found in ( [2], p. 31).

LEMMA. *Let* $f \in C_0^k(\mathbf{R}^n)$, $0 \leqq k < \infty$. *For* $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ *and* $\lambda > 0$ *set*,

$$G_\lambda(f)(z) = \pi^{-(n/2)}\lambda^{(n/2)} \int_{\mathbf{R}^n} f(t) \exp\left[-\lambda \sum_{i=1}^{n} (z_i - t_i)^2\right] dt.$$

*Then we have*

(a) $$G_\lambda(f) \in H(\mathbf{C}^n)$$

(b) $$\|G_\lambda(f) - f\|_k^{\mathbf{R}^n} \to 0, \text{ as } \lambda \to \infty.$$

PROOF OF THE THEOREM. Set $Ap = \overline{K_{p+1} \setminus K_p}$, for $p \geqq 0$. There exists a sequence of positive real numbers $(\epsilon_p)$ such that $\epsilon_p < \epsilon(x)$ for $x \in A_p$ and $\epsilon_{p+1} \leqq \epsilon_p/2$ for every $p \geqq 0$. It suffices then to prove that there exists $g \in H(\mathbf{C}^n)$ so that:

$$\|f - g\|_{k_p}^{A_p} < \delta_p, \text{ for every } p \geqq 0, \text{ where } \delta_p = \frac{\epsilon_p}{k_p!}.$$

We may suppose $k_{p+1} \geqq k_p$. Then we have

(1) $$\delta_{p+1} < \frac{1}{2}\delta_p.$$

Let $\varphi_p \in C_0^\infty(\mathbf{R}^n)$, $\varphi_p \equiv 0$ in a neighborhood of $K_{p-1}$ and $\varphi_p \equiv 1$ in a neighborhood of $A_p$. Set

$$C_p = 1 + \|\varphi_p\|_{k_p}^{\mathbf{R}^n}.$$

By the lemma, there exists $\lambda_0 > 0$ such that if $g_0 = G_{\lambda_0}(\varphi_0 f)$ we have

$$\|g_0 - \varphi_0 f\|_{k_0}^{K_1} < \frac{\delta_0}{5C_1}.$$

By induction, we define numbers $\lambda_i$ and functions $g_i$ in the following way. Suppose we have defined $\lambda_i$ and $g_i$ for $i \leqq p - 1$; then using the lemma, there is a function $l_p(\lambda_i, g_i)i \leqq p - 1$ such that if $\lambda_p > l_p(\lambda_i, g_i)$ and if we set $g_p = G_{\lambda_p}[\varphi_p(f - g_0 - \ldots - g_{p-1})]$, then

(2) $$\left\|g_p - \varphi_p\left(f - \sum_{i=0}^{p-1} g_i\right)\right\|_{k_p}^{K_{p+1}} < \frac{\delta_p}{5C_{p+1}}.$$

The function $g_p$ depends on $\lambda_p$ and $g_i$, $i \leqq p - 1$; this implies that the function $l_p$ depends only on $\lambda_i$, $i \leqq p - 1$. Since $\varphi_p \equiv 0$ (respectively $\varphi_p \equiv 1$)

https://doi.org/10.4153/CMB-1988-071-1 Published online by Cambridge University Press

in a neighborhood of $K_{p-1}$ (respectively in a neighborhood of $A_p$) we get from (2):

$$\|g_p\|_{k_p}^{K_{p-1}} < \frac{\delta_p}{5C_{p+1}}, \tag{3}$$

$$\left\| f - \sum_{i=0}^{p} g_i \right\|_{k_p}^{A_p} < \frac{\delta_p}{5C_{p+1}}. \tag{4}$$

Replacing $p$ by $p + 1$ in (2) and, using (1), (4) and the fact that $C_p \geqq 1$, we have

$$\begin{aligned}\|g_{p+1}\|_{k_p}^{A_p} &\leqq \left\| \varphi_{p+1}\left(f - \sum_{i=0}^{p} g_i\right) \right\|_{k_p}^{A_p} + \left\| g_{p+1} - \varphi_{p+1}\left(f - \sum_{i=0}^{p} g_i\right) \right\|_{k_p}^{A_p} \\ &< \|\varphi_{p+1}\|_{k_p}^{\mathbf{R}^n} \frac{\delta_p}{5C_{p+1}} + \frac{\delta_{p+1}}{5C_{p+2}} \\ &< C_{p+1}\frac{\delta_p}{5C_{p+1}} + \frac{\delta_{p+1}}{5C_{p+2}} < \frac{3\delta_p}{10}.\end{aligned}$$

With (3), this gives

$$\|g_{p+1}\|_{k_p}^{K_{p+1}} < \frac{2}{5}\delta_p. \tag{5}$$

We show now that the $\lambda_p$ can be chosen so that the series $\sum_{i \geqq 0} g_i(z)$ defines an entire function. Since $\cup K_p = \mathbf{R}^n$, there is a $p_0$ such that $0 \in K_{p_0}$. Let $(B_{r_p})$ be a sequence of balls with radius $r_p$ centred at the origin such that $B_{r_p} \subset K_p$ for $p \geqq p_0$ and $r_p \to \infty$ as $p \to \infty$. Let $z \in \mathbf{C}^n$ and $R \geqq |z|$. Choose $p \geqq p_0$ such that

$$1 - \frac{R^2}{r_p^2} - \frac{2R}{r_p} > \frac{1}{2}.$$

For $q > p$, supp $\varphi_q \subset \mathbf{R}^n \backslash K_p \subset \mathbf{R}^n \backslash B_{r_p}$.

By definition

$$g_q(z) = \pi^{-n/2}\lambda^{n/2} \int_{\mathrm{supp}\varphi_q} \varphi_q\left(f - \sum_{i=0}^{q-1} g_i\right) e^{-\lambda(z-t)^2} dt,$$

where

$$z = (z_1, \ldots, z_n),\ t = (t_1, \ldots, t_n) \quad \text{and} \quad (z - t)^2 = \sum_{i=1}^{n} (z_i - t_i)^2.$$

From the inequality

$$|e^{-\lambda_q(z-t)^2}| \leqq e^{-\lambda_q(|t|^2-|z|^2-2|z||t|)}$$

and since $|t| \geqq r_p > 0$ for $q > p$ we obtain

$$|g_q(z)| \leqq \pi^{-n/2}\lambda_q^{n/2} \int_{\text{supp}\varphi_q} \left|\varphi_q\left(f - \sum_{i=0}^{q-1} g_i\right)\right| e^{-\lambda_q r_p^2(1-(R^2/r_p^2)-(2R/r_p))}dt.$$

Thus for $q > p$ we have

$$|g_q(z)| \leqq \pi^{-n/2}\lambda_q^{n/2}e^{-\lambda_q(r_p^2/2)}M_q,$$

where

$$M_q = \int_{\text{supp}\varphi_q} \left|\varphi_q\left(f - \sum_{i=0}^{q-1} g_i\right)\right| dt.$$

Since $M_q$ depends only on $\lambda_1, \ldots, \lambda_{q-1}$, we can choose inductively the $\lambda_q$ such that

$$\lambda_q^{n/2}e^{-\lambda_q/q}M_q < \frac{1}{q^2}$$

and this implies that

$$\sum \lambda_q^{n/2}e^{-\lambda_q\sigma}M_q < \infty \text{ for any } \sigma > 0.$$

Thus the series $\sum_{i\geqq 0} g_i(z)$ converges uniformly on compact sets of $\mathbf{C}^n$.

Let $g(z) = \sum_{i\geqq 0} g_i(z)$. Then using (1), (4), (5) and $C_p \geqq 1$, we have

$$\begin{aligned} \|f - g\|_{k_p}^{A_p} &\leqq \left\|f - \sum_{i=0}^{p} g_i\right\|_{k_p}^{A_p} + \sum_{i>p} \|g_i\|_{k_p}^{A_p} \\ &\leqq \frac{\delta_p}{5} + \sum_{i\geqq p} \frac{2}{5}\delta_i \\ &\leqq \frac{\delta_p}{5} + \frac{2}{5}\delta_p \sum_{i=0}^{\infty} \frac{1}{2^i} \\ &\leqq \delta_p. \end{aligned}$$

COROLLARY. *Let $f \in C^k(\mathbf{R}^n)$, $0 \leqq k < \infty$, $\epsilon \in C(\mathbf{R}^n)$ and $\epsilon$ positive. Then there exists $g \in H(\mathbf{C}^n)$ satisfying*

$$|D^\alpha f(x) - D^\alpha g(x)| < \epsilon(x) \text{ for all } x \in \mathbf{R}^n \text{ and } |\alpha| \leqq k.$$

PROOF. Take $n_0 = k$ in the theorem.

REMARKS. (1) The corollary would, of course, fail for $k = +\infty$. The counterexample given by Hoischen [1] in one variable works also in several variables. Take $x_0 \in \mathbf{R}^n$. There is a function $f \in C^\infty(\mathbf{R}^n)$ such that $D^\alpha f(x_0) = (\alpha!)^2 + 1$. If there is $g \in H(C^n)$ satisfying $|D^\alpha f(x) - D^\alpha g(x)| < 1$ for all $x \in \mathbf{R}^n$ and all $\alpha$, we get $|D^\alpha g(x_0)| > |D^\alpha f(x_0)| - 1 = (\alpha!)^2$ and then

$$\frac{|D^\alpha g(x_0)|}{\alpha!} > \alpha!.$$

This implies that $g$ cannot be representable as an absolutely convergent power series in a neighbourhood of $x_0$, and this is a contradiction.

(2) If $f$ is real valued, in the theorem and the corollary, $g$ can be chosen to take real values on $\mathbf{R}^n$.

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