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MICROLOCAL REGULARITY ON STEP TWO NILPOTENT LIE GROUPS

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Introduction

A necessary and sufficient condition for a homogeneous left invariant partial differential operator P on a nilpotent Lie group G to be hypoelliptic is that $\pi(P)$ be injective in $\mathcal{S}_\pi$ for every nontrivial irreducible unitary representation π of G. This was conjectured by Rockland in **[18]**, where it was also proved in the case of the Heisenberg group. The necessity of the condition in the general case was proved by Beals **[2]** and the sufficiency by Helffer and Nourrigat **[4]**. In this paper we present a microlocal version of this theorem when G is step two nilpotent. The operator may be homogeneous with respect to any family of dilations on G, not just the natural dilations. We may also consider pseudodifferential operators as well as partial differential operators.

Throughout the paper G is assumed to be a connected, simply connected nilpotent Lie group and is step two nilpotent unless otherwise stated. Let $\mathcal{G}$ be the Lie algebra of G, let $\mathcal{G}^*$ be the dual of $\mathcal{G}$ and $\dot{\mathcal{G}}^*=\mathcal{G}^*-\{0\}$. If Ω is an open subset of G, a pseudodifferential operator P on Ω is said to be microhypoelliptic (or microlocally hypoelliptic) at $(g,\xi)\in\Omega\times\dot{\mathcal{G}}^*$ if $(g,\xi)\in WF(Pu)$ for every $u\in\mathcal{D}'(\Omega)$ for which $(g,\xi)\in WF(u)$. In this paper, for the most part, rather than considering the usual wave front set, $WF(u)$, we will consider a variant of the wave front set, $WF_\delta(u)$, based on sets which are conic with respect to some family $\delta=\{\delta_r : r>0\}$ of dilations on G, and the corresponding notion of δ-microhypoellipticity. On $\mathbb{R}^n$ similar types of wave front have been considered by Lascar **[8]** and Parenti and Rodino **[15]**, among others. On a Lie group the definitions need to be modified somewhat to take into account the action of G on $\mathcal{G}^*$. Details are in Section 2.

Given $\xi\in\mathcal{G}^*$ let $\mathcal{O}_\xi$ be the orbit of ξ under the coadjoint action of G on $\mathcal{G}^*$ and let π_ξ be the irreducible unitary representation corresponding to ξ in the Kirillov theory. Let $\tilde{\mathcal{G}}^*$ be the set of $\xi\in\mathcal{G}^*$ for which $\dim\mathcal{O}_\xi$ is maximal. $\tilde{\mathcal{G}}^*$ is an open dense subset of $\mathcal{G}^*$. If δ is a family of dilations on G, a subset Γ of $\dot{\mathcal{G}}^*$ is said to be a δ-cone if it is invariant under δ. $\Gamma\subset\mathcal{G}^*$ is said to be G-invariant if it is invariant under the coadjoint action of G on $\mathcal{G}^*$, i.e. if Γ is a union of orbits. Let Γ_ξ be the smallest G-invariant δ-cone containing ξ.

If P is a left invariant operator which is homogeneous with respect to the dilations δ, then the injectivity of $\pi_\xi(P)$ implies the injectivity of $\pi_\eta(P)$ for all $\eta\in\Gamma_\xi$. One might therefore expect that the injectivity of $\pi_\xi(P)$ is related to δ-microhypoellipticity on Γ_ξ. Rather than considering just Γ_ξ it is helpful to consider $\bar{\Gamma}_\xi=(\text{closure of }\Gamma_\xi)-\{0\}$. We note (Proposition 3.2) that $\partial\Gamma_\xi=\bar{\Gamma}_\xi-\Gamma_\xi\subset[\mathcal{G},\mathcal{G}]^\perp$, (for standard dilations $\partial\Gamma_\xi=R_\xi^\perp$, where R_ξ is the radical of the bilinear form associated with ξ), and hence π_η is a one dimensional representation for $\eta\in\partial\Gamma_\xi$. The following two theorems are the main results of the paper. In both G is a step two group with a family of dilations δ.

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Theorem. *Let P be a left invariant pseudodifferential operator on G which is homogeneous with respect to δ. If P is δ-microhypoelliptic at (e,η) for every $\eta \in \bar{\Gamma}_\xi$, then $\pi_\xi(P)$ is injective in $\mathcal{S}_\pi$.*

Theorem. *Let P be a pseudodifferential operator on $\Omega \subset G$. Let P_g^0 be the principal part of the invariant operator P_g obtained by "freezing the coefficients" of P at g. Let Γ be an open δ-conic subset of $\Omega \times \dot{\mathcal{G}}^*$. If, for every $(g,\xi) \in \Gamma$, $\pi_\eta(P_g^0)$ is injective for all $\eta \in \bar{\Gamma}_\xi$, then P is δ-microhypoelliptic on Γ.*

The first theorem is proved in Section 3, the second is stated more precisely and proved in Section 4. These results were announced in slightly less generality in [**14**].

The second result can be improved somewhat when the dilations are natural dilations on G, i.e. when $\delta_r x = r^2 x$ for $x \in [\mathcal{G},\mathcal{G}]$ and $\delta_r x = rx$ for x in some supplement $\mathcal{G}_1$ of $[\mathcal{G},\mathcal{G}]$. In Corollary 4.5 it is shown that for natural dilations in order to prove δ-microhypoellipticity near (g_0,ξ_0) it is not necessary to assume injectivity of $\pi_\xi(P_g^0)$ for all (g,ξ) in a δ-conic neighborhood of (g_0,ξ_0), but simply to assume the injectivity of $\pi_\eta(P_{g_0}^0)$ for all $\eta \in \bar{\Gamma}_{\xi_0}$. This generalizes a result proved by Grigis [**3**] for operators of order 2. Corollary 4.6 gives sufficient conditions for microhypoellipticity with respect to standard conic sets.

At least for partial differential operators the parametrix construction of Melin [**10**] shows that if P is left invariant and homogeneous on a nilpotent Lie group G (with no restriction on the nilpotence step) and if $\pi(P)$ is injective for all nontrivial irreducible unitary representations π of G, then P is globally microhypoelliptic in the standard sense, i.e. microhypoelliptic at (g,ξ) for all $g \in G$ and $\xi \in \dot{\mathcal{G}}^*$. The parametrix construction in [**10**] makes use of global *a priori* estimates proved by Helffer and Nourrigat [**4**] under the assumption of the injectivity of $\pi(P)$ for all nontrivial π. In order to construct a microlocal parametrix under the weaker hypotheses stated above, we use a different method. The construction, which is a refinement of that used in [**13**], makes use of the fact that the calculus for invariant operators on a step two group "fibres" over the orbits and the orbit level calculus is the Weyl calculus [**12**]. This allows us to construct the parametrix on the orbits individually in terms of the symbols of the inverses of the operators $\pi(P)$.

1. Dilations and pseudodifferential operators

A family of dilations on $\mathcal{G}$ is a continuous one parameter family $\delta = \{\delta_r : r > 0\}$ of simultaneously diagonalizable automorphisms of $\mathcal{G}$ with positive eigenvalues such that $\delta_r \delta_s = \delta_{rs}$ for all $r,s > 0$ and such that $\lim_{r \to 0} \delta_r x = 0$ for all $x \in \mathcal{G}$. For each r define $\delta_r : G \to G$ by $\delta_r g = \exp \delta_r \log g$ and define $\delta_r : \mathcal{G}^* \to \mathcal{G}^*$ to be the transpose of $\delta_r : \mathcal{G} \to \mathcal{G}$.

If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis of eigenvectors for $\{\delta_r : r > 0\}$, then there are $\mu_j > 0$ such that

$$\delta_r e_j = r^{\mu_j} e_j. \tag{1.1}$$

Without loss of generality we may assume that $\min \mu_j = 1$. Let $\bar{\mu} = \max \mu_j$. It can be easily shown (Lemma 1.2 of [**11**]) that there is a set $S = \{e_1, \ldots, e_N\}$ of linearly independent eigenvectors for $\{\delta_r : r>0\}$ which generate $\mathscr{G}$ and such that $\mathscr{G}_1=\operatorname{span} S$ intersects $\mathscr{G}_2=[\mathscr{G},\mathscr{G}]$ trivially. Let $\{e_{N+1},\dots,e_n\}$ be a basis for $\mathscr{G}_2$ chosen so that each $e_k, k>N$, is a multiple of $[e_i,e_j]$ for some $i<j\leqq N$. Since $\{\delta_r : r>0\}$ is a family of automorphisms, each $e_k, k>N$, is also an eigenvector for δ_r. If the numbers γ_{ij}^k are defined by

$$[e_i,e_j]=\sum \gamma_{ij}^k e_k \tag{1.2}$$

and if the numbers μ_j are defined by (1.1), then

$$\gamma_{ij}^k\neq 0 \quad \text{implies} \quad \mu_i+\mu_j=\mu_k. \tag{1.3}$$

For $x\in\mathscr{G}$ let $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$, where $(x_1,\dots,x_n)$ are the coordinates of x with respect to the basis $\{e_1,\dots,e_n\}$. By replacing each e_k, $N<k\leqq n$, by ce_k for sufficiently large c we may assume that

$$|[x,y]|\leqq|x|\,|y|, \quad \text{for all } x \text{ and } y \text{ in } \mathscr{G}. \tag{1.4}$$

We fix a basis $\mathscr{B}=\{e_1,\dots,e_n\}$ for $\mathscr{G}$ having the properties just described. Coordinates and norms on $\mathscr{G}$ and G^* will always be with respect to this basis or its dual $\{e_1^*,\dots,e_n^*\}$. If α is a multi-index, let $\mu\alpha=\sum\mu_j\alpha_j$.

For $\xi\in\mathscr{G}^*-\{0\}$, define $[\xi]$ by $[\xi]=r$ if $|\delta_r^{-1}\xi|=1$. Note that in terms of the chosen coordinate system

$$[\xi]\approx\sum|\xi_j|^{1/\mu_j}. \tag{1.5}$$

Let $\chi:\mathscr{G}^*\to\mathbb{R}$ be a smooth function such that $\chi(\xi)\approx[\xi]+1$.

Definition. Let δ be a family of dilations on $\mathscr{G}$ and let $m\in\mathbb{R}$. $S^m(\mathscr{G}^*,\delta)$ is the set of $p\in C^\infty(\mathscr{G}^*)$ such that for every multi-index α there is a C_α such that

$$|D^\alpha p(\xi)|\leqq C_\alpha\chi(\xi)^{m-\mu\alpha} \tag{1.6}$$

for all $\xi\in\mathscr{G}^*$. If Ω is an open subset of $G, S^m(\Omega\times\mathscr{G}^*,\delta)$ is the set of $p\in C^\infty(\Omega\times\mathscr{G}^*)$ such that for every compact $K\subset\Omega_1=\log\Omega$ and all multi-indices α and β there is a $C_{\alpha\beta K}$ such that

$$|D_x^\beta D_\xi^\alpha p(\exp x,\xi)|\leqq C_{\alpha\beta K}\chi(\xi)^{m-\mu\alpha}, \quad \text{for all } (x,\xi)\in K\times\mathscr{G}^*.$$

If $p\in S^m(\mathscr{G}^*,\delta)$, define the left invariant operator $P=\operatorname{Op}(p)$ by

$$Pu=u*F_1^{-1}p, \quad u\in\mathscr{S}(G), \tag{1.7}$$

where $*$ denotes convolution on G and $F_1^{-1}p=F^{-1}p\circ\log$, $F:\mathscr{S}^*(\mathscr{G})\to\mathscr{S}^*(\mathscr{G}^*)$ the

Euclidean Fourier transform. If $p \in S^m(\Omega \times \mathscr{G}^*, \delta)$ define

$$P = \mathrm{Op}(p): \mathscr{D}(\Omega) \to \mathscr{E}(\Omega) \quad \text{by}$$

$$Pu(g) = \mathrm{Op}(p_g)u(g),$$

where $p_g(\xi) = p(g, \xi)$ and $\mathrm{Op}(p_g)$ is defined by (1.7).

Note that if $p \in C^\infty(\mathscr{G}^*)$ is homogeneous of degree m with respect to δ for large ξ, i.e. $p(\delta_r \xi) = r^m p(\xi)$ for $|\xi| \geqq C$, then $p \in S^m(\mathscr{G}^*, \delta)$.

There are two asymptotic expansions which are important in the parametrix construction given in Section 4. The first is for the symbol $p \sqcap q = F_1(F_1^{-1}p * F_1^{-1}q)$ of a product $\mathrm{Op}(p)\,\mathrm{Op}(q)$ where $p \in S^{m_1}(\mathscr{G}^*, \delta)$, $q \in S^{m_2}(\mathscr{G}^*, \delta)$. Given $\xi \in \mathscr{G}^*$ let $\xi' = \xi|_{\mathscr{G}_2}$ and let

$$h(\xi) = |\delta_{\chi(\xi)}^{-1} \xi'|. \tag{1.8}$$

Following Melin [9], given $m \in \mathbb{R}$ and $k \geqq 0$ we define $S^{m,k}(\mathscr{G}^*, \delta)$ to be the set of $p \in C^\infty(\mathscr{G}^*)$ such that

$$|D^\alpha p(\xi)| \leqq C_\alpha h(\xi)^{\max\{k - |\alpha'|, 0\}} \chi(\xi)^{m - \mu\alpha}, \quad \xi \in \mathscr{G}^*. \tag{1.9}$$

Define the higher order brackets $\{p, q\}_j$ as in [12]. The following theorem can be proved using the Weyl Calculus of Hörmander [6] as in [12].

Theorem 1.1. *Let $p \in S^{m_1, k_1}(\mathscr{G}^*, \delta)$, $q \in S^{m_2, k_2}(\mathscr{G}^*, \delta)$. For any integer $J \geqq 0$,*

$$p \sqcap q = \sum_{j < J} (i/2)^j (j!)^{-1} \{p, q\}_j + r_J$$

where $r_J \in S^{m_1 + m_2, k_1 + k_2 + J}(\mathscr{G}^*, \delta)$.

The second asymptotic expansion, due to Michael Taylor [19], is for the symbol $p \# q$ of the product $\mathrm{Op}(p)\,\mathrm{Op}(q)$ where $p \in S^m(\Omega \times \mathscr{G}^*, \delta)$, $q \in S^k(\Omega \times \mathscr{G}^*, \delta)$. If α is a multi-index let $t_\alpha(\xi) = \xi^\alpha$, ξ^α being defined in terms of the chosen coordinate system, and let $T^\alpha = \mathrm{Op}(t_\alpha)$. If $p \in S^m(\Omega \times \mathscr{G}^*, \delta)$, let $T_{gl}^\alpha p$ refer to the function obtained by applying T^α to p as a function on Ω as ξ is held fixed. Define $p \sqcap q(g, \xi) = (p_g \sqcap q_g)(\xi)$.

Theorem 1.2 ([19]). *Let $p \in S^m(\Omega \times \mathscr{G}^*, \delta)$, $q \in S^k(\Omega \times \mathscr{G}^*, \delta)$ with $\mathrm{Op}(q)$ properly supported. Then*

$$p \# q = \sum_{|\alpha| < J} (i^\alpha / \alpha!) D_\xi^\alpha p \sqcap T_g^\alpha q + r_J$$

where $r_J \in S^{m+k-J}(\Omega \times \mathscr{G}^*, \delta)$.

Let $\xi \in \mathscr{G}^*$, let $\tilde{V}$ be a subspace of $\mathscr{G}$ maximally subordinate to ξ, V a supplement to $\tilde{V}$ in $\mathscr{G}$, and let $\pi = \pi_{\xi, V, \tilde{V}}$ be the irreducible unitary representation of G on $L^2(V)$ as defined in [4] or [13]. Let $\mathcal{O}_\pi$ be the orbit of the coadjoint action corresponding to π in

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the Kirillov theory. Let ψ_π be the symplectomorphism from $V\times V^*$ onto $\mathcal{O}_\pi$ defined in [12]. (ψ_π will sometimes be denoted ψ_ξ). If $p\in S^m(\mathcal{G}^*,\delta)$ and $P=\mathrm{Op}(p)$, define $\pi(P)$ to be the pseudodifferential operator on V with Weyl symbol p_π where

$$p_\pi = p\circ\psi_\pi. \tag{1.10}$$

It was shown in [12] that (*a*) $\pi(PQ)=\pi(P)\pi(Q)$; (*b*) $\pi(P)$ agrees with the usual definition of $\pi(P)$ when P is an invariant differential operator and (c) if $p\in\mathcal{S}(\mathcal{G}^*)$ then

$$\pi(P)=\pi(F_1 p). \tag{1.11}$$

Given $\xi\in\mathcal{G}_2^*$ let $d=d(\zeta)$ be the rank of the bilinear from $B_\zeta(x,y)=\langle\zeta,[x,y]\rangle$ on $\mathcal{G}\times\mathcal{G}$. As shown in [4] and [11], the irreducible unitary representations of G can be parametrized by $\zeta\in\mathcal{G}_2^*$ and $\rho\in\mathbb{R}^{N-2d}$, $N=\dim\mathcal{G}_1$: Every irreducible unitary representation of $\mathcal{G}$ is equivalent to exactly one of the representations $\pi_{\rho\zeta}$ as defined in [11]. Given a family of dilations it is convenient to replace $\pi_{\rho\zeta}$ for $\zeta\neq 0$ by $\pi_{\rho\zeta_0}\circ\delta_r$ where $r=[\zeta]$ and $\zeta_0=\delta_r^{-1}\zeta$, for we then have

$$\pi_{\rho\zeta}(P)=[\zeta]^m\pi_{\rho\zeta_0}(P) \tag{1.12}$$

if P is homogeneous with respect to δ. We will let $p_{\rho\zeta}=p_\pi$ and $\mathcal{O}_{\rho\zeta}=\mathcal{O}_\pi$ where $\pi=\pi_{\rho\zeta}$.

2. Microlocal analysis on G

If $k\in\mathbb{R}$ and $g\in G$, define $kg\in G$ by $kg=\exp(k\log g)$. For $a\in G$, define $\lambda^a\colon G\to G$ by $\lambda^a g=a^{-1}g$. We identify the tangent bundle TG with $G\times\mathcal{G}$. If G is step two nilpotent, then for any a and b in G, $(d\lambda^a)_b=\mathrm{Ad}(-\tfrac{1}{2}a)\colon\mathcal{G}\to\mathcal{G}$. If $\phi\colon M\to N$ is a diffeomorphism let ϕ_* denote the naturally induced map $\phi_*\colon T^*M\to T^*N$. Since G is step two nilpotent, if $a\in G$, $g\in G$ and $\xi\in\mathcal{G}^*$, then

$$\lambda_*^a(g,\xi)=(a^{-1}g,(\mathrm{Ad}\tfrac{1}{2}a)^*\xi),$$

since $(\mathrm{Ad}-\tfrac{1}{2}a)^{-1}=\mathrm{Ad}\tfrac{1}{2}a$.

Let $\dot{\mathcal{G}}^*=\mathcal{G}^*-\{0\}$. Let $\{\delta_r\colon r>0\}$ be a family of dilations on $\mathcal{G}$. A subset Γ of $\mathcal{G}^*$ is called a δ-cone if $\xi\in\Gamma$ implies $\delta_r\xi\in\Gamma$ for all $r>0$. If Γ is a δ-cone, then $\bar{\Gamma}$ will denote (closure Γ)-$\{0\}$. The notation $\Gamma_1\subset\subset\Gamma_2$ for δ-cones means $\bar{\Gamma}_1\subset\Gamma_2$.

Definition. A set $\Gamma\subset G\times\dot{\mathcal{G}}^*$ is *δ-conic* if for every $g\in G$, $\{(\mathrm{Ad}\tfrac{1}{2}g)^*\xi\colon(g,\xi)\in\Gamma\}$ is a δ-cone. In other words, Γ is δ-conic if and only if for all $g\in G$, $\lambda_*^g(\Gamma_g)$ is a δ-cone in $T_e^*G\cong\mathcal{G}^*$, where Γ_g is the fibre of Γ over g.

Note that if $\Gamma\subset G\times\mathcal{G}^*$ is δ-conic and $a\in G$, then $\lambda_*^a\Gamma$ is also δ-conic. It should also be noted that if $\Gamma\subset\dot{\mathcal{G}}^*$ is a δ-cone and $a\in G$, then $(\mathrm{Ad}\,a)^*\Gamma$ is not usually a δ-cone—even in the case of the natural dilations on the Heisenberg group. That is why we did not define a set $\Gamma\subset G\times\dot{\mathcal{G}}^*$ to be δ-conic if Γ_g is a δ-cone for every g.

If $\omega\subset G$ and $\Gamma'\subset\dot{\mathcal{G}}^*$ is a δ-cone, let $\omega^*\Gamma'=\{(g,\xi)\colon g\in\omega$ and $(\mathrm{Ad}\tfrac{1}{2}g)^*\xi\in\Gamma'\}$. If $g\in G$, let

$g^*\Gamma' = \{\xi : (\mathrm{Ad}\tfrac{1}{2}g)^*\xi \in \Gamma'\}$. Note that if $\Gamma \subset G \times \dot{\mathscr{G}}^*$ is an open δ-conic set and $(g_0, \xi_0) \in \Gamma$, then there is an open neighborhood ω of g_0 and an open δ-cone Γ' containing $(\mathrm{Ad}\tfrac{1}{2}g_0)^*\xi_0$ such that $\omega^*\Gamma' \subset \Gamma$. Let $F_1u = F(u \circ \exp)$ where F is the Fourier transform.

Definition. Let Ω be an open subset of G, let $pr_2 : \Omega \times \dot{\mathscr{G}}^* \to \dot{\mathscr{G}}^*$ be the projection and let $u \in \mathscr{D}'(\Omega)$. Define $WF_\delta(u) \subset \Omega \times \dot{\mathscr{G}}^*$ by $(a, \eta) \notin WF_\delta(u)$ if there is a δ-conic neighborhood Γ of (a, η) and a $\phi \in C_c^\infty(\Omega)$, $\phi \equiv 1$ in some neighborhood of a, such that $F_1(\phi u)$ is rapidly decreasing in $pr_2\Gamma$, i.e. for every N

$$|F_1(\phi u)(\xi)| \leqq C_N(1 + |\xi|)^{-N}, \quad \text{for } \xi \in pr_2\Gamma.$$

Definition. Let $p \in S^m(\Omega \times \mathscr{G}^*, \delta)$. $P = \mathrm{Op}(p)$ is said to be *regularizing* on an open δ-conic set Γ if given $(a, \eta) \in \Gamma$, there is a ϕ as in the preceding definition and an open δ-cone Γ' with $\eta \in a^*\Gamma' \subset \Gamma_a$ such that

$$|D_x^\alpha D_\xi^\beta \phi(\exp x) p(\exp x, \xi)| \leqq C_{\alpha\beta N}(1 + |\xi|)^{-N}$$

for all $\xi \in \Gamma'$. The complement of the union of all δ-conic open sets on which P is regularizing is denoted $\mathrm{microsupp}_\delta(P)$.

If P is left invariant we sometimes refer to $\{\xi : (e, \xi) \in \mathrm{microsupp}_\delta(P)\} \subset \dot{\mathscr{G}}^*$ as simply $\mathrm{microsupp}_\delta(P)$.

Definition. Let $P \in \mathrm{Op}\,S^m(\Omega \times \mathscr{G}^*, \delta)$. Let Γ be a δ-conic subset of $\Omega \times \mathscr{G}^*$. P is said to be *δ-microhypoelliptic* on Γ if $WF_\delta(u) \cap \Gamma \subset WF_\delta(Pu) \cap \Gamma$ for every $u \in \mathscr{D}'(\Omega)$. Let $(g, \xi) \in \Omega \times \dot{\mathscr{G}}^*$. P is said to be δ-microhypoelliptic at (g, ξ) if it is δ-microhypoelliptic on the smallest δ-conic set containing (g, ξ). P is said to be δ-microhypoelliptic near (g, ξ) if it is δ-microhypoelliptic on a δ-conic neighborhood of (g, ξ).

If $a \in G$ and $P \in \mathrm{Op}\,S^m(\lambda^a\Omega \times \mathscr{G}^*, \delta)$ define $P^a \in \mathrm{Op}\,S^m(\Omega \times \mathscr{G}^*, \delta)$ by $P^a u = P(u \circ \lambda^{a^{-1}}) \circ \lambda^a$. Then $P^a = \mathrm{Op}(p^a)$ where $p^a(g, \xi) = p(\lambda^a g, \xi)$.

Proposition 2.1. *Let $a \in G, u \in \mathscr{D}'(\Omega)$ and $P \in \mathrm{Op}\,S^m(\lambda^a\Omega \times \mathscr{G}^*, \delta)$. Then*

(a) $\lambda_*^a WF_\delta(u \circ \lambda^a) = WF_\delta(u)$;

(b) $\lambda_*^a \mathrm{microsupp}_\delta(P^a) = \mathrm{microsupp}_\delta(P)$;

(c) *For any δ-conic set Γ, P^a is δ-microhypoelliptic on Γ if and only if P is δ-microhypoelliptic on $\lambda_*^a\Gamma$.*

Proof. Since

$$F_1(u \circ \lambda^a)(\xi) = e^{-i\langle \xi, \log a\rangle} F_1 u(\mathrm{Ad}\tfrac{1}{2}a^*\xi) \tag{2.1}$$

it follows that $F_1(\phi u \circ \lambda^a)$ is rapidly decreasing in $pr_2\Gamma$ if and only if $F_1(\phi u)$ is rapidly decreasing in $pr_2\lambda_*^a\Gamma$. This implies (a). The proof of (b) is trivial and (c) follows from (a).

Corollary 2.2. If $P \in \mathrm{Op}\,S^m(\mathscr{G}^*, \delta)$ *and P is δ-microhypoelliptic at (near) (e, ξ) for all*

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$\xi \in \mathcal{O}$, where $\mathcal{O} \subset \dot{\mathcal{G}}^*$ is some orbit, then P is δ-microhypoelliptic at (near) (g,ξ) for all $g \in G$ and all $\xi \in \mathcal{O}$.

Corollary 2.3. *If $P \in \mathrm{Op}\, S^m(\mathcal{G}^*, \delta)$ is δ-microhypoelliptic at (near) (g,ξ) for all $g \in G$, then P is δ-microhypoelliptic at (near) (g,η) for all $g \in G$ and all $\eta \in \mathcal{O}_\xi$.*

Proposition 2.4. *Let Ω be an open subset of G* and *Γ an open δ-conic subset of $\Omega \times \dot{\mathcal{G}}^*$. Let $P_j \in \mathrm{Op}\, S^{m_j}(\Omega \times \mathcal{G}^*, \delta)$ for $j=1,2$, with P_1 and P_2 either both properly supported or both left invariant. If* $\mathrm{microsupp}_\delta P_1 \cap \mathrm{microsupp}_\delta P_2 \cap \bar{\Gamma} = \phi$, *then $P_1 P_2$ is regularizing on Γ.*

Proof. First consider the case when P_1 and P_2 are left invariant. Then Γ and $\mathrm{microsupp}\, P_j$ may be regarded as δ-cones in $\mathcal{G}^*$. Let p_j be the symbol of P_j. If $p_j \in \mathcal{S}(\mathcal{G}^*)$, then

$$p_1 \sqcap p_2(\eta) = \iint_{\mathcal{G}^* \mathcal{G}} e^{i\langle \eta - \xi, x\rangle} p_1(\xi) p_2(\eta + \mathrm{ad}\tfrac{1}{2}x^*\eta)\, dx\, d\xi$$

$$= \iint_{\mathcal{G}^* \mathcal{G}} e^{i\langle \eta - \xi, x\rangle} p_1(\eta - \mathrm{ad}\tfrac{1}{2}x^*\eta) p_2(\xi)\, dx\, d\xi. \tag{2.2}$$

We shall show that there exist C_N such that

$$|(p_1 \sqcap p_2)(\eta)| \leqq C_N (1+|\eta|^{-N}) \quad \text{for all } \eta \in \Gamma, \tag{2.3}$$

where $C_N = \|p_1\|\,\|p_2\|$ for appropriate seminorms on $S^{m_j}(\mathcal{G}^*, \delta)$.

For $j=1,2$ there exist open δ-cones $\Gamma_j, \Gamma_j', \Gamma_j''$ and a $c>0$ such that $\mathrm{microsupp}_\delta P_j \subset \Gamma_j$, $\mathrm{microsupp}_\delta P_j \cap \Gamma_j' = \phi$, $\Gamma_j \cup \Gamma_j' = \dot{\mathcal{G}}^*$, $\Gamma_1'' \cup \Gamma_2'' = \Gamma$ and if $\xi \in \Gamma_j$, $\eta \in \Gamma_j''$, then $[\xi - \eta] \geqq c[\eta]$.

If $\eta \in \Gamma_1''$ we apply integration by parts to the first formula in (2.2) to obtain

$$p_1 \sqcap p_2(\eta) = \iint e^{i\langle \eta - \xi, x\rangle} \langle x\rangle^{-2M} (I - \Delta_\xi)^M (p_1(\xi)\langle \eta - \xi\rangle^{-2M}).$$

$$(I - \Delta_x)^M p_2(\eta + \mathrm{ad}\tfrac{1}{2}x^*\eta)\, dx\, d\xi \tag{2.4}$$

for all $M>0$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. For sufficiently large M, (2.4) is valid for all $p_j \in S^{m_j}(\mathcal{G}^*, \delta)$, $j=1,2$. Note that $\langle \eta + \mathrm{ad}\, x^*\eta\rangle \leqq C\langle x\rangle\langle \eta\rangle$. Hence if M is sufficiently large

$$|p_1 \sqcap p_2(\eta)| \leqq C\langle \eta\rangle^{|m_2|} \int |(I-\Delta_\xi)^M (p_1(\xi)\langle \eta - \xi\rangle^{-2M})|\, d\xi \tag{2.5}$$

for $\eta \in \Gamma_1''$. We estimate this integral over Γ_1 and Γ_1' separately. Since P_1 is regularizing on Γ_1', it follows from Peetre's inequality that

$$\int_{\Gamma_1'} |(I-\Delta_\xi)^M (p_1(\xi)\langle \eta - \xi\rangle^{-2M})|\, d\xi \leqq C_M \langle \eta\rangle^{-2M}.$$

If $\xi \in \Gamma_1$, then $\langle \eta - \xi\rangle \geqq c(1+[\eta - \xi]) \geqq c(1+[\eta]) \geqq c\langle \eta\rangle^{1/\bar{\mu}}$, and

$$|(I-\Delta_\xi)^M (p_1(\xi)\langle \eta - \xi\rangle^{-2M})| \leqq C_M \chi(\eta)^{m_1} \langle \eta - \xi\rangle^{-2M+|m_1|}. \tag{2.6}$$

Thus

$$\int_{\Gamma_1} |(I-\Delta_\xi)^M(p_1(\xi)\langle\eta-\xi\rangle^{-2M})|d\xi \leqq C_M\langle\eta\rangle^{2|m_1|+n+1-2M/\bar{\mu}}$$

if M is sufficiently large. Since $\Gamma_1 \cup \Gamma_1' = \dot{\mathscr{G}}^*$, (2.3) is verified for $\eta \in \Gamma_1''$. To prove (2.3) for $\eta \in \Gamma_2''$ apply a similar argument using the second formula in (2.2). By using (4.1) below, it follows that

$$|D^\alpha(p_1 \dot{\sqcup} p_2)(\eta)| \leqq C_{\alpha N}(1+|\eta|^{-N}) \quad \text{for } \eta \in \Gamma_1,$$

which proves the proposition when P_1 and P_2 are left invariant.

For general P_1 and P_2 the preceding argument shows that for every α, $\mathrm{Op}(D_\xi^\alpha p_1 \sqcup T_g^\alpha p_2)$ is regularizing on Γ. By Theorem 1.2 P_1P_2 is regularizing on Γ if P_1 and P_2 are properly supported.

Propositions 2.5 and 2.6 now follow by standard arguments.

Proposition 2.5. *Let Ω be an open subset of G, $u \in \mathscr{D}'(\Omega)$. Then $(a,\eta) \notin WF_\delta(u)$ if and only if there is a δ-conic neighborhood Γ of (a,η) such that $Pu \in C^\infty(\Omega)$ for every properly supported $P \in \mathrm{Op}\, S^m(\Omega \times \mathscr{G}^*, \delta)$ for which* microsupp$_\delta P \subset \Gamma$.

Proposition 2.6. *If $Q \in \mathrm{Op}\, S^m(\Omega \times \mathscr{G}^*, \delta)$ and $u \in \mathscr{D}'(\Omega)$, then*

$$WF_\delta(Qu) \subset WF_\delta(u) \cap \mathrm{microsupp}_\delta Q.$$

3. A necessary condition for δ-microhypoellipticity

If $\xi \in \dot{\mathscr{G}}^* = \mathscr{G}^* - \{0\}$, let Γ_ξ be the smallest G-invariant δ-cone containing ξ. Let $\bar{\Gamma}_\xi =$ (closure of Γ_ξ) $- \{0\}$, and $\partial\Gamma_\xi = \bar{\Gamma}_\xi - \Gamma_\xi$. Later in this section an algebraic description of $\partial\Gamma_\xi$ will be given, but our primary concern is the following necessary condition for δ-microhypoellipticity on $\bar{\Gamma}_\xi$.

Theorem 3.1. *Let G be a step two nilpotent Lie group with a family of dilations δ. Let $p \in C^\infty(\mathscr{G}^*)$ be homogeneous of degree m with respect to δ for large ξ. Let $\xi \in \dot{\mathscr{G}}^*$. If $P = \mathrm{Op}(p)$ is δ-microhypoelliptic at (e,η) for every $\eta \in \bar{\Gamma}_\xi$, then $\pi_\xi(P)$ is injective in $\mathscr{S}_{\pi_\xi}$.*

Proof. Suppose there is a $v \in \mathscr{S}_\pi$, $v \neq 0$, such that $\pi(P)v = 0$, where $\pi = \pi_\xi$. Define the function $u(g) = (\pi(g)v, v)$ where $(,)$ denotes the L^2 inner product. For all $r > 0$ let $u_r = u \circ \delta_r$, $\pi_r = \pi \circ \delta_r$ and let $\mathcal{O}_r$ be the orbit corresponding to π_r. Then $u_r(g) = (\pi_r(g)v, v)$ for all $r > 0$. If $p \in \mathscr{S}(\mathscr{G}^*)$, then

$$\begin{aligned} Pu_r(g) &= u_r * F_1^{-1}p(g) \\ &= \int \pi_r(g)\pi_r(h)F_1(p)(h)v(t)\overline{v(t)}\, dh\, dt \\ &= (\pi_r(P)v, \pi_r(g^{-1})v) \end{aligned}$$

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by (1.11). By an approximation argument $Pu_r(g)=(\pi_r(P)v, \pi_r(g^{-1})v)$ for all $p\in S^m(\mathscr{G}^*,\delta)$. By redefining p on a set with compact support we may assume that $p(\delta_r\eta)=r^m p(\eta)$ for all $\eta\in\mathcal{O}_\xi$ and all $r\geqq 1$. It follows that $\pi_r(P)=r^m\pi(P)$ for all $r\geqq 1$. Therefore $Pu_r=0$ for all $r\geqq 1$.

Let $\Gamma_1\supset\supset\Gamma_2\supset\supset\ldots$ be a sequence of open δ-cones containing $\bar{\Gamma}_\xi$ such that $\cap\Gamma_j=\bar{\Gamma}_\xi$, and let $\{\phi_j\}$ be a sequence of functions in $C_0^\infty(G)$ such that $e\in\operatorname{supp}\phi_{j+1}\subset\subset\operatorname{supp}\phi_j$ and $\varphi_j\geqq 0$. Let $C_b^0(G)$ denote the space of continuous bounded functions on G with the supremum norm. Let $R_j=\{u\in C_b^0(G): F_1(u\phi_j)$ is rapidly decreasing on $\Gamma_j\}$. R_j is a Frechet space with the seminorms

$$\sigma_{kj}(u)=\sup\{\langle\eta\rangle^k|F_1(\phi_j u)(\eta)|:\eta\in\Gamma_j\},$$

together with the $C_b^0(G)$ norm. It follows from the proof of Lemma 8.1.1 of [5] that $R_j\subset R_{j+1}$ and that the inclusion is continuous. Thus $R(\bar{\Gamma}_\xi)=\cup R_j$ can be given the corresponding LF topology.

Since P is δ-microhypoelliptic at (e,η) for every point of $\bar{\Gamma}_\xi$ and since $\{\eta\in\bar{\Gamma}_\xi:[\eta]=1\}$ is compact, it follows that $\{u\in C_b^0(G): Pu=0\}\subset R(\bar{\Gamma}_\xi)$. The closed graph theorem implies that the inclusion is continuous. The set $\{u_r: r\geqq 1\}$ is a bounded subset of $\{u\in C_b^0(G)$: $Pu=0\}$ and hence is a bounded subset of $R(\bar{\Gamma}_\xi)$. Thus there is a j such that $\{u_r: r\geqq 1\}$ is contained and bounded in R_j. Let $\phi=\phi_j$. Then for every $k>0$, there is a C_k such that

$$|F_1(\phi u_r)(\eta)|\leqq C_k\langle\eta\rangle^{-k} \tag{3.1}$$

for all $\eta\in\bar{\Gamma}_\xi$ and all $r\geqq 1$.

Given $\eta\in\mathscr{G}^*$ define $e_\eta\in C^\infty(G)$ by $e_\eta(\exp x)=e^{-i\langle\eta,x\rangle}$. Note that

$$F_1(\phi u_r)(\eta)=\left(\int e^{-i\eta x}\phi(\exp x)\pi_r(\exp x)v\,dx, v\right)=(\pi_r(e_\eta\phi)v,v).$$

By (1.11), if $\phi\in C_0^\infty(G)$, then $\pi(\phi)$ is a pseudodifferential operator on $\mathbb{R}^d$ with Weyl symbol $F_1^{-1}(\phi)\circ\psi_\pi$. Since $F_1^{-1}(e_\eta\phi)(\zeta)=F_1^{-1}\phi(\zeta-\eta)$,

$$\int_{\mathcal{O}_\pi}(\pi(e_\eta\phi)v,v)\,d\eta=\int_{\mathbb{R}^{3d}}\int_{\mathcal{O}_\pi}e^{i(t-s)\tau}F_1^{-1}\phi(\psi_\pi((t+s)/2,\tau)-\eta)v(s)\overline{v(t)}\,d\eta\,ds\,dt\,d\tau.$$

For fixed t,s and τ, as η varies over $\mathcal{O}_\pi$, $\psi_\pi((t+s)/2,\tau)-\eta$ varies over $T\mathcal{O}_\pi$, the linear space parallel to $\mathcal{O}_\pi$. Thus by the Plancherel Theorem

$$\int_{\mathcal{O}_\pi}(\pi(e_\eta\phi)v,v)\,d\eta=\|v\|^2\int_{T\mathcal{O}_\pi}F_1^{-1}\phi(\eta)\,d\eta,$$

where $\|v\|$ is the L^2 norm of v. Assuming that the value of $\phi(\exp x)$ depends only on $|x|$, Lemma 3.3 below implies

$$\int_{T\mathcal{O}_{\pi_r}}F_1^{-1}\phi(\eta)\,d\eta=\int_{R(\delta_r\xi)}\phi(\exp x)\,dx=c$$

where $c>0$ is independent of r, since the subspaces $R(\delta_r\xi)$ have the same dimension for all $r>0$. Thus $\int_{\mathcal{O}_r} F_1(\phi u_r)(\eta)\,d\eta = c$ is independent of r and positive. But by (3.1),

$$\left|\int_{\mathcal{O}_r} F_1(\phi u_r)(\eta)\,d\eta\right| \leqq C \int_{\mathcal{O}_r} \langle \eta \rangle^{-k}\,d\eta \to 0 \quad \text{as} \quad r \to \infty,$$

if k is sufficiently large. This contradiction implies that $\pi_\xi(P)$ is injective in $\mathcal{S}_\pi$.

We now give a more explicit description of $\partial\Gamma_\xi$. Let $R(\xi)=\{x\in\mathcal{G}_1 : \langle \xi, [x,y]\rangle = 0$ for all $y\in\mathcal{G}\}$. Let $2d(\xi)$ be the codimension of $R(\xi)$ in $\mathcal{G}_1$. If δ is a family of dilations, then $d(\delta_r\xi)=d(\xi)$ for all $r>0$. For fixed $\xi\in\dot{\mathcal{G}}^*$, the map $\phi: r\to R(\delta_r\xi)$ is an algebraic function from $(0,\infty)$ into the Grassmann manifold $GR_{2d}(\mathcal{G}_1)$ of subspaces of $\mathcal{G}_1$ of codimension $2d=2d(\xi)$. Since $GR_{2d}(\mathcal{G}_1)$ is compact and ϕ is an algebraic function of one variable, ϕ extends to a continuous function $\phi:[0,\infty)\to GR_{2d}(\mathcal{G}_1)$. Let $R_0(\xi)=\phi(0)$, i.e. $R_0(\xi)=\lim_{r\to 0} R(\delta_r\xi)$.

Proposition 3.2. *Let G be a step two nilpotent Lie group with a family of dilations δ. If $\xi\in\dot{\mathcal{G}}^*$, then $\partial\Gamma_\xi = R_0(\xi)^\perp - \{0\}$. In particular, $\partial\Gamma_\xi \subset \mathcal{G}_1^*$.*

Lemma 3.3. *Define $\phi_\xi:\mathcal{G}_1\to\mathcal{G}_1^*$ by $\phi_\xi(x)=\operatorname{ad} x^*\xi$. Let $V\subset\mathcal{G}_1$ be a subspace such that $V\oplus R(\xi)=\mathcal{G}_1$. Then ϕ_ξ is a bijection of V onto $R(\xi)^\perp$. Consequently the linear space $T\mathcal{O}_\xi$ parallel to $\mathcal{O}_\xi$ is $R(\xi)^\perp$.*

Proof. Clearly, the range of ϕ_ξ is contained in $R(\xi)^\perp \cong V^*$. Also $\phi_\xi|_V$ is injective, therefore by a dimension argument the range of ϕ_ξ equals $R(\xi)^\perp$. The last statement follows from the observations that $T\mathcal{O}_\xi=\{\operatorname{ad} x^*\xi : x\in\mathcal{G}_1\}$.

Proof of Proposition 3.2. Write $\xi=\eta+\zeta$, where $\eta\in\mathcal{G}_1^*$ and $\zeta\in\mathcal{G}_2^*$. If $\zeta=0$, then $\partial\Gamma_\xi=\phi=R_0(\xi)^\perp-\{0\}$, so we may assume $\zeta\neq 0$.

Suppose $\xi_j\in\Gamma_\xi$ and $\lim\xi_j=\xi_0$. Then

$$\xi_j=\delta_{r_j}\xi+\delta_{r_j}\operatorname{ad} x_j^*\xi=\delta_{r_j}\eta+\delta_{r_j}\operatorname{ad} x_j^*\zeta+\delta_{r_j}\zeta,$$

where $\delta_{r_j}\eta+\delta_{r_j}\operatorname{ad} x_j^*\zeta\in\mathcal{G}_1^*$ and $\delta_{r_j}\zeta\in\mathcal{G}_2^*$. Since $\delta_{r_j}\zeta$ converges, r_j must converge to $r_0\geqq 0$. Suppose $r_0\neq 0$. Then $\delta_{r_j}\operatorname{ad} x_j^*\zeta\to\eta'\in\mathcal{G}_1^*$, and hence $\operatorname{ad} x_j^*\zeta=\operatorname{ad} x_j^*\xi\to\delta_{r_0}^{-1}\eta'$. Let V and ϕ_ξ be as in Lemma 3.3. Since $\operatorname{ad} x_j^*\xi\in R(\xi)^\perp$, we may assume that $x_j\in V$. Since ϕ_ξ is a linear bijection, the convergence of $\phi_\xi(x_j)$ implies that $x_j\to x_0\in V$. Thus $\xi_0=\delta_{r_0}\xi+\delta_{r_0}\operatorname{ad} x_0^*\xi\in\Gamma_\xi$. Hence if $\xi_0\in\partial\Gamma_\xi$, then $r_0=0$. We may also write $\xi_j=\delta_{r_j}\xi+\operatorname{ad} y_j^*(\delta_{r_j}\xi)$, where $\operatorname{ad} y_j^*(\delta_{r_j}\xi)\in R(\delta_{r_j}\xi)^\perp$. If $\xi_0\in\partial\Gamma_\xi$, then $r_j\to 0$, and hence $R(\delta_{r_j}\xi)\to R_0(\xi)$. Since $\delta_{r_j}\xi\to 0$, it follows that $\xi_0\in R_0(\xi)^\perp$.

Conversely, if $\eta'\in R_0(\xi)^\perp$, $\eta'\neq 0$, choose $\eta_r\in R(\delta_r\xi)^\perp$ such that $\eta_r\to\eta'$ as $r\to 0$. Then $\delta_r\xi+\eta_r\in\mathcal{O}_{\delta_r\xi}\subset\Gamma_\xi$. Hence $\eta'\in\bar{\Gamma}_\xi$. Since $\mathcal{G}_1^*\cap\Gamma_\xi=\phi$, $\eta'\in\partial\Gamma_\xi$.

Definition. If G is a step two nilpotent Lie group a family of dilations $\{\delta_r : r>0\}$ is called *natural* if $\delta_r x=r^2x$ for all $x\in\mathcal{G}_2$ and $\delta_r x=rx$ for all $x\in\mathcal{G}_1$.

Corollary 3.4. *If G is a step two nilpotent group with natural dilations, or more generally, if $\mu_j=\bar{\mu}$ for all $j>N=\dim\mathcal{G}_1$, then $\partial\Gamma_\xi=R(\xi)^\perp-\{0\}$.*

Proof. Let $\zeta=\xi|_{\mathscr{G}_2}$. If $\mu_j=\bar{\mu}$ for all $j>N$, then $\delta_r\zeta=r^{\bar{\mu}}\zeta$, and $R(\delta_r\xi)=R(\xi)$ for all $r>0$. Hence $R_0(\xi)=R(\xi)$.

Note that for any family of dilations, given ξ such that $\xi'=\xi|_{\mathscr{G}_2}\neq 0$, there is a unique $s>0$ such that $\xi''=\lim_{r\to 0}r^{-s}\delta_r\xi'$ exists and is non-zero. In fact $s=\min\{\mu_j\colon j>N$ and $\xi_j\neq 0\}$. Since $R(r^{-s}\delta_r\xi')=R(\delta_r\xi')$, $R_0(\xi)\subset R(\xi'')$ in all cases. If $d(\xi'')=d(\xi)$, then $R_0(\xi)=R(\xi'')$ and $\partial\Gamma_\xi=R(\xi'')^\perp-\{0\}$.

Let G satisfy the condition $d(\xi)=d(\mathscr{G})$ for all $\xi\notin\mathscr{G}_1^*$, i.e. $\tilde{\mathscr{G}}^*=\mathscr{G}^*\backslash\mathscr{G}_1^*$. Such groups were said to be typed H in [13]. Then $R_0(\xi)=R(\xi'')$ as in the preceding paragraph. One might expect that $R_0(\xi)$ would depend continuously on $\xi\in\tilde{\mathscr{G}}^*$ in such a case. However, there are examples of dilations on the free step two nilpotent group on three generators (a six dimensional group of type H), where $R_0(\xi)$ does not depend continuously on $\xi\in\tilde{\mathscr{G}}^*$.

4. A sufficient condition for microhypoellipticity

Let $p\in S^m(\Omega\times\mathscr{G}^*,\delta)$, where Ω is an open subset of G. If p can be written in the form $p=p^0+p^1$ where $p^1\in S^{m-\varepsilon}(\Omega\times\mathscr{G}^*,\delta)$ for some $\varepsilon>0$ and p^0 is homogeneous of degree m with respect to δ in the ξ variables, for large ξ, then p^0 is called the principal symbol of p.

Theorem 4.1. *Let G be a step two nilpotent Lie group with dilations δ. Let Ω be an open subset of G and let $p\in S^m(\Omega\times\mathscr{G}^*,\delta)$ have principal symbol p^0. Let $P=\mathrm{Op}(p)$ and $P_g^0=\mathrm{Op}(p_g^0)$ for $g\in\Omega$. Let Γ be an open δ-conic subset of $\Omega\times\tilde{\mathscr{G}}^*$. Assume that there is a C such that if $(g,\xi)\in\Gamma$ and $[\xi]\geqq C$, then the following holds: $\pi_\xi(P_g^0)$ is injective on $\mathscr{S}_{\pi_\xi}$ and $\pi_\eta(P_g^0)\neq 0$ for all $\eta\in\partial\Gamma_\xi$, $[\eta]\geqq C$. Then P* is *δ-microhypoelliptic on Γ.*

Proof. The theorem will be proved first under the assumption that $p\in S^m(\mathscr{G}^*,\delta)$, i.e. that P is left invariant. In that case, by Corollary 2.2 we need only consider microhypoellipticity over the identity element e of G and may consider Γ to be a subset of $\mathscr{G}^*$. Let $\xi_0\in\Gamma$ and let Γ_1 be an open δ-cone containing ξ_0, $\bar{\Gamma}_1\subset\Gamma$. Let $\Gamma'=\cup\{\mathcal{O}_\xi\colon\xi\in\Gamma_1\}$. Define $h(\xi)$ by (1.8).

Lemma 4.2. *If $p\in S^m(\mathscr{G}^*,\delta)$ satisfies the hypotheses of Theorem* 4.1 *then there exist C and $c>0$ such that $|p(\xi)|\geqq c\chi(\xi)^m$ for all $\xi\in\Gamma'$ such that $[\xi]\geqq C$ and $h(\xi)\leqq c$.*

Proof. Since $p^0(\eta)\neq 0$ for $\eta\in\partial\Gamma_\xi$, $\xi\in\Gamma_1$, η large, there exist C_1 and $c_1>0$ such that if $\xi\in\Gamma'$, $[\xi]=C_1$ and $|\xi'|\leqq c_1$, then $|p^0(\xi)|\geqq c_1$. As before, $\xi'=\xi|_{\mathscr{G}_2}$. There is a C_2 such that $|\delta_{C_1}\xi|\leqq C_2|\xi|$ for all $\xi\in\mathscr{G}^*$. Let $c=c_1/\max\{C_2,C_1^m\}$. Given $\xi\in\Gamma$ such that $[\xi]\geqq C_1$ and $h(\xi)\leqq c$, let $r=C_1^{-1}[\xi]$ and $\xi_0=\delta_r^{-1}\xi$. Then $[\xi_0]=C_1$ and $[\xi_0']\leqq C_2|\delta_{[\xi]}^{-1}\xi'|=C_2h(\xi)\leqq c_1$. Thus $|p^0(\xi)=r^m|p^0(\xi_0)|\geqq c\chi(\xi)$. The lemma follows by choosing C large enough that $\chi(\xi)^{-m}|p^1(\xi)|\leqq c/2$ if $[\xi]\geqq C$.

A special type of cut-off function, as described in the next lemma, will be needed in dealing with open G-invariant δ-cones $\Gamma_2\subseteq\Gamma_1\subseteq\tilde{\mathscr{G}}^*$. Note that for such Γ_1 and $\Gamma_2\neq 0$ it is impossible to have $\bar{\Gamma}_2\subseteq\Gamma_1$, since $\bar{\Gamma}_2\cap\mathscr{G}^*\neq\phi$ by Proposition 3.2.

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Definition. Given a G-invariant δ-cone $\Gamma\subseteq\mathscr{G}^*$, let $i\Gamma=\{(\rho,\zeta)\in R^{N-2d}\times\mathscr{G}_2^*:\mathcal{O}_{\rho\zeta}\subset\Gamma$ and $[\zeta]=1\}$, $\mathcal{O}_{\rho\zeta}$ defined as at the end of Section 1. If Γ_1 and Γ_2 are open G-invariant δ-cones such that $\Gamma_2\subseteq\Gamma_1\subseteq\mathscr{G}^*$ and $(i\Gamma_2)^-$ is a compact subset of $i\Gamma_1$, then we write $\Gamma_2\subseteq\Gamma_1$ *properly*.

Definition. $S^m_{00}(\mathscr{G}^*,\delta)$ is the set of functions p for which estimate (1.6) is required only for derivatives in directions parallel to the orbits of the coadjoint action (see [**12**]). $S^m_0(\mathscr{G}^*,\delta)$ is the set of p such that $D^\alpha p\in S^m_{00}(\mathscr{G}^*,\delta)$ for all α such that $\alpha'=(\alpha_{N+1},\dots,\alpha_n)=0$. If Γ is an open subset of $\mathscr{G}^*$, $S^m(\Gamma,\delta)$ is the set of $p\in C^\infty(\Gamma)$ such that p has an extension in $S^M(\mathscr{G}^*,\delta)$ for some M and such that (1.6) holds for $\xi\in\Gamma$.

Lemma 4.3. *Let Γ_1 and Γ_2 be open G-invariant δ-cones such that $\Gamma_1\subseteq\mathscr{G}^*$ and $\Gamma_2\subseteq\Gamma_1$ properly. Given $c_1>c_2\geqq 0$, there is a $\varphi\in S^0_0(\mathscr{G}^*,\delta)$ such that* $\operatorname{supp}\varphi\subset\Gamma_1$, *$\varphi(\xi)=1$ if $\xi\in\Gamma_2$ and $[\xi']\geqq c_1$, $\varphi(\xi)=0$ if $[\xi]\leqq c_2$.*

Proof. If Γ is a G-invariant δ-cone and $\mathcal{O}_{\rho\zeta}\subset\Gamma$, then $\mathcal{O}_{\rho,\delta_r\zeta}\subset\Gamma$ for all $r>0$. Thus one can find a function $\varphi_0\in C^\infty(R^{N-2d}\times\mathscr{G}_2^*)$ such that $\varphi_0(\rho,\zeta)=0$ if $\mathcal{O}_{\rho\zeta}\cap\Gamma_1=\phi$, $\varphi_0(\rho,\zeta)=0$ if $[\zeta]\leqq c_2$, $\varphi_0(\rho,\zeta)=1$ if $\mathcal{O}_{\rho\zeta}\subset\Gamma_2$ and$[\zeta]\geqq c_1$, $\varphi_0(\rho,\delta_r\zeta)=\varphi_0(\rho,\zeta)$ if $[\zeta]\geqq c_1$ and $r\geqq 1$ and $|D^\alpha_\rho\varphi_0(\rho,\zeta)|\leqq C_\alpha$ on $R^{N-2d}\times\mathscr{G}_2^*$. Define $\varphi\in C^\infty(\mathscr{G}^*)$ so that $\varphi(\xi)=\varphi_0(\rho,\zeta)$ if $\pi_\xi=\pi_{\rho\zeta}$, $\varphi(\xi)=0$ if $\xi\notin\Gamma_1$. Since φ is constant on orbits it follows that $\varphi\in S^0_0(\mathscr{G}^*,\delta)$.

Note that if $\xi\in\Gamma_1\subseteq\mathscr{G}^*$ with Γ_1 an open G-invariant δ-cone, then there is an open G-invariant δ-cone Γ_2 such that $\xi\in\Gamma_2$ and $\Gamma_2\subseteq\Gamma_1$ properly. To simplify notation we will, without specific mention, occasionally replace the G-invariant δ-cone Γ', chosen before Lemma 4.2, by a properly contained G-invariant δ-cone, also written Γ', still containing ξ_0. If $c>0$, then $\Gamma'_c=\{\xi\in\Gamma':[\xi']>c\}$. The constant c may also change from statement to statement.

Lemma 4.4. *If $p\in S^m(\mathscr{G}^*,\delta)$ satisfies the hypotheses of Theorem 4.1, then there exist $b\in S^{-m}_{00}(\mathscr{G}^*,\delta)$ and $c>0$ such that $b\,\sqcap\, p-1$ and $p\,\sqcap\, b-1$ are in $S^{0,k}_{00}(\Gamma'_c,\delta)$ for all k.*

Proof. Let c and C be as in Lemma 4.2 and φ as in Lemma 4.3. Let $F\in C^\infty(R)$ satisfy $F(r)=1$ if $r\geqq 2$ and $F(r)=0$ if $r\leqq 1$. Define $b_0(\xi)=\varphi(\xi)F(C^{-1}[\xi])F(ch(\xi)^{-1})p(\xi)^{-1}$. Then $b_0\in S^{-m}(\mathscr{G}^*,\delta)$ and $1-(b_0\,\sqcap\, p)\in S^{0,1}_{00}(\Gamma'_{2C},\delta)$ by Theorem 1.1. The desired symbol b is now constructed by the standard parametrix method.

Returning to the proof of Theorem 4.1, the injectivity of $\pi_\xi(P^0)$ on $\mathscr{S}(R^d)$ implies, by Theorem 7.7 of [**1**], that $\pi_\xi(P^0)$ has an inverse which is a pseudodifferential operator. There is a $c>0$ such that if $[\xi]>c$, $\xi\in\Gamma'$, then $\pi_\xi(P)$ also has an inverse which is a pseudodifferential operator, the Weyl symbol of which we denote by q_ξ. Define q on Γ'_c by $q|_{\mathcal{O}_\xi}=q_\xi\circ\psi_\xi^{-1}$, where $\psi_\xi:R^d\times R^d\to\mathcal{O}_\xi$ is the symplectomorphism described in Section 1. Using Lemma 4.4 it follows by arguments given in [**13**] that $q\in S^{-m}_{00}(\Gamma'_c,\delta)$. Replace q by φq for φ as in Lemma 4.3. Then $q\in S^{-m}_{00}(\mathscr{G}^*,\delta)$ and $q\,\sqcap\, p=1$ on Γ'_c.

In order to obtain estimates for derivatives transverse the orbits we will look at difference quotients. Given $\tau\in\mathscr{G}^*$ and $p\in S^m_{00}(\mathscr{G}^*,\delta)$, define $p_\tau(\xi)=p(\xi+\tau)$. If $\tau\in\mathscr{G}_1^*$, then $(p\,\sqcap\, q)_\tau=p_\tau\,\sqcap\, q_\tau$. For general τ and for $p\in S^m_0(\mathscr{G}^*,\delta)$, $q\in S^k_0(\mathscr{G}^*,\delta)$ we use formula (2.2) and Taylor's Theorem to find that

$$(p\,\sqcap\, q)_\tau=p_\tau\,\sqcap\, q_\tau+(2i)^{-1}B^{\sqcap}_\tau(p_\tau,q_\tau)+R_\tau(p,q) \tag{4.1}$$

where

$$B_\tau^{\sqcup}(p,q)(\xi)=i\int e^{i\langle\eta-\xi,y\rangle}(\operatorname{ad}y^*\tau)\cdot\nabla p(\xi+\tfrac{1}{2}\operatorname{ad}y^*\xi)q(\eta)\,dy\,d\eta$$

and where $\lim_{\tau\to 0}|\tau|^{-1}R_\tau(p,q)=0$ uniformly on compact subsets of $\mathscr{G}^*$. By (1.2) the jth component of $\operatorname{ad}y^*\tau$ is $\sum\gamma_{lj}^k\tau_k y_l$, and consequently by (2.2)

$$B_\tau^{\sqcup}(p,q)=\sum\gamma_{jl}^k\tau_k\partial_j p\sqcup\partial_l q. \tag{4.2}$$

Given $\xi\in\Gamma_c'$ there is a $t_0>0$ such that if $\tau=te_k^*$, where e_k^* is one of the chosen basis vectors for $\mathscr{G}^*$ and $|t|<t_0$, then

$$(p\sqcup q)_\tau(\xi)=p\sqcup q(\xi)=1.$$

If $k\leqq N=\dim\mathscr{G}_1$, then

$$p\sqcup(q_\tau-q)=-(p_\tau-p)\sqcup q_\tau.$$

Without loss of generality we may assume that $\pi_\xi(Q)$ is a two sided inverse for $\pi_\xi(P)$, for if not P can be multiplied on the left by its adjoint. Therefore, $q_\tau-q=-q\sqcup(p_\tau-p)\sqcup q_\tau$ and hence $D_kq=-q\sqcup D_kp\sqcup q\in S_{00}^{-m-\mu_k}(\Gamma_c',\delta)$. Proceeding by induction on $|\alpha|$, we find that $D^\alpha q\in S_{00}^{-m-\mu\alpha}(\Gamma_c',\delta)$ for all α such that $\alpha'=0$. Multiplying q by a cut-off function as in Lemma 4.3 we may also assume that $q\in S_0^{-m}(\mathscr{G}^*,\delta)$.

Similarly, for $k>N$ we use (4.1) to find that $D_kq(\xi)$ exists for $\xi\in\Gamma_c'$ and

$$D_kq=-q\sqcup D_kp\sqcup q+\tfrac{1}{2}q\sqcup B_k^{\sqcup}(p,q), \tag{4.3}$$

where $B_k^{\sqcup}=B_\tau^{\sqcup}$ for $\tau=e_k^*$. By (1.3), (4.2) and (4.3) we find that

$$|D_kq(\xi)|<C\chi(\xi)^{-m-\mu_k},\quad \xi\in\Gamma_c'.$$

By an inductive argument $q\in S^{-m}(\Gamma_c',\delta)$. Multiplying q by a cut-off function $\varphi\in S^0(\mathscr{G}^*,\delta)$ which is 1 on a δ-cone containing ξ_0, the proof of Theorem 4.1 for left invariant P now follows from Propositions 2.4 and 2.6.

Turning to the proof of the theorem for non-invariant P, let $(g_0,\xi_0)\in\Gamma$. As noted in Section 2 there is a neighborhood ω_1 of g_0 and an open δ-cone Γ_1 containing $\xi_1=\operatorname{Ad}\tfrac{1}{2}g_0^*\xi_0$ such that $\omega_1^*\Gamma_1\subset\Gamma$. If $\eta\in\Gamma_1$ and $g\in\omega_1$, then $\eta=\operatorname{Ad}\tfrac{1}{2}g^*\xi\in\mathcal{O}_\xi$ for some ξ such that $(g,\xi)\in\Gamma$ and therefore $\pi_\eta(P_g^0)\cong\pi_\xi(P_g^0)$ is injective and $\pi_{\bar\eta}(P_g^0)\neq 0$ for all $\bar\eta\in\partial\Gamma_\eta=\partial\Gamma_\xi$, $[\bar\eta]\geqq C$. The theorem having been proved in the invariant case, for each $g\in\omega_1$ there is a q_g^0 such that $q_g^0\sqcup p_g=1$ for all $\eta\in\Gamma_1$ such that $[\eta]\geqq C$. Define $q^0(g,\xi)=\varphi(g)q_g^0(\xi)$ where $\varphi\in C_0^\infty(\omega_1)$, $\varphi=1$ on a neighborhood ω_2 of g_0. Let $r=1-q^0\sqcup p$. Then $q^0\in S^{-m}(\Omega\times\mathscr{G}^*,\delta)$ and $r\in S^{-M}(\omega_2\times\Gamma_1,\delta)$ for all M, the appropriate estimates for derivatives along G being established by the same methods as above. For positive integers k define

$$q^k=-\sum_{0<|\alpha|\leq k}(\alpha!)^{-1}\partial_\xi^\alpha q^{k-|\alpha|}\sqcup T_g^\alpha p\sqcup q^0$$

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and chooses $q\in S^{-m}(\Omega\times\mathscr{G}^*,\delta)$ such that $q\sim\sum_{k=0}^{\infty}q^k$. Using Theorem 1.2 it follows by standard arguments that $q\#p-1\in S^{-M}(\omega_2\times\Gamma_1,\delta)$ for all M. Thus P is δ-microhypoelliptic in a neighborhood of (g_0,ξ_0).

Corollary 4.5. *Let $\delta=\{\delta_r:r>0\}$ be natural dilations on a step two nilpotent Lie group G, i.e. $\delta_r x=r^2x$ for $x\in\mathscr{G}_2$, $\delta_r x=rx$ for $x\in\mathscr{G}_1$. Let $p\in S^m(\Omega\times\mathscr{G}^*,\delta)$ have principal symbol p^0. Let $(g_0,\xi_0)\in\Omega\times\tilde{\mathscr{G}}^*$. If $\pi_{\xi_0}(P^0_{g_0})$ is injective and $\pi_\eta(P^0_{g_0})\neq 0$ for all $\eta\in R^\perp_{\xi_0}$ such that $[\eta]\geqq C$, then P is δ-microhypoelliptic in a δ-conic neighborhood of (g_0,ξ_0).*

Proof. For $\xi\in\mathscr{G}^*$ let $\chi_\xi=\chi\circ\psi_\xi,\psi_\xi$ as is Section 1, and let $H^m_\xi(R^d)$ be the Sobolev space as defined in [1] corresponding to the weight functions $\Phi=\chi_\xi$, $\varphi=1$ and order $m\log\chi_\xi$. If ξ is near ξ_0, then $\chi_\xi\approx\chi_{\xi_0}$ and hence $H^m_\xi=H^m_{\xi_0}$. By using the Mean Value Theorem and a simple perturbation argument one sees that there is a neighborhood U of (g_0,ξ_0) such that $\pi_\xi(P^0_g):H^m_\xi(R^d)\to L^2(R^d)$ is injective for all $(g,\xi)\in U$. Furthermore, since the subspace R_ξ depends continuously on ξ for $\xi\in\tilde{\mathscr{G}}^*$, there is a neighborhood U of (g_0,ξ_0) such that $\pi_\eta(P^0_g)\neq 0$ for all $(g,\xi)\in U$, $\eta\in R^\perp_\xi$, $[\eta]\geqq C$. Hence the corollary follows from Theorem 4.1 and Corollary 3.4.

The proof of the corollary breaks down for arbitrary dilations since χ_ξ is not necessarily equivalent to χ_{ξ_0} for ξ near ξ_0. Furthermore, $\partial\Gamma_\xi$ does not necessarily depend continuously on $\xi\in\tilde{\mathscr{G}}^*$, as mentioned at the end of Section 3.

The next corollary gives a sufficient condition for microhypoellipticity in the standard sense. For simplicity the result will be stated for left invariant operators only. A subset $\Lambda\subset\dot{\mathscr{G}}^*$ is a cone if $\xi\in\Lambda$ implies $r\xi\in\Lambda$ for all $r>0$. Define the standard wave front set $WF(u)$ of a distribution as in [5] or [20] and define microhypoellipticity analogously. If $P\in\mathrm{Op}S^m(\mathscr{G}^*,\delta)$ and $\pi_\xi(P^0)$ is injective for all $\xi\in\Lambda$, then $\pi_\eta(P^0)$ is automatically injective for all η in the smallest G-invariant δ-cone Γ' containing Λ. In order to apply the parametrix construction of Theorem 4.1 we will need to assume injectivity on a δ-cone Γ containing Γ' properly.

Corollary 4.6. *Let $p\in S^m(\mathscr{G}^*,\delta)$ have principal symbol p^0. Let $\Gamma\in\tilde{\mathscr{G}}^*$ be a G-invariant δ-cone such that for $\xi\in\Gamma$ and all $\eta\in\bar{\Gamma}_\xi$, $\pi_\eta(P^0)$ is injective. Let Λ be an open standard cone such that $\Lambda\subseteq\Gamma'$ where Γ' is a G-invariant δ-cone with $\Gamma'\subseteq\Gamma$ properly. Then P is microhypoelliptic on Λ.*

Proof. By the construction in the proof of Theorem 4.1 we obtain $q\in S^{-m}(\Gamma'_C,\delta)\cap S_0^{-m}(\mathscr{G}^*,\delta)$ with $q\,\square\, p=1$ on Γ'_C. There is a C_1 such that if $\xi\in\Lambda$ and $|\xi|\geqq C_1$, then $[\xi']\geqq C$. Given $\xi_0\in\Lambda$ there is a function $\varphi\in C^\infty(\mathscr{G}^*)$ and a conic neighborhood Λ_1 of ξ_0 such that $\varphi(\xi)=1$ if $\xi\in\Lambda_1$ and $|\xi|\geqq 2C_1$, $\operatorname{supp}\varphi\subset\Gamma'_C$, $\varphi(r\xi)=\varphi(\xi)$ if $|\xi|\geqq 2C_1$ and $r\geqq 1$. Let $q'=\varphi q$. Then $Q'\in\mathrm{Op}S^{-m}_{\rho,1-\rho}$ with $\rho=1/\bar{\mu}$, and $Q'P-I$ is regularizing on Λ_1. Since $Q\in\mathrm{Op}S^{-m}_{\rho,1-\rho}$, it follows from Theorem VI.1.6 of [20] that $WF(Q'Pu)\subseteq WF(Pu)$. Thus P is microhypoelliptic on Λ.

As a simple example, let $\mathscr{G}$ be the Heisenberg algebra with basis vectors satisfying $[e_j,e_{j+n}]=e_{2n+1}$ if $j\leqq n$. Let $\Lambda=\Gamma'=\Gamma=\{\xi:\xi_{2n+1}>0\}$. Let $\xi_0=(0,\dots,0,1)$. If $\pi_{\xi_0}(P^0)$ is injective and $\pi_\eta(P^0)\neq 0$ for all $\eta\in\mathscr{G}^*_1$, then P is microhypoelliptic on Λ. For example, if $p(\xi)=\xi_1^2+\cdots+\xi_{2n}^2+\xi_{2n+1}$, then $\mathrm{Op}(p)$ is microhypoelliptic on Λ, but is not hypoelliptic on G.

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